

MATHEMATICAL ANALYSIS OF A DIMENSIONALLY REDUCED VISCOELASTIC PROBLEM WITH LOGARITHMIC SOURCE AND NONLINEAR FRICTION

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ABSTRACT: This paper studies the dimensional reduction of a nonlinear viscoelastic problem posed in a thin domain $\mathcal{D}^\xi \subset \mathbb{R}^3$, featuring a logarithmic source term and mixed boundary conditions. The model combines a Dirichlet boundary condition on one portion of the boundary with non-linear slip conditions described by a Tresca friction law on the remaining parts. Using asymptotic analysis, we rigorously derive the limiting behavior of the solution as the thickness parameter ξ approaches zero. Our results show that the three-dimensional solution converges to a well-defined two-dimensional limit problem, where the effects of the Tresca friction are preserved in the reduced formulation. Additionally, we prove the uniqueness of the solution to the limiting problem, ensuring the well-posedness of the derived model.

KEY WORDS: Logarithmic source term; Viscoelastic contact problem; Asymptotic behavior; Tresca friction law.

1 INTRODUCTION

In the field of continuum mechanics, materials frequently defy simple classification. While purely elastic solids conserve mechanical energy during deformation and ideal viscous fluids irreversibly dissipate it, many real-world substances exhibit a hybrid response: they partially store and partially lose energy when subjected to external forces. This intermediate behaviour, known as viscoelasticity, has been observed in a wide spectrum of physical systems, ranging from biological tissues such as tendons and cartilage to synthetic polymers and even certain molten alloys. In all of these systems, the balance between resilience and damping plays a critical role in function and performance.

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Mathematically, this duality can be modelled using equations that incorporate both memory effects and energy dissipation. These equations are often described as partial differential equations with convolution-type viscoelastic terms. A particularly rich class of such models includes nonlinear source terms with logarithmic dependence. These have emerged as powerful tools for representing complex interactions in contexts ranging from quantum field theory to geophysical dynamics and nonlinear optics. Though subtle in form, these logarithmic nonlinearities introduce significant analytical challenges due to their slow growth and sign-changing nature. However, they offer greater fidelity in modelling long-range interactions and phase transitions.

Recent investigations have explored the well-posedness and asymptotic properties of viscoelastic systems featuring logarithmic sources. Notably, studies by Mezouar [1], Choucha [2], Cordeiro [3] and others (see [4–9]) have provided evidence of the existence of solutions, determined the rates at which energy decays, and identified stabilisation mechanisms for various structural equations, including Kirchhoff, Lamé and pseudo-parabolic systems. These studies have often employed techniques such as the potential well method, Nakao's inequality or semigroup analysis. Extensions to stochastic settings and fractional dissipation have further broadened the scope of these studies, revealing the influence of noise and nonlocal effects on solution behaviour.

Alongside these developments, another significant area of research involves reducing the dimensions of physical models in domains with singular geometries, particularly thin structures where one spatial dimension becomes asymptotically small. These configurations naturally arise in micro-electromechanical systems, layered composites and biological membranes. The challenge in these settings is to derive effective lower-dimensional models that accurately reflect the full three-dimensional physics. This is typically achieved through asymptotic analysis as the thickness parameter ξ approaches zero, leading to limit problems defined on surfaces or lines with modified constitutive laws and boundary interactions.

Several authors, including Dilmi [10–14], Bayada [15] and Rodríguez-Arós [16, 17] and their collaborators, have developed this area of research by analysing the convergence of solutions in elastic and viscoelastic thin media under various contact conditions. They have treated problems involving frictional interfaces, nonlinear dissipation and mixed boundary constraints using asymptotic expansion methods and energy estimates. This has yielded justified reduced models with emergent transmission conditions.

Building on this foundation, the present work investigates a viscoelastic system in a three-dimensional thin domain $\mathcal{D}^\xi \subset \mathbb{R}^3$, where $0 < \xi < 1$ denotes a small thickness parameter. The model incorporates a logarithmic source term and is subject to Tresca friction on part of the boundary, a condition that introduces additional

non-smoothness and requires careful treatment within a variational framework. Our objective is twofold: first, to prove the existence and uniqueness of a weak solution for each fixed ξ ; and second, to carry out a rigorous asymptotic analysis as ξ approaches 0, in order to identify the limiting two-dimensional problem and prove the convergence of the rescaled solutions.

In line with previous studies, the present work may be regarded as the first to rigorously analyse the combined effects of the Tresca friction law and a logarithmic source term within the framework of dimension reduction for viscoelastic systems. In contrast to earlier contributions, the proposed scaling is specifically designed to ensure that both the frictional effects and the logarithmic nonlinearity remain effective in the asymptotic limit. This leads to a non-trivial two-dimensional limit problem in which the Tresca friction condition is preserved on the reduced boundary, thereby capturing the essential mechanical features of the original three-dimensional model.

From an application perspective, the proposed model is directly motivated by the study of adhesive materials and thin viscoelastic layers arising in biomechanical and industrial settings, such as articular cartilage in joints and micro-electromechanical systems (MEMS). In these contexts, the logarithmic nonlinearity provides an effective representation of long-range interactions and nonlinear internal responses, while the Tresca friction law describes threshold-dependent slip behaviour at contact interfaces. The resulting reduced model offers a computationally efficient yet physically reliable framework for simulating thin viscoelastic structures, allowing engineers and applied scientists to accurately capture the essential behaviour of such systems without resolving the full three-dimensional geometry.

The paper unfolds as follows. Section 2 introduces the governing equations and the geometric setting. Section 3 derives the weak formulation and proves well-posedness via a combination of Faedo–Galerkin approximation and energy arguments. It also applies a suitable scaling to reformulate the problem on a fixed reference domain. Section 4 develops uniform estimates with respect to ξ and establishes the compactness necessary for passing to the limit. Finally, Section 5 derives the effective limit problem, characterises its structure and demonstrates the uniqueness of its solution.

2 MATHEMATICAL FORMULATION OF THE PROBLEM

In order to formulate the problem of nonlinear viscoelasticity with a logarithmic nonlinearity source term, we start with the geometry of the domain. Let $\mathcal{D}^\xi$ be a bounded domain of $\mathbb{R}^3$, where $0 < \xi < 1$ is a small positive parameter. The boundary of $\mathcal{D}^\xi$ will be denoted $\partial\mathcal{D}^\xi = \Lambda \cup \partial\mathcal{D}_1^\xi \cup \partial\mathcal{D}_L^\xi$, where: Λ is a fixed bounded domain in $\mathbb{R}^2$, characterized by $x_3 = 0$, which represents the bottom of $\mathcal{D}^\xi$, $\partial\mathcal{D}_1^\xi$ is the upper

surface defined by $x_3 = \xi G(y)$ with $y = (x_1, x_2) \in \mathbb{R}^2$ and the function $G(\cdot)$ is defined on Λ of class C^1 , such that

$$0 < \underline{G} = G_{\min} \leq G(y) \leq G_{\max} = \bar{G}, \forall y \in \Lambda.$$

$\mathcal{D}_L^\xi$ is the lateral surface of the domain $\mathcal{D}^\xi$. The thin domain $\mathcal{D}^\xi$ is given by

$$\mathcal{D}^\xi = \{(y, x_3) \in \mathbb{R}^3 : y \in \Lambda, 0 < x_3 < \xi G(y)\}.$$

Let $\omega^\xi(x, t) = \left(\omega_i^\xi(x, t)\right)_{1 \leq i \leq 3}$ denote the displacement of the viscoelastic body at position x and time $t \in \mathcal{I} = [0, T]$. The strain tensor is denoted by $\mathbb{D}_{ij}(u^\xi)$, where

$$\mathbb{D}_{ij}(\omega^\xi) = \frac{1}{2} \left(\frac{\partial \omega_i^\xi}{\partial x_j} + \frac{\partial \omega_j^\xi}{\partial x_i} \right), i, j \in \{1, 2, 3\}.$$

The relationship between stress and strain can be expressed as

$$\sigma^\xi(\omega^\xi) = 2\mu \mathbb{D}(\omega^\xi) + \vartheta \mathbb{D} \left(\frac{\partial \omega^\xi}{\partial t} \right).$$

Here μ is Lamé's coefficient and ϑ is the viscosity coefficient.

The normal and the tangential components of ω^ξ along the boundary can be expressed as

$$\omega_n^\xi = \omega^\xi \cdot n, \quad \omega_\tau^\xi = \omega^\xi - \omega_n^\xi \cdot n.$$

Similarly, the normal and tangential components of σ^ξ on the boundary are defined as:

$$\sigma_n^\xi = (\sigma_n^\xi \cdot n) \cdot n, \quad \sigma_\tau^\xi = \sigma^\xi \cdot n - (\sigma_n^\xi) \cdot n,$$

where n the exterior normal on $\partial \mathcal{D}^\xi$.

The dynamics of a three-dimensional viscoelastic body, including source effects, is described by the following governing equations

$$(1) \quad \rho \frac{\partial^2 \omega_i^\xi}{\partial t^2} - \sum_{j=1}^3 \frac{\partial \sigma_{ij}^\xi(\omega^\xi)}{\partial x_j} - \alpha^\xi \Upsilon_i(\omega_i^\xi) \ln(|\omega_i^\xi|) = \mathfrak{F}_i^\xi, 1 \leq i \leq 3, \text{ in } \mathcal{D}^\xi \times \mathcal{I},$$

where ρ is the volumetric mass density, $\mathfrak{F}^\xi = \left(\mathfrak{F}_i^\xi\right)_{1 \leq i \leq 3}$ represents a density of external forces and α^ξ is positive constant.

Our boundary conditions are described as follows:

- The displacement is on $\partial \mathcal{D}_1^\xi \times \mathcal{I}$, $\partial \mathcal{D}_L^\xi \times \mathcal{I}$ satisfies the Dirichlet condition

$$(2) \quad \omega^\xi = 0.$$

- On $\Lambda \times \mathcal{I}$ the normal component of the velocity satisfies

$$(3) \quad \frac{\partial \omega^\xi}{\partial t} \cdot n = 0.$$

- The tangential viscosity on $\Lambda \times \mathcal{I}$ is unknown and governed by the Tresca friction law

$$(4) \quad \left. \begin{aligned} & \left| \sigma_\tau^\xi \right| < k^\xi \Rightarrow \left(\frac{\partial \omega^\xi}{\partial t} \right)_\tau = 0; \\ & \left| \sigma_\tau^\xi \right| = k^\xi \Rightarrow \exists \kappa > 0, \text{ such that } \left(\frac{\partial \omega^\xi}{\partial t} \right)_\tau = -\kappa \sigma_\tau^\xi \end{aligned} \right\},$$

where $|\cdot|$ represents the Euclidean norm in $\mathbb{R}^2$, and k^ξ is a prescribed function.

The problem (1) – (4) is complemented by the following initial condition

$$(5) \quad \omega^\xi(y, x_3, 0) = \omega_0(y, x_3), \quad \frac{\partial \omega^\xi(y, x_3, 0)}{\partial t} = \omega_1(y, x_3) \quad \forall (y, x_3) \in \mathcal{D}^\xi.$$

Finally, we present some conditions satisfied by the logarithmic source term

$$\Upsilon(\omega^\xi) \ln(|\omega^\xi|) = \left(\Upsilon_i(\omega_i^\xi) \ln(|\omega_i^\xi|) \right)_{1 \leq i \leq 3}.$$

where $\Upsilon_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{1, 2, 3\}$ are continuous functions that satisfies the following conditions:

- $\Upsilon_i(\xi) \ln(|\xi|)$, $i \in \{1, 2, 3\}$ are a continuous functions for all $\xi \in \mathbb{R}$.
- There exists a positive constant C_Υ such that

$$|\Upsilon(\zeta) \ln(|\zeta|)|^2 \leq C_\Upsilon (|\zeta|^2 + 1), \text{ for all } \zeta \in \mathbb{R}^3.$$

3 NOTATIONS AND WEAK FORMULATION

To obtain the variational formulation of the problem. We will write $\|\cdot\|_{L^2(\mathcal{D}^\xi)}$ and $\|\cdot\|_{L^p(\mathcal{D}^\xi)}$, ($1 < p < +\infty$) for the usual $L^2(\mathcal{D}^\xi)$ norm and $L^p(\mathcal{D}^\xi)$ norm, respectively. We will use the standard Lebesgue space $L^2(\mathcal{D}^\xi)$ with the inner product

$$(\omega, v) = \int_{\mathcal{D}^\xi} \omega \cdot v \, dy \, dx_3,$$

and the norm

$$\|\omega\|_{L^2(\mathcal{D}^\xi)} = \left(\int_{\mathcal{D}^\xi} |\omega|^2 \, dy \, dx_3 \right)^{\frac{1}{2}},$$

$H^1(\mathcal{D}^\xi)^3$ Sobolev space defined by

$$H^1(\mathcal{D}^\xi)^3 = \left\{ \omega \in L^2(\mathcal{D}^\xi)^3 : \frac{\partial \omega_i}{\partial x_j} \in L^2(\mathcal{D}^\xi), i, j = 1, 2, 3 \right\},$$

$H_0^1(\mathcal{D}^\xi)^3$ the closure of $\mathcal{D}(\mathcal{D}^\xi)^3$ in $H^1(\mathcal{D}^\xi)^3$, and $H^{-1}(\mathcal{D}^\xi)^3$ the dual space of $H_0^1(\mathcal{D}^\xi)^3$.

Let $\mathfrak{B}$ a Banach space endowed with the norm $\|\cdot\|_{\mathfrak{B}}$. We denote by $L^p(\mathcal{I}; \mathfrak{B})$ the space of functions $\omega : \mathcal{I} \rightarrow \mathfrak{B}$ such that $\|\omega\|_{\mathfrak{B}} \in L^p(\mathcal{I})$, which is a Banach space equipped with the norm:

$$\|\omega\|_{L^p(0,T;\mathfrak{B})} = \left(\int_{\mathcal{I}} \|\omega(s)\|_{\mathfrak{B}}^p ds \right)^{\frac{1}{p}}.$$

$L^\infty(\mathcal{I}; \mathfrak{B})$ denotes the space of functions $\omega : \mathcal{I} \rightarrow \mathfrak{B}$ which are measurable and $\|\omega\|_{\mathfrak{B}} \in L^\infty(\mathcal{I})$. This space is a Banach space endowed with the norm

$$\|\omega\|_{L^\infty(\mathcal{I}; \mathfrak{B})} = \operatorname{ess\,sup}_{s \in]0,T[} \|\omega(s)\|_{\mathfrak{B}}.$$

Lemma 1. [18]. *The condition of Tresca (4) is equivalent to*

$$\sigma_{\tau^\xi}^\xi \left(\omega^\xi \right)_\tau + k^\xi \left| \left(\frac{\partial \omega^\xi}{\partial t} \right)_\tau \right| = 0 \text{ on } \Lambda \times \mathcal{I}.$$

We multiply the equation (1) by $(\varphi - \omega^\xi)$ and integrating over $\mathcal{D}^\xi$, then using (2) – (4) and the integration by parts, we get the following variational result:

Problem ($\mathcal{P}_v$). Find $\omega^\xi \in \mathbb{W}^\xi$, $\forall t \in \mathcal{I}$, such that

$$\begin{aligned} (6) \quad & \rho \left(\frac{\partial^2 \omega^\xi}{\partial t^2}, \varphi - \frac{\partial \omega^\xi}{\partial t} \right) + \mathcal{A}_\mu \left(\omega^\xi, \varphi - \frac{\partial \omega^\xi}{\partial t} \right) + \mathcal{A}_\vartheta \left(\frac{\partial \omega^\xi}{\partial t}, \varphi - \frac{\partial \omega^\xi}{\partial t} \right) \\ & - \alpha^\xi \left(\Upsilon(\omega^\xi) \ln(|\omega^\xi|), \varphi - \frac{\partial \omega^\xi}{\partial t} \right) + j^\xi(\varphi) \\ & - j^\xi \left(\frac{\partial \omega^\xi}{\partial t} \right) \geq \left(\mathfrak{F}^\xi, \varphi - \frac{\partial \omega^\xi}{\partial t} \right), \quad \forall \varphi \in \mathbb{W}^\xi, \end{aligned}$$

$$\omega^\xi(y, x_3, 0) = \omega_0(y, x_3), \quad \frac{\partial \omega^\xi}{\partial t}(y, x_3, 0) = \omega_1(y, x_3),$$

where

$$\mathbb{W}^\xi = \left\{ v \in H^1(\mathcal{D}^\xi)^3 : v = 0 \text{ on } \partial \mathcal{D}_1^\xi \cup \partial \mathcal{D}_L^\xi, v.n = 0 \text{ on } \Lambda \right\};$$

$$\mathcal{A}_{\Phi}(\omega, v) = 2\Phi \int_{\mathcal{D}^{\xi}} \mathbb{D}(\omega) : \mathbb{D}(v) dy dx_3; \mathbb{D}(\omega) : \mathbb{D}(v) := \sum_{i,j=1}^3 \mathbb{D}_{ij}(\omega) \mathbb{D}_{ij}(v);$$

$$j^{\xi}(v) = \int_{\Lambda} k^{\xi} |v| dy; (\partial, v) = \int_{\mathcal{D}^{\xi}} \partial \cdot v dy dx_3.$$

The existence and uniqueness results of (6) are established in the following theorem.

Theorem 1. *Under the following assumptions*

$$\partial^{\xi}, \frac{\partial \partial^{\xi}}{\partial t} \in L^2 \left(\mathcal{I}; L^2 \left(\mathcal{D}^{\xi} \right)^3 \right),$$

$$k^{\xi} \in L^{\infty}(\Lambda), k^{\xi} > 0,$$

$$\omega_0, \omega_1 \in H^1(\mathcal{D}^{\xi})^3,$$

there exists a unique solution ω^{ξ} to problem (6) with

$$\omega^{\xi}, \frac{\partial \omega^{\xi}}{\partial t} \in L^{\infty} \left(\mathcal{I}; H^1(\mathcal{D}^{\xi})^3 \right); \frac{\partial^2 \omega^{\xi}}{\partial t^2} \in L^2 \left(\mathcal{I}; L^2(\mathcal{D}^{\xi})^3 \right).$$

Proof. The proof relies on the regularization method, which approximates the non-differentiable term $j^{\xi}(\cdot)$ by a family of differentiable functions $j_{\zeta}^{\xi}(\cdot)$, where

$$j_{\zeta}^{\xi}(v) = \int_{\Lambda} k^{\xi}(y) \sqrt{|v_{\tau}|^2 + \zeta^2}, \text{ where } 0 < \zeta < 1.$$

It is clear that $j_{\zeta}^{\xi}(\cdot)$ is convex and Gateaux differentiable with its Gateaux derivative $(j_{\zeta}^{\xi})'$ defined on $\mathbb{W}^{\xi}$ and given by

$$((j_{\zeta}^{\xi})'(v), \varphi) = \int_{\Lambda} k^{\xi}(y) \frac{v_{\tau} \cdot \varphi_{\tau}}{\sqrt{|v_{\tau}|^2 + \zeta^2}} dy.$$

Since $j_{\zeta}^{\xi}(\cdot)$ is differentiable, adopting the classical arguments in [18], one can state that (6) is equivalent to:

Find $\omega_{\zeta}^{\xi}(t) \in L^2(\mathcal{I}; \mathbb{W}^{\xi})$, such that for almost all t

$$(7) \quad \rho \left(\frac{\partial^2 \omega_{\zeta}^{\xi}(t)}{\partial t^2}, \varphi \right) + \mathcal{A}_{\mu} \left(\omega_{\zeta}^{\xi}(t), \varphi \right) + \mathcal{A}_{\vartheta} \left(\frac{\partial \omega_{\zeta}^{\xi}(t)}{\partial t}, \varphi \right)$$

$$\begin{aligned}
& -\alpha^\xi \left(\Upsilon \left(\omega_\zeta^\xi(t) \right) \ln \left(\left| \omega_\zeta^\xi(t) \right| \right), \varphi \right) + \left(\left(j_\zeta^\xi \right)' \left(\frac{\partial \omega_\zeta^\xi(t)}{\partial t} \right), \varphi \right) \\
& = \left(\tilde{\partial}^\xi(t), \varphi \right), \quad \forall \varphi \in \mathbb{W}^\xi.
\end{aligned}$$

Now, using the Faedo-Galerkin procedures as in [18] and [19] to prove that the problem (7) has a solution. $\square$

3.1 THE PROBLEM IN A FIXED DOMAIN

The aim of the asymptotic analysis is to ascertain the behaviour of the solution ω^ξ of the problem (6) as ξ approaches zero. For that it is significant that the domain on which is posed the problem does not depend of ξ . According to the change of variables $z = x_3/\xi$, we define the fixed domain

$$\mathcal{D} = \{(y, z) \in \mathbb{R}^3 : y \in \Lambda, 0 < z < G(y)\},$$

and we denote by $\partial\mathcal{D} = \Lambda \cup \partial\mathcal{D}_1 \cup \partial\mathcal{D}_L$ its boundary. We define on $\mathcal{D}$ the new unknowns

$$\omega_i^\xi(y, z, t) = \omega_i^\xi(y, x_3, t), \quad i = 1, 2, \quad \omega_3^\xi(y, z, t) = \xi^{-1} \omega_3^\xi(y, x_3, t).$$

The data of problem (6) are assumed to depend on ξ as follows

$$\hat{\partial}(y, z, t) = \xi^2 \tilde{\partial}^\xi(y, x_3, t), \quad \hat{k} = \xi k^\xi, \quad i = 1, 2; \quad \hat{\alpha} = \xi^2 \alpha^\xi,$$

with $\tilde{\partial}$, $\hat{k}$ and $\hat{\alpha}$ does not depend on ξ .

We now introduce the functional framework on $\mathcal{D}$ as follows:

$$\mathbb{W} = \left\{ \varphi \in H^1(\mathcal{D})^3 : \varphi = 0 \text{ on } \partial\mathcal{D}_1 \cup \partial\mathcal{D}_L \text{ and } \varphi \cdot n = 0 \text{ on } \Lambda \right\},$$

$$\Sigma = \left\{ \varphi \in H^1(\mathcal{D})^2 : \varphi = (\varphi_1, \varphi_2), \varphi_i = 0 \text{ on } \partial\mathcal{D}_1 \text{ for } i = 1, 2 \right\}.$$

The problem (6) is reformulated on the fixed domain $\mathcal{D}$ as follows:

Problem $(\hat{\mathcal{P}}_v)$. Find $\omega^\xi \in \mathbb{W}, \forall t \in \mathcal{I}$ such that

$$\begin{aligned}
(8) \quad & \rho_\xi^2 \left(\frac{\partial^2 \omega_i^\xi}{\partial t^2}, \varphi_i - \frac{\partial \omega_i^\xi}{\partial t} \right) + \rho_\xi^4 \left(\frac{\partial^2 \omega_3^\xi}{\partial t^2}, \varphi_3 - \omega_3^\xi \right) + \mathcal{A}_\mu \left(\omega^\xi, \hat{\varphi} - \frac{\partial \omega^\xi}{\partial t} \right) \\
& + \mathcal{A}_\vartheta \left(\frac{\partial \omega^\xi}{\partial t}, \hat{\varphi} - \frac{\partial \omega^\xi}{\partial t} \right) + \hat{\alpha} \sum_{i=1}^2 \left(\Upsilon_i \left(\omega_i^\xi \right) \ln \left(\left| \omega_i^\xi \right| \right), \varphi_i - \frac{\partial \omega_i^\xi}{\partial t} \right)
\end{aligned}$$

$$\begin{aligned}
& -\hat{\alpha}\xi^2 \left(\Upsilon_3 \left(\xi \omega_3^\xi \right) \ln \left(\left| \xi \omega_3^\xi \right| \right), \varphi_3 - \frac{\partial \omega_3^\xi}{\partial t} \right) + \mathcal{J}(\hat{\varphi}) - \mathcal{J} \left(\frac{\partial \omega^\xi}{\partial t} \right) \\
& \geq \sum_{i=1}^2 \left(\hat{\partial}_i, \hat{\varphi}_i - \frac{\partial \omega_i^\xi}{\partial t} \right) + \xi \left(\hat{\partial}_3, \hat{\varphi}_3 - \frac{\partial \omega_3^\xi}{\partial t} \right), \forall \varphi \in \mathbb{W},
\end{aligned}$$

$$\omega^\xi(x, 0) = \omega_0(x); \frac{\partial \omega^\xi(x, 0)}{\partial t} = \omega_1(x),$$

where

$$\mathcal{J}(\hat{\varphi}) = \int_{\Lambda} \hat{k} |\hat{\varphi}| dy,$$

and

$$\begin{aligned}
\mathcal{A}_{\Phi} \left(\omega^\xi, \hat{\varphi} - \omega^\xi \right) &= \Phi \xi^2 \sum_{i,j=1}^2 \int_{\mathcal{D}} \left(\frac{\partial \omega_i^\xi}{\partial x_j} + \frac{\partial \omega_j^\xi}{\partial x_i} \right) \frac{\partial (\hat{\varphi}_i - \omega_i^\xi)}{\partial x_j} dy dz \\
&+ \Phi \sum_{i=1}^2 \int_{\mathcal{D}} \left(\frac{\partial \omega_i^\xi}{\partial z} + \xi^2 \frac{\partial \omega_3^\xi}{\partial x_i} \right) \left[\frac{\partial}{\partial z} (\hat{\varphi}_i - \omega_i^\xi) + \xi^2 \frac{\partial}{\partial x_i} (\hat{\varphi}_3 - \omega_3^\xi) \right] dy dz \\
&+ 2\Phi \xi^2 \int_{\mathcal{D}} \frac{\partial \omega_3^\xi}{\partial z} \cdot \frac{\partial (\hat{\varphi}_3 - \omega_3^\xi)}{\partial z} dy dz.
\end{aligned}$$

4 A PRIORI ESTIMATES AND CONVERGENCE RESULTS

In order to derive a priori estimates and convergence results for the solutions to the problem 8, it is necessary to consider the following two inequalities:

Korn's inequality

$$(9) \quad C_K \left\| \nabla \omega^\xi \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}} \leq \left\| \mathbb{D} \left(\omega^\xi \right) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}, (C_K \text{ independent of } \xi).$$

Poincaré's inequality

$$(10) \quad \left\| \omega_i^\xi \right\|_{L^2(\mathcal{D}^\xi)^3} \leq \xi \bar{G} \left\| \frac{\partial \omega_i^\xi}{\partial x_3} \right\|_{L^2(\mathcal{D}^\xi)^3}, i = 1, 2, 3.$$

Theorem 2. *There exists a constant c that does not depend on ξ , such that the following estimates*

$$\begin{aligned}
(11) \quad \sum_{i=1}^2 \left(\left\| \frac{\partial \omega_i^\xi}{\partial z} \right\|_{L^2(\mathcal{D})}^2 + \left\| \xi^2 \frac{\partial \omega_3^\xi}{\partial x_i} \right\|_{L^2(\mathcal{D})}^2 + \left\| \xi \frac{\partial \omega_i^\xi}{\partial t} \right\|_{L^2(\mathcal{D})}^2 \right) &+ \sum_{i,j=1}^2 \left\| \xi \frac{\partial \omega_i^\xi}{\partial x_j} \right\|_{L^2(\mathcal{D})}^2 \\
&+ \left\| \xi \frac{\partial \omega_3^\xi}{\partial z} \right\|_{L^2(\mathcal{D})}^2 + \left\| \xi \frac{\partial \omega_3^\xi}{\partial t} \right\|_{L^2(\mathcal{D})}^2 \leq c,
\end{aligned}$$

$$(12) \quad \sum_{i=1}^2 \left(\left\| \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^\xi}{\partial t} \right) \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2 + \left\| \xi^2 \frac{\partial}{\partial x_i} \left(\frac{\partial \omega_3^\xi}{\partial t} \right) \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2 \right) \\ + \sum_{i,j=1}^2 \left\| \xi \frac{\partial}{\partial x_j} \left(\frac{\partial \omega_i^\xi}{\partial t} \right) \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2 + \left\| \xi \frac{\partial}{\partial z} \left(\frac{\partial \omega_3^\xi}{\partial t} \right) \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2 \leq c,$$

$$(13) \quad \sum_{i=1}^2 \left\| \xi^2 \frac{\partial^2 \omega_i^\xi}{\partial t^2} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2 + \left\| \xi^3 \frac{\partial^2 \omega_3^\xi}{\partial t^2} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}))}^2.$$

Proof. Let ω^ξ be the solution of problem (6). Setting $\varphi = (0, 0, 0)$, we obtain

$$\rho \left(\frac{\partial^2 \omega^\xi}{\partial t^2}, \frac{\partial \omega^\xi}{\partial t} \right) + \mathcal{A}_\mu \left(\omega^\xi, \frac{\partial \omega^\xi}{\partial t} \right) + \mathcal{A}_\vartheta \left(\frac{\partial \omega^\xi}{\partial t}, \frac{\partial \omega^\xi}{\partial t} \right) \\ - \alpha^\xi \left(\Upsilon(\omega^\xi) \ln |\omega^\xi|, \frac{\partial \omega^\xi}{\partial t} \right) + j \left(\frac{\partial \omega^\xi}{\partial t} \right) \leq \left(\vartheta^\xi, \frac{\partial \omega^\xi}{\partial t} \right),$$

as $j \left(\frac{\partial \omega^\xi}{\partial t} \right)$ is positive, we have

$$\frac{d}{dt} \left[\frac{\rho}{2} \left\| \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + 2\mu \left\| \mathbb{D}(\omega^\xi) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \right] + 2\vartheta \left\| \mathbb{D} \left(\frac{\partial \omega^\xi}{\partial t} \right) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\ \leq \alpha^\xi \left(\Upsilon(\omega^\xi) \ln |\omega^\xi|, \frac{\partial \omega^\xi}{\partial t} \right) + \left(\vartheta^\xi, \frac{\partial \omega^\xi}{\partial t} \right).$$

By integrating the above inequality over $(0, t)$, then using Korn's inequality (9), we find

$$\frac{\rho}{2} \left\| \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + 2\mu C_K \left\| \nabla \omega^\xi \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + 2\vartheta C_K \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds \\ \leq \alpha^\xi \int_0^t \left(\Upsilon(\omega^\xi(s)) \ln |\omega^\xi(s)|, \frac{\partial \omega^\xi(s)}{\partial t} \right) ds + \int_0^t \left(\vartheta^\xi(s), \frac{\partial \omega^\xi(s)}{\partial t} \right) ds.$$

Now we estimate the right-hand terms of this inequality. Using Poincaré's inequality (10), we see that

$$\int_0^t \left| \alpha^\xi \left(\Upsilon(\omega^\xi(s)) \ln |\omega^\xi(s)|, \frac{\partial \omega^\xi(s)}{\partial t} \right) \right| ds \\ \leq \int_0^t \left(\left[\int_{\mathcal{D}^\xi} \left(\alpha^\xi |\Upsilon(\omega^\xi(s)) \ln |\omega^\xi(s)| \right)^2 dx \right]^{\frac{1}{2}} \left\| \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3} \right) ds$$

$$\begin{aligned}
&\leq \int_0^t \left(\xi \bar{G} \left[\int_{\mathcal{D}^\xi} \left(\alpha^\xi \Upsilon \left(\omega^\xi(s) \right) \ln |\omega^\xi(s)| \right)^2 dx \right]^{\frac{1}{2}} \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}} \right) ds \\
&\leq \int_0^t \frac{4(\alpha^\xi \xi \bar{G})^2}{\vartheta C_K} \left[\int_{\mathcal{D}^\xi} \left(\Upsilon \left(\omega^\xi(s) \right) \ln |\omega^\xi(s)| \right)^2 dx \right] ds \\
&\quad + \frac{\vartheta C_K}{4} \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds.
\end{aligned}$$

We know that

$$\left(\Upsilon \left(\omega^\xi \right) \ln |\omega^\xi| \right)^2 \leq 2C_\Upsilon (|\omega^\xi|^2 + 1),$$

so, using Young's inequality, we conclude that

$$\begin{aligned}
&\int_0^t \left| \alpha^\xi \left(\Upsilon \left(\omega^\xi(s) \right) \ln |\omega^\xi(s)|, \frac{\partial \omega^\xi(s)}{\partial t} \right) \right| ds \\
&\leq \int_0^t \frac{8C_\Upsilon (\alpha^\xi \xi \bar{G})^2}{\vartheta C_K} \left[(\xi \bar{G})^2 \left\| \nabla \omega^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + |\mathcal{D}^\xi| \right] ds \\
&\quad + \frac{\vartheta C_K}{4} \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds,
\end{aligned}$$

this implies that

$$\begin{aligned}
(14) \quad &\int_0^t \left| \alpha^\xi \left(\Upsilon \left(\omega^\xi(s) \right) \ln |\omega^\xi(s)|, \frac{\partial \omega^\xi(s)}{\partial t} \right) \right| ds \\
&\leq \frac{2C_\Upsilon (\hat{\alpha} \bar{G}^2)^2}{\vartheta C_K} \int_0^t \left\| \nabla \omega^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{2C_\Upsilon (\bar{G} (\hat{\alpha}/\sqrt{\xi}))^2}{\vartheta C_K} T |\mathcal{D}| \\
&\quad + \vartheta C_K \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds.
\end{aligned}$$

The term $\int_0^t \left(\bar{\partial}^\xi(s), \frac{\partial \omega^\xi(s)}{\partial t} \right) ds$ is estimate by the Cauchy-Schwarz inequality, Poincaré's inequality (10) and Young's inequality as follows:

$$\begin{aligned}
(15) \quad &\int_0^t \left| \left(\bar{\partial}^\xi(s), \frac{\partial \omega^\xi(s)}{\partial t} \right) \right| ds \leq \frac{\vartheta C_K}{4} \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds \\
&\quad + \frac{4(\xi \bar{G})^2}{\vartheta C_K} \int_0^t \left\| \bar{\partial}^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds.
\end{aligned}$$

According to (14) and (15), we get

$$\begin{aligned}
 (16) \quad & \frac{\rho}{2} \left\| \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + \mu C_K \left\| \nabla (\omega^\xi) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
 & + \frac{\vartheta C_K}{2} \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds \leq \mu C_K \|\omega_0\|_{H^1(\mathcal{D}^\xi)}^2 \\
 & + \frac{\rho}{2} \|\omega_1\|_{L^2(\mathcal{D}^\xi)^3}^2 + \frac{2C_\Upsilon (\hat{\alpha} \bar{G}^2)^2}{\vartheta C_K} \int_0^t \left\| \nabla \omega^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds \\
 & + \frac{2C_\Upsilon ((\hat{\alpha}/\sqrt{\xi}) \bar{G})^2}{\vartheta C_K} T |\mathcal{D}| + \frac{4(\xi \bar{G})^2}{\vartheta C_K} \int_0^t \left\| \tilde{\partial}^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds.
 \end{aligned}$$

As

$$\xi^2 \left\| \tilde{\partial}^\xi \right\|_{L^2(\mathcal{D}^\xi)^3}^2 = \xi^{-1} \left\| \hat{\tilde{\partial}} \right\|_{L^2(\mathcal{D})^3}^2,$$

we multiply this inequality by ξ , we deduce

$$\begin{aligned}
 & \frac{\rho}{2} \xi \left\| \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + \mu C_K \xi \left\| \nabla \omega^\xi \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
 & + \frac{\vartheta C_K}{2} \int_0^t \xi \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds \\
 & \leq \tilde{A} + \frac{2C_\Upsilon (\hat{\alpha} \bar{G}^2)^2}{\vartheta C_K} \int_0^t \xi \left\| \nabla \omega^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds,
 \end{aligned}$$

where $\tilde{A}$ is a constant does not depend of ξ with

$$\tilde{A} = \mu C_K \|\omega_0\|_{H^1(\mathcal{D})}^2 + \frac{\rho}{2} \|\omega_1\|_{L^2(\mathcal{D})^3}^2 + \frac{4\bar{G}^2}{\vartheta C_K} \left\| \hat{\tilde{\partial}} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D})^3)}^2.$$

Now, using Gronwall's lemma, we obtain that there exists a constant c does not depend of ξ such that

$$(17) \quad \xi \left\| \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + \xi \left\| \nabla \omega^\xi \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \leq c,$$

with

$$(18) \quad \xi \left\| \nabla \frac{\partial \omega^\xi}{\partial t} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D}^\xi)^{3 \times 3})}^2 \leq c,$$

thus we get the first estimate (11) – (12).

To show the a priori estimate (13), we take $\varphi = \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2}$ in (7), we obtain

$$\begin{aligned} & \rho \left(\frac{\partial^2 \omega_\zeta^\xi}{\partial t^2}, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) + \mathcal{A}_\mu \left(\omega_\zeta^\xi, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) + \mathcal{A}_\theta \left(\frac{\partial \omega_\zeta^\xi}{\partial t}, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) \\ & + \left((j_\zeta^\xi)' \left(\frac{\partial \omega_\zeta^\xi}{\partial t}, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) - \alpha^\xi \left(\Upsilon \left(\omega_\zeta^\xi \right) \ln \left(|\omega_\zeta^\xi| \right), \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) \right) \\ & = \left(\partial^\xi, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right). \end{aligned}$$

This leads us to

$$\begin{aligned} & \rho \left\| \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + \frac{d}{dt} \left((j_\zeta^\xi) \left(\frac{\partial \omega_\zeta^\xi}{\partial t} \right) + \vartheta \left\| \frac{\partial \omega_\zeta^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \right) \\ & \leq \left(\partial^\xi, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) - \mathcal{A}_\mu \left(\omega_\zeta^\xi, \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) + \alpha^\xi \left(\Upsilon \left(\omega_\zeta^\xi \right) \ln \left(|\omega_\zeta^\xi| \right), \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right). \end{aligned}$$

Integrating this inequality over $(0, t)$, use the Korn inequality (9), we obtain

$$\begin{aligned} (19) \quad & \rho \int_0^t \left\| \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds + \vartheta C_K \left\| \nabla \frac{\partial \omega_\zeta^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + (j_\zeta^\xi) \left(\frac{\partial \omega_\zeta^\xi}{\partial t} \right) \\ & \leq \int_\Lambda k^\xi \sqrt{|(\omega_1)_\tau|^2 + 1} dy + 2 \int_0^t \left(\partial^\xi(s), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds \\ & - \int_0^t \mathcal{A}_\mu \left(\omega_\zeta^\xi(s), \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right) ds + \alpha^\xi \left(\Upsilon \left(\omega_\zeta^\xi \right) \ln \left(|\omega_\zeta^\xi| \right), \frac{\partial^2 \omega_\zeta^\xi}{\partial t^2} \right). \end{aligned}$$

Using integration by parts, we notice that

$$\begin{aligned} \int_0^t \left(\partial^\xi(s), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds & = \left(\partial^\xi(t), \frac{\partial \omega_\zeta^\xi(t)}{\partial t} \right) - \left(\partial^\xi(0), \omega_1 \right) \\ & - \int_0^t \left(\frac{\partial \partial^\xi(s)}{\partial t}, \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right) ds. \end{aligned}$$

Now, using Cauchy-Schwarz's inequality, Poincaré's inequality (10) and Young's inequality, we obtain

$$\begin{aligned}
(20) \quad & \left| \int_0^t \left(\bar{\partial}^\xi(s), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds \right| \\
& \leq \frac{\vartheta C_K}{4} \int_0^t \left\| \nabla \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \int_0^t \left\| \frac{\partial \bar{\partial}^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds \\
& \quad + \frac{\vartheta C_K}{4} \left\| \nabla \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \left\| \bar{\partial}^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^3}^2 \\
& \quad + \left\| \nabla \omega_1 \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + 4\xi^2 \bar{G}^2 \left\| \bar{\partial}^\xi(0) \right\|_{L^2(\mathcal{D}^\xi)^3}^2.
\end{aligned}$$

We have

$$\begin{aligned}
\int_0^t \mathcal{A}_\mu \left(\omega^\xi(s), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds &= \mathcal{A}_\mu \left(\omega^\xi(t), \frac{\partial \omega_\zeta^\xi(t)}{\partial t} \right) - \mathcal{A}_\mu(\omega_0^\xi, \omega_1^\xi) \\
&\quad - \int_0^t \mathcal{A}_\mu \left(\frac{\partial \omega_\zeta^\xi(s)}{\partial t}, \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right) ds.
\end{aligned}$$

Using Cauchy-Schwarz's inequality and Young's inequality, we get

$$\begin{aligned}
(21) \quad & \int_0^t \left| \mathcal{A}_\mu \left(\omega^\xi(s), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) \right| ds \leq \frac{4\mu^2}{\vartheta C_K} \left\| \nabla \omega^\xi(t) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
& \quad + \frac{\vartheta C_K}{4} \left\| \nabla \frac{\partial \omega_\zeta^\xi(t)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + \mu \left\| \nabla \omega_0 \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
& \quad + \mu \left\| \nabla \omega_1 \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + \mu \int_0^t \left\| \nabla \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds.
\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}
& \left| \int_0^t \left(\alpha^\xi \Upsilon(\omega_\zeta^\xi(s)) \ln(|\omega_\zeta^\xi(s)|), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds \right| \\
& \leq \frac{\rho}{2} \int_0^t \left\| \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{4C_\Upsilon e^4}{\rho} (\alpha^\xi)^2 \int_0^t \left(\left\| \omega_\zeta^\xi(s) \right\|_{L^2(\mathcal{D}^\xi)^3}^2 + |\mathcal{D}^\xi| \right) ds,
\end{aligned}$$

then, we deduce that

$$\begin{aligned}
 (22) \quad & \left| \int_0^t \left(\alpha^\xi \Upsilon(\omega_\zeta^\xi(s)) \ln(|\omega_\zeta^\xi(s)|), \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right) ds \right| \\
 & \leq \frac{4C_{\Upsilon} e^4 \alpha^\xi \bar{G}^2}{\rho} \int_0^t \xi^2 \|\nabla \omega_\zeta^\xi(s)\|_{L^2(\mathcal{D}^\xi)^3}^2 ds \\
 & \quad + \frac{4C_{\Upsilon} e^4 (\alpha^\xi)^2}{\rho} T_\xi |\mathcal{D}| + \frac{\rho}{2} \int_0^t \left\| \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds.
 \end{aligned}$$

Now, from (20) – (22) and (19), we conclude that

$$\begin{aligned}
 (23) \quad & \frac{\rho}{2} \int_0^t \xi^3 \left\| \frac{\partial^2 \omega_\zeta^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds + \frac{\vartheta C_K}{2} \left\| \nabla \frac{\partial \omega_\zeta^\xi}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
 & \leq \frac{4C_{\Upsilon} e^4 (\alpha^\xi \bar{G})^2}{\rho} \int_0^t \xi^2 \|\nabla \omega_\zeta^\xi\|_{L^2(\mathcal{D}^\xi)^3}^2 ds + \frac{4C_{\Upsilon} e^4 (\alpha^\xi)^2}{\rho} T_\xi |\mathcal{D}| \\
 & \quad + \left(\frac{\vartheta C_K}{4} + \mu \right) \int_0^t \left\| \nabla \frac{\partial \omega_\zeta^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \|\bar{\partial}^\xi(s)\|_{L^2(\mathcal{D}^\xi)^3}^2 \\
 & \quad + \frac{4\mu^2}{\vartheta C_K} \|\nabla \omega_\zeta^\xi(t)\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + \mu \|\nabla \omega_0\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
 & \quad + (\mu + 1) \|\nabla \omega_1\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + 4\xi^2 \bar{G}^2 \|\bar{\partial}^\xi(0)\|_{L^2(\mathcal{D}^\xi)^3}^2 \\
 & \quad + \int_\Lambda k^\xi \sqrt{|(\omega_1)_\tau|^2 + 1} dy + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \int_0^t \left\| \frac{\partial \bar{\partial}^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds.
 \end{aligned}$$

We pass to the limit in (23) as ζ tends to zero, we deduce

$$\begin{aligned}
 (24) \quad & \frac{\rho}{2} \int_0^t \xi^3 \left\| \frac{\partial^2 \omega^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds \\
 & \leq \frac{4C_{\Upsilon} e^4 (\alpha^\xi \bar{G})^2}{\rho} \int_0^t \xi^2 \|\nabla \omega^\xi\|_{L^2(\mathcal{D}^\xi)^3}^2 ds + \frac{4C_{\Upsilon} e^4 (\alpha^\xi)^2}{\rho} T_\xi |\mathcal{D}| \\
 & \quad + \frac{\vartheta C_K}{4} \int_0^t \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \|\bar{\partial}^\xi(s)\|_{L^2(\mathcal{D}^\xi)^3}^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{4\mu^2}{\vartheta C_K} \left\| \nabla \omega^\xi(t) \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + \mu \left\| \nabla \omega_0 \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 \\
& + (\mu + 1) \left\| \nabla \omega_1 \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 + 4\xi^2 \bar{G}^2 \left\| \tilde{\partial}^\xi(0) \right\|_{L^2(\mathcal{D}^\xi)^3}^2 \\
& + \int_\Lambda k^\xi \sqrt{|(\omega_1)_\tau|^2 + 1} dy + \frac{4\xi^2 \bar{G}^2}{\vartheta C_K} \int_0^t \left\| \frac{\partial \tilde{\partial}^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds.
\end{aligned}$$

Multiplying now the inequality (24) by ξ^3 and using the estimations (17)–(18), we obtain

$$(25) \quad \int_0^t \xi^3 \left\| \frac{\partial^2 \omega^\xi(s)}{\partial t^2} \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds \leq \frac{2}{\rho} \tilde{B},$$

where $\tilde{B}$ is a constant does not depend of ξ such that

$$\begin{aligned}
\tilde{B} = & \frac{4C_{\Upsilon} e^4 (\hat{\alpha} \bar{G})^2}{\rho} \xi T \left\| \nabla \omega^\xi \right\|_{L^2(\mathcal{D}^\xi)^3}^2 ds + \frac{4C_{\Upsilon} e^4 (\hat{\alpha})^2}{\rho} T |\mathcal{D}| \\
& + \left(\frac{\vartheta C_K}{4} + \mu \right) \int_0^t \xi \left\| \nabla \frac{\partial \omega^\xi(s)}{\partial t} \right\|_{L^2(\mathcal{D}^\xi)^{3 \times 3}}^2 ds + \frac{4\bar{G}^2}{\vartheta C_K} \left\| \hat{\partial}(t) \right\|_{L^2(\mathcal{D})^3}^2 \\
& + (\mu + 1) \left\| \nabla \omega_1 \right\|_{L^2(\mathcal{D})^{3 \times 3}}^2 + 4\bar{G}^2 \left\| \hat{\partial}(0) \right\|_{L^2(\mathcal{D})^3}^2 \\
& + \int_\Lambda \hat{k} \sqrt{|(\omega_0)_\tau|^2 + 1} dy + \frac{4\bar{G}^2}{\vartheta C_K} \left\| \frac{\partial \hat{\partial}}{\partial t} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D})^3)}^2.
\end{aligned}$$

Thus, we get

$$(26) \quad \xi^3 \left\| \frac{\partial^2 \omega^\xi}{\partial t^2} \right\|_{L^2(\mathcal{I}; L^2(\mathcal{D})^3)}^2 \leq \frac{2}{\rho} \tilde{B}.$$

From (26), we deduce (13). $\square$

Next, we aim to establish the convergence theorem as $\xi \rightarrow 0$. To achieve this, we introduce the Banach space

$$\mathcal{V}_z = \left\{ v = (v_1, v_2) \in L^2(\mathcal{D})^2 : \frac{\partial v_i}{\partial z} \in L^2(\mathcal{D}), i = 1, 2 \text{ and } v = 0 \text{ on } \partial \mathcal{D}_1 \right\}$$

with the norm

$$\|v\|_{\mathcal{V}_z} = \left(\sum_{i=1}^2 \left(\|v_i\|_{L^2(\mathcal{D})}^2 + \left\| \frac{\partial v_i}{\partial z} \right\|_{L^2(\mathcal{D})}^2 \right) \right)^{\frac{1}{2}}.$$

Theorem 3. Under the hypotheses of Theorem (2), there exist functions $\omega_i^* \in L^2(\mathcal{I}; \mathcal{V}_z)$, $i = 1, 2$ such that

$$(27) \quad \omega_i^\xi \rightharpoonup \omega_i^* \text{ and } \frac{\partial \omega_i^\xi}{\partial t} \rightharpoonup \frac{\partial \omega_i^*}{\partial t}, i = 1, 2 \left\} \text{ in } L^2(\mathcal{I}; \mathcal{V}_z) ;$$

$$(28) \quad \Upsilon_i(\omega_i^\xi) \ln(|\omega_i^\xi|) \longrightarrow \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|), i = 1, 2 \text{ a.e in } \mathcal{D};$$

$$(29) \quad \left. \begin{aligned} \xi^2 \frac{\partial \omega_3^\xi}{\partial t} \rightharpoonup 0, \xi^2 \frac{\partial \omega_3^\xi}{\partial z} \rightharpoonup 0 \\ \xi^2 \frac{\partial \omega_3^\xi}{\partial x_i} \rightharpoonup 0, \xi \frac{\partial \omega_i^\xi}{\partial x_j} \rightharpoonup 0, i, j = 1, 2 \end{aligned} \right\} \text{ in } L^2(\mathcal{I}; L^2(\mathcal{D})) ;$$

$$(30) \quad \left. \begin{aligned} \xi \frac{\partial^2 \omega_i^\xi}{\partial x_j \partial t} \rightharpoonup 0, i, j = 1, 2 \\ \xi^2 \frac{\partial^2 \omega_3^\xi}{\partial x_i \partial t} \rightharpoonup 0, \xi^2 \frac{\partial^2 \omega_i^\xi}{\partial t^2} \rightharpoonup 0, i = 1, 2 \\ \xi^4 \frac{\partial^2 \omega_3^\xi}{\partial t^2} \rightharpoonup 0 \end{aligned} \right\} \text{ in } L^2(\mathcal{I}; L^2(\mathcal{D})).$$

Proof. By the Theorem (2), there exists a constant c independent of ξ such that

$$\left\| \frac{\partial \omega_i^\xi}{\partial z} \right\|_{L^2(\mathcal{D})}^2 \leq c, i = 1, 2.$$

Using this estimate with the Poincaré inequality in the domain $\mathcal{D}$, we get

$$\|\omega_i^\xi\|_{L^2(\mathcal{D})}^2 \leq \left\| \frac{\partial \omega_i^\xi}{\partial z} \right\|_{L^2(\mathcal{D})}^2 \leq c, i = 1, 2.$$

So $(\omega_1^\xi, \omega_2^\xi)_\xi$ is bounded in $L^2(\mathcal{I}; \mathcal{V}_z)$, which implies that there exists an element (ω_1^*, ω_2^*) in $L^2(\mathcal{I}; \mathcal{V}_z)$ with $(\omega_1^\xi, \omega_2^\xi)_\xi$ converges weakly in $L^2(\mathcal{I}; \mathcal{V}_z)$ to (ω_1^*, ω_2^*) , the same for $(\frac{\partial \omega_1^\xi}{\partial t}, \frac{\partial \omega_2^\xi}{\partial t})_\xi$, thus we obtain (27).

We know that $(\omega_1^\xi, \omega_2^\xi)$ is bounded in $L^2(\mathcal{I}; \mathcal{V}_z)$ and $(\frac{\partial \omega_1^\xi}{\partial t}, \frac{\partial \omega_2^\xi}{\partial t})$ is bounded in $L^2(\mathcal{I}; \mathcal{V}_z)$. Using the Lions Lemma [19] and the compact injection of $L^2(\mathcal{I}; \mathcal{V}_z)$ in $L^2(\mathcal{I}; L^2(\mathcal{D})^2)$, we deduce that $(\omega_1^\xi, \omega_2^\xi) \rightarrow (\omega_1^*, \omega_2^*)$ strongly in $L^2(\mathcal{I}; L^2(\mathcal{D})^2)$. Now, since $g(\cdot) = \Upsilon_i(\cdot) \ln |\cdot|$ is continuous, we have the convergence (28).

For (29) and (30) through (12)–(11) and (27). $\square$

5 EXAMINATION OF THE LIMIT PROBLEM

In this section, we aim to determine the limit system satisfied by ω^ξ as $\xi \rightarrow 0$.

Proposition 1. *Under the same assumptions as in Theorem 3, ω^* the following limit variational problem:*

$$(31) \quad \mu \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial \omega_i^*}{\partial z} \frac{\partial}{\partial z} \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right) \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz \\ - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz + \mathcal{J}(\hat{\varphi}) - \mathcal{J}\left(\frac{\partial \omega_i^*}{\partial t}\right) \\ \geq \sum_{i=1}^2 \int_{\mathcal{D}} \hat{\partial}_i \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz, \quad \forall \hat{\varphi} \in \Sigma$$

and the limit problem:

$$(32) \quad -\mu \frac{\partial^2 \omega_i^*}{\partial z^2} - \vartheta \frac{\partial^2}{\partial z^2} \left(\frac{\partial \omega_i^*}{\partial t} \right) - \hat{\alpha} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) = \hat{\partial}_i(t), \quad i \in \{1, 2\} \text{ in } L^2(\mathcal{D}), \\ \omega_i^*(y, z, 0) = \omega_{0,i}(y, z), \quad i \in \{1, 2\}.$$

Proof. Taking the limit as $\xi \rightarrow 0$ in the variational inequality (8) and applying the convergence results from Theorem 3, we deduce

$$(33) \quad \mu \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial \omega_i^*}{\partial z} \frac{\partial}{\partial z} \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right) \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz \\ - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz + \mathcal{J}(\hat{\varphi}) - \mathcal{J}\left(\frac{\partial \omega_i^*}{\partial t}\right) \\ \geq \sum_{i=1}^2 \int_{\mathcal{D}} \hat{\partial}_i \left(\hat{\varphi}_i - \frac{\partial \omega_i^*}{\partial t} \right) dydz, \quad \forall \hat{\varphi} \in \Sigma.$$

We choose now in (33)

$$\hat{\varphi}_i = \omega_i^* \pm \psi_i, \psi_i \in H_0^1(\mathcal{D}), i \in \{1, 2\},$$

and using the Green's formula, then choosing $\psi_1 = 0$ and $\psi_2 \in H_0^1(\mathcal{D})$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\mathcal{D})$, we obtain

$$\begin{aligned} - \int_{\mathcal{D}} \mu \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial z} \right) \psi_i dy dz - \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial^2 \omega_i^*}{\partial z \partial t} \right) \psi_i dy dz \\ - \hat{\alpha} \int_{\mathcal{D}} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) \psi_i dy dz = \int_{\mathcal{D}} \hat{\delta}_i \psi_i dy dz. \end{aligned}$$

Thus

$$(34) \quad -\mu \frac{\partial^2 \omega_i^*}{\partial z^2} - \vartheta \frac{\partial}{\partial z} \left(\frac{\partial^2 \omega_i^*}{\partial z \partial t} \right) - \hat{\alpha} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) = \hat{\delta}_i, \quad i \in \{1, 2\} \text{ in } H^{-1}(\mathcal{D}),$$

since $\hat{\delta}_i \in L^2(\mathcal{D})$, $i \in \{1, 2\}$, it follows that (34) holds in $L^2(\mathcal{D})$. $\square$

Proposition 2. We denote

$$\varrho^*(y, t) = \frac{\partial \omega^*}{\partial z}(y, 0, t), \quad \pi^*(y, t) = \omega^*(y, 0, t),$$

the trace of the displacement. Then ϱ^*, π^* satisfy the inequality

$$(35) \quad \int_{\Lambda} \hat{k} (|\psi + \pi^*| - |\pi^*|) dy - \int_{\Lambda} \left(\mu \varrho^* + \vartheta \frac{\partial \varrho^*}{\partial t} \right) \psi dy \geq 0, \quad \forall \psi \in L^2(\Lambda)^2,$$

and the following limit form of the Tresca boundary conditions

$$(36) \quad \left. \begin{aligned} & \left| \mu \varrho^* + \vartheta \frac{\partial \varrho^*}{\partial t} \right| < \hat{k} \Rightarrow \pi^* = 0, \\ & \left| \mu \varrho^* + \vartheta \frac{\partial \varrho^*}{\partial t} \right| = \hat{k} \Rightarrow \exists \varkappa > 0 : \pi^* = \varkappa \left(\mu \varrho^* + \vartheta \frac{\partial \varrho^*}{\partial t} \right), \end{aligned} \right\} \text{ a.e on } \Lambda \times \mathcal{I}.$$

Also ω^* satisfy the weak equation

$$(37) \quad \int_{\Lambda} \left(\tilde{F}_i - \frac{G(y)}{2} \left(\mu \pi_i^* + \vartheta \frac{\partial \pi_i^*}{\partial t} \right) + \int_0^G \left(\mu \omega_i^* + \vartheta \frac{\partial \omega_i^*}{\partial t} \right) d\zeta + \tilde{R}_i(y, G, t) \right) \frac{\partial \psi}{\partial x_i}(y) dx = 0$$

$$i = 1, 2, \quad \forall \psi \in H^1(\Lambda),$$

where

$$\begin{aligned}\tilde{F}_i(y, G, t) &= \int_0^G F_i(y, z, t) dz - \frac{G}{2} F_i(y, G, t), \\ F_i(y, z, t) &= \int_0^z \int_0^\zeta \hat{\delta}_i(y, \eta, t) d\eta d\zeta, \\ \tilde{R}_i(y, G, t) &= \hat{\alpha} \int_0^G \hat{R}_i(y, z, t) dz - \frac{\hat{\alpha} G}{2} \hat{R}_i(y, G, t), \\ \hat{R}_i(y, z, t) &= \int_0^z \int_0^\zeta \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) d\eta d\zeta.\end{aligned}$$

Proof. To prove (35) and (36), we follow the same steps as in [20]. To prove (37) by integrating (32) from 0 to z , we see that

$$\begin{aligned}-\mu \frac{\partial \omega_i^*}{\partial z}(y, z, t) + \mu \frac{\partial \omega_i^*}{\partial z}(y, 0, t) - \vartheta \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right)(y, z, t) \\ + \vartheta \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right)(y, 0, t) - \hat{\alpha} \int_0^z \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) d\eta = \int_0^z \hat{\delta}_i(y, \eta, t) d\eta.\end{aligned}$$

The second integration between 0 and z gives

$$\begin{aligned}(38) \quad \left(\mu \omega_i^* + \vartheta \frac{\partial \omega_i^*}{\partial t} \right)(y, z, t) &= \mu z \varrho_i^* + \vartheta z \frac{\partial \varrho_i^*}{\partial t} + \mu \pi_i^* + \vartheta \frac{\partial \pi_i^*}{\partial t} \\ &\quad - \hat{\alpha} \int_0^z \int_0^\zeta \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) d\eta d\zeta - \int_0^z \int_0^\zeta \hat{\delta}_i(y, \eta, t) d\eta d\zeta.\end{aligned}$$

We replace z by $G(y)$, hence

$$\begin{aligned}(39) \quad G(y) \left[\mu \varrho_i^* + \vartheta \frac{\partial \varrho_i^*}{\partial t} \right] + \mu \pi_i^* + \vartheta \frac{\partial \pi_i^*}{\partial t} - \hat{\alpha} \int_0^z \int_0^\zeta \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) d\eta d\zeta \\ = \int_0^G \int_0^\zeta \hat{\delta}_i(y, \eta, t) d\eta d\zeta.\end{aligned}$$

Integrating (38) from 0 to $G(y)$, we obtain

$$\begin{aligned}(40) \quad \int_0^G \mu \omega^*(y, \zeta, t) d\zeta + \vartheta \int_0^G \frac{\partial \omega^*(y, \zeta, t)}{\partial t} d\zeta \\ = G(y) \left[\mu \pi_i^* + \vartheta \frac{\partial \pi_i^*}{\partial t} \right] + \frac{G(y)^2}{2} \left[\mu \varrho_i^* + \vartheta \frac{\partial \varrho_i^*}{\partial t} \right] \\ - \hat{\alpha} \int_0^G \int_0^z \int_0^\zeta \Upsilon_i(\omega_i^*) \ln |\omega_i^*| d\eta d\zeta dz - \int_0^G \int_0^z \int_0^\zeta \hat{\delta}_i(y, \eta, t) d\eta d\zeta dz.\end{aligned}$$

From (39) and (40), we deduce (37). $\square$

Theorem 4. We assume that the functions $\Upsilon_i(\cdot)$ are differentiable and there exists a positive constant $C^\Upsilon > 0$ such that, the following inequality holds

$$(41) \quad \Upsilon'_i(\xi) \ln |\xi| + \frac{\Upsilon_i(\xi)}{\xi} \leq C^\Upsilon, \quad i \in \{1, 2, 3\}. \quad \text{For all } \xi \in \mathbb{R}.$$

Then the solution (ω_1^*, ω_2^*) of the limit problem (31)-(32) is unique in $L^2(\mathcal{I}; \mathcal{V}_z)$.

Proof. First we take $\hat{\varphi} = -\hat{\psi} + \frac{\partial \omega^*}{\partial t}$ in (31) we deduce the solution (ω_1^*, ω_2^*) that satisfies

$$(42) \quad \begin{aligned} & \mu \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial \omega_i^*}{\partial z} \frac{\partial \hat{\psi}_i}{\partial z} dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right) \hat{\psi}_i dydz \\ & - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) \hat{\psi}_i dydz - \mathcal{J}(\hat{\psi}) \\ & \leq \sum_{i=1}^2 \int_{\mathcal{D}} \hat{\partial}_i \hat{\psi}_i dydz, \quad \forall \hat{\psi} \in \Sigma. \end{aligned}$$

Now, let ω^*, ω^{**} solutions for the limit problem (31)-(32) with initial data $\omega(0) = \omega_0^* = \omega_0^{**}$. So we have

$$(43) \quad \begin{aligned} & \mu \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial \omega_i^*}{\partial z} \frac{\partial \hat{\psi}_i}{\partial z} dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^*}{\partial t} \right) \hat{\psi}_i dydz \\ & - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} \Upsilon_i(\omega_i^*) \ln(|\omega_i^*|) \hat{\psi}_i dydz - \mathcal{J}(\hat{\psi}) \\ & \leq \sum_{i=1}^2 \int_{\mathcal{D}} \hat{\partial}_i \hat{\psi}_i dydz, \quad \forall \hat{\psi} \in \Sigma, \end{aligned}$$

and

$$(44) \quad \begin{aligned} & \mu \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial \omega_i^{**}}{\partial z} \frac{\partial \hat{\psi}_i}{\partial z} dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial \omega_i^{**}}{\partial t} \right) \hat{\psi}_i dydz \\ & - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} \Upsilon_i(\omega_i^{**}) \ln(|\omega_i^{**}|) \hat{\psi}_i dydz - \mathcal{J}(\hat{\psi}) \\ & \leq \sum_{i=1}^2 \int_{\mathcal{D}} \hat{\partial}_i \hat{\psi}_i dydz, \quad \forall \hat{\psi} \in \Sigma. \end{aligned}$$

Denoting $U = \omega^* - \omega^{**}$, we choose $\hat{\psi} = U$ in (43) and in (44) respectively, then subtract the resulting inequalities to get

$$\begin{aligned} & \mu \sum_{i=1}^2 \int_{\mathcal{D}} \left(\frac{\partial U_i}{\partial z} \right)^2 dydz + \vartheta \sum_{i=1}^2 \int_{\mathcal{D}} \frac{\partial}{\partial z} \left(\frac{\partial U_i}{\partial t} \right) U_i dydz \\ & - \hat{\alpha} \sum_{i=1}^2 \int_{\mathcal{D}} (\Upsilon_i(\omega_i^*) \ln |\omega_i^*| - \Upsilon_i(\omega_i^{**}) \ln |\omega_i^{**}|) U_i dydz \\ & \leq 0, \end{aligned}$$

this implies that

$$\frac{\vartheta}{2} \int_{\mathcal{D}} \left| \frac{\partial U(t)}{\partial z} \right|^2 dydz \leq \hat{\alpha} \sum_{i=1}^2 \int_0^t \int_{\mathcal{D}} (\Upsilon_i(\omega_i^*) \ln |\omega_i^*| - \Upsilon_i(\omega_i^{**}) \ln |\omega_i^{**}|) U_i dydz.$$

By the Mean Value Theorem, there exists $\tilde{U}_i = \omega_i^* + \eta(\omega_i^{**} - \omega_i^*)$ where $\eta \in [0, 1]$ such that

$$\frac{\vartheta}{2} \int_{\mathcal{D}} \left| \frac{\partial U(t)}{\partial z} \right|^2 dydz \leq \hat{\alpha} \sum_{i=1}^2 \int_0^t \int_{\mathcal{D}} \left(\Upsilon_i'(\tilde{U}_i) \ln |\tilde{U}_i| + \frac{\Upsilon_i(\tilde{U}_i)}{\tilde{U}_i} \right) |U_i^*|^2 dydz.$$

Now, we use Poincaré inequality and (41), we get

$$\frac{\vartheta}{2} \int_{\mathcal{D}} \left| \frac{\partial U}{\partial z} \right|^2 dydz \leq \hat{\alpha} C^{\Upsilon} \bar{G}^2 \int_0^t \left\| \frac{\partial U(s)}{\partial z} \right\|_{L^2(\mathcal{D})^2}^2 ds.$$

It follows from Gronwall's inequality that

$$\left\| \frac{\partial U(t)}{\partial z} \right\|_{L^2(\mathcal{D})^2}^2 \leq 0.$$

Using Poincaré inequality, we conclude

$$U = 0, \text{ a.e in } \mathcal{D} \times \mathcal{I}.$$

□

6 CONCLUSION AND PERSPECTIVES

This work provides a rigorous asymptotic analysis of a nonlinear viscoelastic problem posed in a thin three-dimensional domain with a logarithmic source term and Tresca friction boundary conditions. By means of an appropriate scaling of the data and careful uniform estimates, we proved the existence and uniqueness of weak solutions for each fixed thickness and established the convergence of the three-dimensional solution toward a well-defined two-dimensional limit problem. The resulting reduced model preserves both the frictional effects and the logarithmic nonlinearity, leading to a system of two-dimensional viscoelastic equations coupled with the corresponding limit form of the Tresca boundary conditions. These results offer a mathematically justified dimension reduction and provide a reliable theoretical foundation for simplified modeling of thin viscoelastic structures.

As a natural continuation of this theoretical study, a complementary work devoted to the numerical validation of the asymptotic model is currently in progress. The forthcoming study will develop a complete computational framework for both the original three-dimensional formulation and its reduced limit counterpart. A consistent finite element discretization in space, combined with suitable time-integration schemes adapted to viscoelastic memory effects and the nonlinear logarithmic source term, is being implemented. Particular attention is devoted to the nonlinear coupling between displacement, hereditary terms, and the source contribution, through stable and robust staggered or fully coupled algorithms. The numerical investigation will include convergence analyses with respect to the thickness parameter, mesh refinement, and time-step size, as well as systematic comparisons between the three-dimensional and reduced solutions in terms of relative errors and computational cost. Moreover, extensive simulations will explore stability properties, long-time behavior, parameter sensitivity, and the influence of the logarithmic source term on the mechanical response and energy evolution. This second contribution aims to provide a complete numerical confirmation of the asymptotic results and to demonstrate that the reduced model constitutes an accurate, stable, and computationally efficient alternative to the full three-dimensional viscoelastic problem.

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