

SYNTHESIS OF SLIDER-CRANK MECHANISMS WITH APPLICATION OF THE THEORY OF STATIONARY CURVATURE

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ABSTRACT: A linear mathematical model that applies the cubic of stationary curvature theory and, in particular, the Carter-Hall's circle to generate Burmester's curves is adapted and further development for the synthesis of grippers with crank-fork mechanisms. The model also includes a condition for achieving a certain value of the pressure angle is considered, in which the kinematic diagram of the mechanism is unambiguously determined. The application of the mathematical model is illustrated by the synthesis of mechanisms of two grippers with practical application. The introduced linear mathematical model enables automated computer synthesis by four infinitely close relative positions of slider-crank mechanisms, which was considered impossible for decades. The proposed new model significantly facilitates the work of engineers in the synthesis of slider-crank mechanisms, which are very often found in modern devices such as e.g. some grippers for robots in microelectronics. The new method avoids the shortcoming of the known methods of synthesis by infinitely close positions, namely the non-linear mathematical models for synthesis that are reached with them. The developed linear model is easy to implement with any computer program for engineering calculations (eg. MathCAD, Matlab, etc.).

KEY WORDS: Carter-Hall circle, Burmester curves, pressure angle, synthesis of slider-crank mechanisms.

1 INTRODUCTION

The end effectors of manipulators, robots, and other technical devices are usually grippers that convert the input force into a required gripping force. A number of authors research and present in many of their publications different types of grippers and mathematical models for their synthesis [1–4]. In some research reports and articles [1–4] it is noted that changing the parameters of the manipulated object requires

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a corresponding change in the gripping force. For this purpose, various methods for the synthesis of grippers with higher kinematic pairs have been applied and compared [5–8].

To achieve the desired gripping force, it is advisable to use grippers with linkages, taking into account their advantages and the relatively small working interval of the opening of the gripper levers. Then it is appropriate to use the Burmester curves for the synthesis of the gripper mechanisms [9–11] or Carter-Hall circle [12–16], since their gripper levers usually move within a relatively small vicinity of their nominal position. The theory of instantaneous motion is a subject of scientific research primarily in planar mechanisms. This theory about spherical motions is reflected in the monographs of Dobrovolsky [17], Suh, and Radcliffe [18], as well as in St. Bachvarov dissertation [19] and related publications.

The aim of the present study is to generate *Burmester curves* (cubic of stationary curvature) by means of the *Carter-Hall circle* in such a way that the mathematical model for the synthesis of grippers with slider-crank mechanisms is linear and easy for computer application with software for engineering calculations, such as MathCAD, Matlab, CalcTree or other. In addition, the model should also include a condition for a certain value of the pressure angle [20], with the kinematic diagram of the mechanism can be unambiguously determined.

In [21], a mathematical model for the synthesis of crank-fork mechanisms was developed, which in the present work has been further developed, adapted, and applied to the synthesis of crank-fork mechanisms for grippers of manipulators, robots, and other technical devices.

2 KINEMATIC INVARIANTS UP TO THIRD-ORDER

In the rotation (φ) of the input link and the rectilinear translation (S) of the output link (Fig. 1), the kinematic invariants are unambiguously determined by the transfer functions $S'(\varphi)$, $S''(\varphi)$ and $S'''(\varphi)$. The first-order kinematic invariant is the abscissa (1) of the relative instantaneous center of velocity (ICV), point P , according to Aronhold-Kennedy and Willis theorems.

The kinematic invariant of the second-order is the angle (2) locked between the direction of the connecting rod (straight line BP) and the collinear axis $q \equiv PQ$, according to the Bobillier's and Freudenstein's theorems [14, 22] (Q is the absolute ICV)

$$(1) \quad x_P = S', \quad (y_P = 0),$$

$$(2) \quad \mu = \arctan(S'/S'')$$

$$(3) \quad x_H = x_P - 3 \frac{S'^2 + S''^2}{S' + S'''}$$

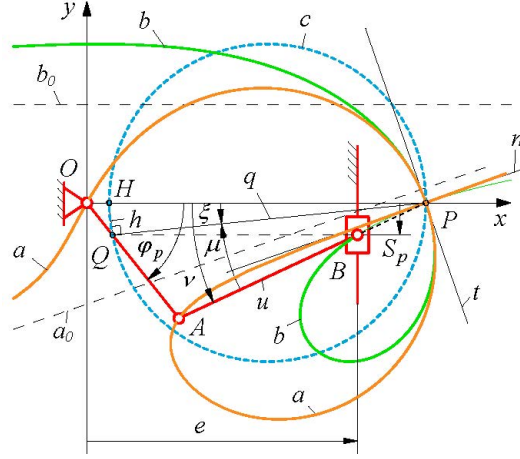


Figure 1. Burmester curves: **a** and **b**. Mathematical model for the synthesis of a slider-crank mechanism in design position.

$$\begin{aligned}
 (4) \quad & \xi \rightarrow k_q = \tan \xi \\
 (5) \quad & k_\nu = \tan(\xi - \mu) \\
 (6) \quad & x_Q = k_q^2 x_P + x_H/k_q^2 + 1; \quad y_Q = k_q(x_Q - x_P) \\
 (7) \quad & k_{OQ} = y_Q/x_Q \\
 (8) \quad & x_A = k_\nu x_P/k_\nu - k_{OQ}; \quad y_A = k_{OQ}x_A \\
 (9) \quad & x_B = x_P + y_Q/k_\nu; \quad y_B = y_Q
 \end{aligned}$$

The third-order kinematic invariant is the abscissa (3) at the intersection $H(x_H, y_H = 0)$ of the circle c of Carter-Hall (with diameter $d_c \equiv l_{PH} = |x_H - x_P|$) with axis Ox [15, 16]. The equation of this circle itself is not used in the mathematical model derived below.

3 MATHEMATICAL MODEL FOR DETERMINING THE BURMESTER CURVES

The mathematical model uses the equations of straight lines in which the kinematic invariants from (1), (2) and (3) are included. Let the angle ξ , which is between the collinear axis q and the axis Ox , be a variable parameter (4).

The *Burmester curves* are obtained entirely in the interval $\xi \in [0, 180^\circ]$. This interval can be significantly reduced if we look for the *Burmester curves* in a pre-determined area β [20].

For each value of the variable parameter ξ , the following are determined:

- the kinematic invariants: (1), (2) and (3);

- the angular coefficient of the collinearity axis q : (4);
- the angular coefficient k_ν of the straight determining the position of the connecting rod (5);
- the coordinates of the absolute ICV point Q as the intersection point of the axis q with the straight line $h \perp q$: (6);
- the angular coefficient k_{OQ} of the straight line: (7);
- the coordinates of the center A of the joint between the crank and the connecting rod, which is the intersection of the straight line OQ and the straight line PQ : (8);
- the coordinates of the center B of the joint between the connecting rod and the slider, which is the intersection of the straight line OQ and a straight line parallel to the Ox axis passing through the ICV (point Q): (9).

By changing the parameter ξ , it is possible to obtain completely the *Burmester curves* **a** and **b**.

Each Burmester curve **a** in the input link plane corresponds to a **b** Burmester curve in the output link plane. Each straight line that passes through the relative ICV point P , intersects the Burmester curves **a** and **b** at the centers of joints A and B , respectively, resulting in a slider-crank mechanism satisfying the set values of the first three transfer functions.

From the theoretically infinite number of kinematically equivalent slider-crank mechanisms, one can be determined that satisfies an additional synthesis condition, such as the pressure angle θ . The reactions of the forces in the kinematic pairs and to determine the dimensions of mechanism depend on it. In this case, the centers $A = a \cap u$ and $B = b \cap u$ are obtained as intersections of the curves **a** and **b** with the straight u , passing through the center P at an angle ν to the axis Ox , since $\nu = 90^\circ + \theta$. The kinematic diagram of the mechanism is determined by the coordinates of the centers A and B .

SYNTHESIS OF A GRIPPER WITH VARIABLE GRIPPING FORCE (FIG. 2)

Three values of the gripping force are set: $F_{(\Delta\varphi=0)}$ at $\Delta\varphi_p = 0$, when the gripping levers are parallel; $F_{(\Delta\varphi=-f)}$ at the angle $\Delta\varphi_p = -f$ rad of the gripping levers and $F_{(\Delta\varphi=+f)}$ at the angle $\Delta\varphi_p = +f$ rad of the gripping levers. Let the gripping force change according to a parabolic law $F = a + b\Delta\varphi + c\Delta\varphi^2$ in accordance with the change of the diameter (respectively the mass) of the manipulated object. Then from

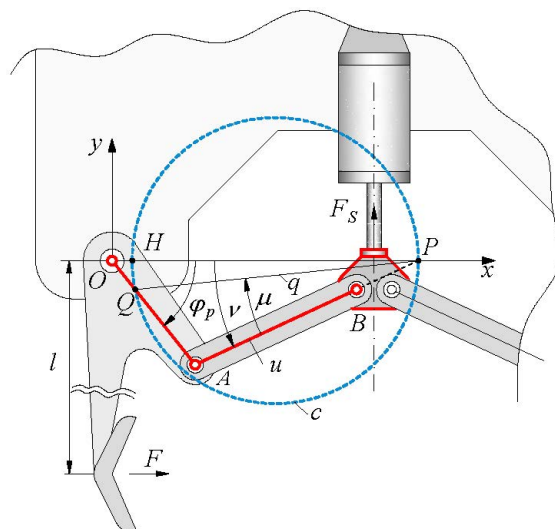


Figure 2. The synthesized gripper with gripping force F changing according to a parabolic law.

the system of equations

$$(10) \quad \begin{cases} F_{(\Delta\varphi=0)} = a, \\ F_{(\Delta\varphi=-f)} = a - bf + cf^2, \\ F_{(\Delta\varphi=+f)} = a + bf + cf^2, \end{cases}$$

the coefficients a , b and c are determined.

From the equality of the powers of the gripping force F and of the constant piston force F_s at the length of the gripping levers l the necessary transfer functions are obtained for the synthesis:

$$(11) \quad S' = dS/d\varphi = c_s F, \quad c_s = 2l/F_s = \text{const.}$$

$$(12) \quad S'' = d^2S/d\varphi^2 = c_s F', \quad S''' = d^3S/d\varphi^3 = c_s F'',$$

where

$$(13) \quad F' = b + 2c \Delta\varphi, \quad F'' = 2c.$$

When $\Delta\varphi = 0$ it derives

$$(14) \quad F = a, \quad F' = b, \quad F'' = 2c$$

and therefore

$$(15) \quad S' = c_s F, \quad S'' = c_s F', \quad S''' = c_s F''.$$

The gripping force generated by the mechanism

$$(16) \quad \tilde{F} = \frac{\tilde{S}'}{c_s} = \frac{l_{OA}}{c_s} \left(\cos \varphi - \frac{(e - l_{OA} \cos \varphi) \sin \varphi}{\sqrt{l_{AB}^2 - (e - l_{OA} \cos \varphi)^2}} \right)$$

is determined by the first transfer function $\tilde{S}'$.

Let the values of $l = 100$ and $F_s = 2000$ N be set. Then $c_s = 2l/F_s = 0.1$. For $f = 0$, $f = -0.1$ rad (-5.73°) and $f = 0.1$ rad ($+5.73^\circ$) from the system of equations (11) for forces, respectively $F = 800$ N, $F = 1145$ N and $F = 685$ N, the coefficients $a = 800$, $b = -2300$ and $c = 11500$ are obtained.

From (13) and (14) are calculated $F = 800$ N, $F' = -2300$, $F'' = 23000$ and respectively $S' = 80$, $S'' = -230$, $S''' = 2300$. From equations (1), (2) and (3), the kinematic invariants are calculated respectively: $x_P = 80$ mm; $\tan \mu = -0.348$ ($\mu = -19.179^\circ$) and $x_H = 5.252$ mm. By changing the value of the parameter k_ν and using equations (1), (2) and (3), we obtain the *Burmester curves* **a** and **b**, shown in Fig. 1.

One of the possible slider-crank mechanisms is obtained at an appropriately selected value of $\nu = 25.00^\circ$ ($k_\nu = 0.4663$), corresponding to the angle of transfer of the movement to the gripping levers $\gamma \approx 80^\circ$.

Sequentially from equations (1), (2) and (3) are calculated: $k_q = 0.1020$, $k_h = -9.8023$, $x_Q = 6.022$ mm, $y_Q = -7.542$ mm, $k_{OQ} = -1.2530$, $x_A = 21.697$ mm, $y_A = -27.186$ mm, $x_B = 63.817$ mm, $y_B = y_Q = -7.546$ mm.

The parameters of the kinematic diagram of the mechanism (Fig. 3) are determined by the coordinates of the centers A and B : $l_{OA} = \sqrt{x_A^2 + y_A^2} = 34.782$ mm, $e = x_B = 63.817$ mm, $l_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = 46.474$ mm at the project position of the lever OA , determined by the angle $\varphi_p = \arctan(y_A/x_A) = -51.408^\circ$. Then the position of the slider is $S_p = y_B = -7.546$ mm.

The first transfer function (15) of the generated gripping force $\tilde{F}$ and the set F must be determined (Fig. 3). For the end positions of the gripping lever, at which $\Delta\varphi = -0.1$ rad $\approx -5.73^\circ$ and $\Delta\varphi = +0.1$ rad $\approx +5.73^\circ$ the corresponding values of the gripping force $\tilde{F} = 1301.48$ N and $\tilde{F} = 644.21$ N are relatively close to the set $F = 1145$ N and $F = 685$ N. Improved approximation of the gripping force $\tilde{F}^*$ to the set F can be obtained by a small change in the angle η locked between the arms OA and the two gripping levers (see Fig. 1).

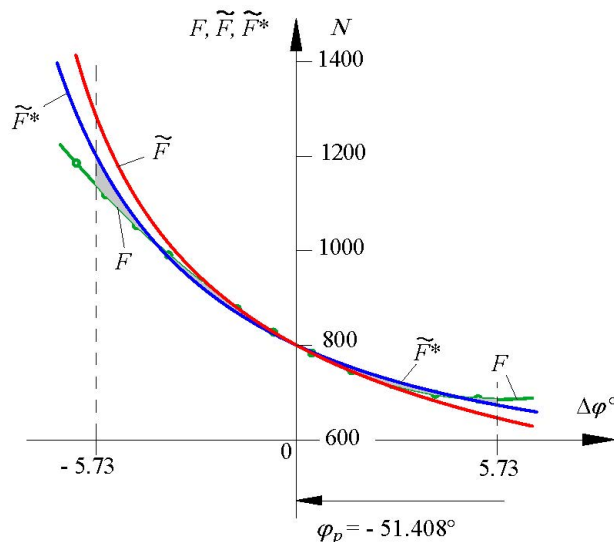


Figure 3. The given function F , generated $\tilde{F}$ and the improved $\tilde{F}^*$ gripping force function.

SYNTHESIS OF A GRIPPER WITH A CONSTANT GRIPPING FORCE (FIG. 4)

The forces $F_s = \text{const}$, $F = \text{const}$, and the length l of the gripping levers are set. Then $c_s = 2l/F_s = 0.1 = \text{const}$; $S' = dS/d\varphi = c_s F$; $S'' = S''' = 0$. In this particular case, the Carter-Hall circle **c**, at a variable angle ν , is described by the equations

$$(17) \quad x_c = S'(1 - 3 \cos^2 \nu), \quad y_c = 1.5 S' \sin(2\nu)$$

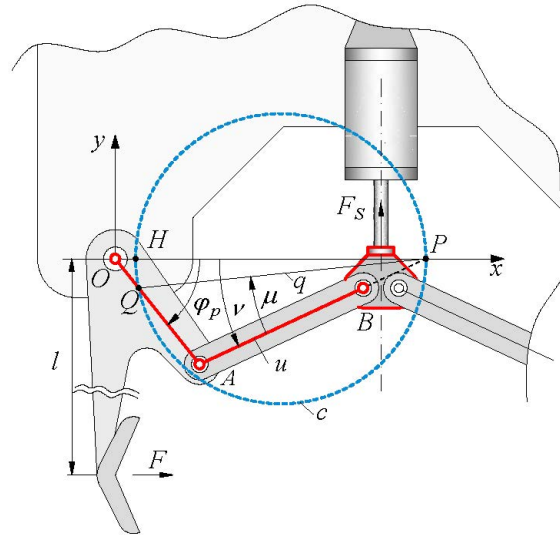
and the Burmester curves disintegrate at the polar normal $n \equiv Ox$ and into circles **a** and **b** equations

$$(18) \quad x_A = S'(1 - 1.5 \cos^2 \nu), \quad y_A = -0.75 S' \sin(2\nu),$$

$$(19) \quad x_B = S'(3 \cos^2 \nu + 1), \quad y_B = 1.5 S' \sin(2\nu).$$

Let $F = 200$ N, $F_s = 2000$ N, and length $l = 100$ mm. Then $c_s = 2l/F_s = 0.1 = \text{const}$; $S' = dS/d\varphi = c_s F = 20$; $S'' = S''' = 0$. When choosing $\gamma = 80^\circ$, it is obtained $\nu = 70.575^\circ$.

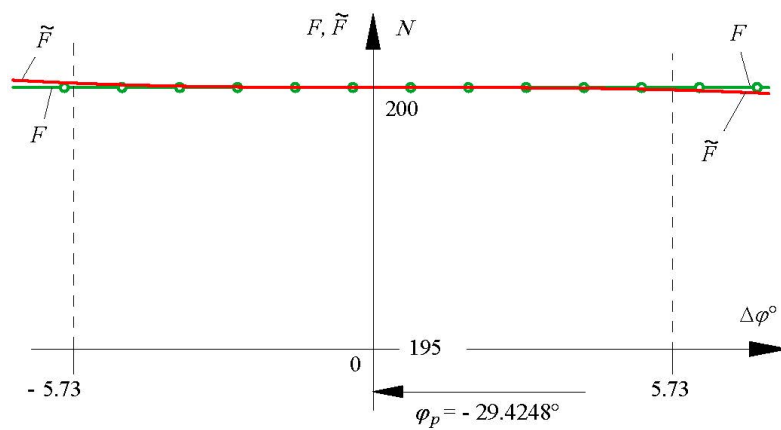
Using equations (18) and (19), $x_A = 16.682$ mm, $y_A = -9.409$ mm, $x_B = 26.636$ mm, $y_B = 18.819$ mm are determined. For the parameters of the kinematic scheme of the mechanism we get: $l_{OA} = \sqrt{x_A^2 + y_A^2} = 19.1525$ mm, $l_{AB} = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = 29.932$ mm, $e = x_B = 26.636$ mm, at the project

Figure 4. The synthesized gripper with constant gripping force F .

position of the lever OA , determined by the angle $\varphi_p = \arctan(y_A/x_A) = -29.425^\circ$. Then the position of the slider is $S_p = y_B = 18.819$ mm.

Next is the determination of the first transfer function $\tilde{S}'$ and the generated gripping force $\tilde{F}$, presented in Fig. 5.

For the end positions of the gripping lever, at which $\Delta\varphi = -0.1$ rad $\approx -5.73^\circ$ and $\Delta\varphi = +0.1$ rad $\approx +5.73^\circ$ the corresponding values of the gripping force and are very close to the set force $F = 200$ N.

Figure 5. The given function F and generated function $\tilde{F}$.

This example clearly illustrates the possibility of realizing a practically certain set gripping force with significant unloading of the gripper levers (in this case $\Delta\varphi = \pm 0.1 \text{ rad} \approx \pm 5.73^\circ$) by using the theory of instantaneous motion.

4 CONCLUSION

The linear mathematical model for the synthesis by four infinitely close relative positions of slider-crank mechanisms is derived, which was considered impossible for decades.

This model is computer implementable with any engineering calculation software (MathCAD, Mathlab, CalcTree, or other). It would facilitate engineers in synthesizing slider-crank mechanisms for grippers in the proximity of a given project position of the gripper levers. The model is based on the theory of cubic of stationary curvature and in particular on the *Carter-Hall circle* for generating *Burmester curves*. The famous Freudenstein formula and the Hall equation, revised by Galabov, are transformed into cases where four-bar mechanisms transform rotation into rectilinear translation. The developed model also includes a condition for achieving a certain value of the angle of force transmission, which unambiguously determines the kinematic diagram of the mechanism.

The application and verification of the mathematical model are illustrated with two examples: synthesis of a gripper with variable gripping force and synthesis of a gripper with constant gripping force.

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