

REVISITING THE STRESS DISTRIBUTION IN A ROTATING ELASTIC DISK: AN ELASTODYNAMIC APPROACH

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ABSTRACT: We present an analytical solution for the steady-state stress field of a rotating elastic disk under gravity, supported by a frictionless localized point contact with a rigid ground. The problem is formulated within two-dimensional linear elastodynamics under plane-stress conditions, with gravity and centrifugal effects treated as body forces and the contact as a singular boundary traction. The steady-state solution is obtained as the long-time limit of the dynamic equations and exhibits a natural modal decomposition. Monopole and dipole modes originate from torque balance and uniform body force, while higher-order modes describe localized stress transmission. The formulation recovers the classical solution for a freely rotating disk in the absence of gravity and contact.

KEY WORDS: Elastodynamics, Gravity, Point contact, Rotating disk, Steady-state limit, Stress singularity.

1 INTRODUCTION

The deformation and stress fields of rotating elastic disks have long been a classical subject in solid mechanics [1–7], motivated by applications ranging from rotating machinery to flywheels and rolling elements. In most traditional analyses, attention has been focused on axisymmetric stress distributions induced by centrifugal forces, for which closed-form solutions are well established [3, 5–7]. In contrast, much less is known about the stress fields generated when a rotating disk is simultaneously subjected to gravity and supported by a localized contact with a rigid ground.

Such a configuration naturally arises in the mechanical modeling of wheels and rolling bodies, where gravitational body forces coexist with highly localized reaction

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forces at the contact point. From a continuum-mechanical perspective, this problem combines three distinct ingredients: uniform body forces due to gravity, inertial effects associated with rotation, and singular boundary tractions originating from point-like contact. The resulting stress field is therefore intrinsically non-axisymmetric and involves multiple length and force scales.

Analytically, this problem presents several theoretical challenges. The localized contact introduces stress singularities that cannot be captured by smooth boundary tractions [8], while gravity acts as a uniform body force that excites non-self-equilibrated stress modes. Moreover, although the final state of interest is steady-state, the presence of rotation and body forces naturally motivates a formulation based on elastodynamics [2, 9] rather than purely steady-state elasticity. In particular, the steady-state solution must be obtained as a limiting case of the two-dimensional elastodynamic equations [10, 11], ensuring consistency with inertia and wave propagation effects.

In this study, we derive an analytical solution for the steady-state stress field of a rotating elastic disk under gravity, supported by a localized contact with a rigid ground. The analysis is carried out within the framework of linear two-dimensional elastodynamics under plane-stress conditions. By introducing appropriate scalar and vector potentials and employing Laplace-transform techniques, we obtain explicit expressions for the displacement and stress fields. The steady-state limit is then evaluated using the final-value theorem, which reveals a natural modal decomposition of the solution. In particular, the monopole and dipole contributions are shown to correspond to torque and gravity-induced body-force effects, while higher-order modes describe self-equilibrated stress transmission emanating from the contact. The present formulation also recovers the classical stress distribution of a freely rotating disk in the appropriate limiting case, thereby providing a unified theoretical description of rotating disks with and without contact.

2 MODEL AND SETUP

We consider a two-dimensional elastic disk of radius R , rotating with angular velocity ω while resting on a rigid horizontal ground, as illustrated in Fig. 1. The disk is characterized by shear modulus G , mass density ρ_0 , and Poisson's ratio ν . The disk is assumed to be thin, so that plane-stress conditions apply.

The contact between the disk and the ground is modeled as a point contact at $\theta = 0$. Here, σ_0 denotes the amplitude of the concentrated normal traction at the contact point. It characterizes the strength of the localized reaction force exerted by the ground, such that the total normal force is proportional to σ_0 . Accordingly, the boundary conditions at $r = R$ are given by

$$(1) \quad \sigma_{rr}|_{r=R} = -\sigma_0 \delta(\theta) = -\frac{\sigma_0}{\pi} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos(m\theta) \right], \quad \sigma_{r\theta}|_{r=R} = 0.$$

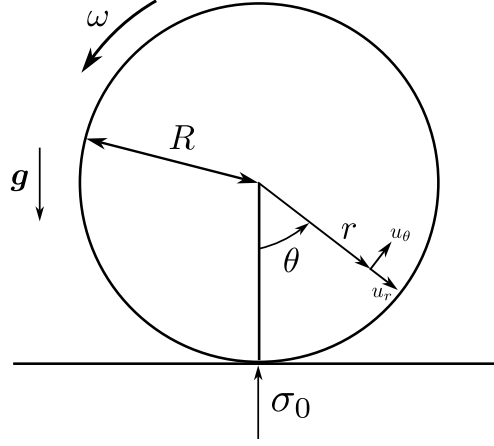


Figure 1. Schematic of the model. An elastic disk of radius R rotates with angular velocity ω and is supported by a point contact with the ground, where a concentrated normal traction of magnitude σ_0 is applied.

Here, we used the Fourier series representation of the Dirac delta function [12].

3 ELASTODYNAMIC FORMULATION

We analyze the problem within the framework of linear elastodynamics. As explained in the previous section, the disk is assumed to be sufficiently thin so that a plane-stress condition is applicable.

In the laboratory frame, the displacement vector $\mathbf{u}$ at position (r, θ) obeys the Navier-Cauchy equation [2, 4, 9]

$$(2) \quad \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = G \left[\nabla^2 \mathbf{u} + \frac{1+\nu}{1-\nu} \nabla (\nabla \cdot \mathbf{u}) \right] + \rho_0 r \omega^2 \hat{\mathbf{e}}_r + \rho_0 g (\cos \theta \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_\theta),$$

where $\hat{\mathbf{e}}_r$ denotes the unit vector in the radial direction. The longitudinal and transverse wave speeds v_L and v_T , respectively, are introduced as

$$(3) \quad v_L \equiv \sqrt{\frac{2}{1-\nu} \frac{G}{\rho_0}}, \quad v_T \equiv \sqrt{\frac{G}{\rho_0}}.$$

For notational simplicity, we introduce the following nondimensional quantities:

$$(4) \quad \rho \equiv \frac{r}{R}, \quad \tau \equiv \sqrt{\frac{g}{R}} t, \quad \tilde{\mathbf{u}} \equiv \frac{\mathbf{u}}{R}, \quad \tilde{\omega} \equiv \sqrt{\frac{R}{g}} \omega,$$

$$M_L \equiv \frac{\sqrt{Rg}}{v_L}, \quad M_T \equiv \frac{\sqrt{Rg}}{v_T}, \quad \tilde{\sigma}_{\alpha\beta} \equiv \frac{\sigma_{\alpha\beta}}{G}, \quad \tilde{\sigma}_0 \equiv \frac{\sigma_0}{G}.$$

Using these definitions, Eq. (2) can be rewritten in dimensionless form as

$$(5) \quad M_T^2 \frac{\partial^2}{\partial \tau^2} \tilde{\mathbf{u}} = \nabla_{\rho}^2 \tilde{\mathbf{u}} + \frac{1+\nu}{1-\nu} \nabla_{\rho} (\nabla_{\rho} \cdot \tilde{\mathbf{u}}) + M_T^2 \tilde{\omega}^2 \rho \hat{\mathbf{e}}_r + M_T^2 (\cos \theta \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_{\theta}),$$

where $\nabla_{\rho} \equiv R \nabla$ and $\hat{\mathbf{e}}_{\rho} \equiv \hat{\mathbf{e}}_r$.

To solve Eq. (5), the nondimensional displacement field $\tilde{\mathbf{u}}$ is decomposed as

$$(6) \quad \tilde{\mathbf{u}} \equiv \tilde{\mathbf{u}}_1 + \tilde{\mathbf{u}}_2,$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfy

$$(7a) \quad 0 = \nabla_{\rho}^2 \tilde{\mathbf{u}}_1 + \frac{1+\nu}{1-\nu} \nabla_{\rho} (\nabla_{\rho} \cdot \tilde{\mathbf{u}}_1) + M_T^2 \tilde{\omega}^2 \rho \hat{\mathbf{e}}_r + M_T^2 (\cos \theta \hat{\mathbf{e}}_r - \sin \theta \hat{\mathbf{e}}_{\theta}),$$

$$(7b) \quad M_T^2 \frac{\partial^2}{\partial \tau^2} \tilde{\mathbf{u}}_2 = \nabla_{\rho}^2 \tilde{\mathbf{u}}_2 + \frac{1+\nu}{1-\nu} \nabla_{\rho} (\nabla_{\rho} \cdot \tilde{\mathbf{u}}_2).$$

For Eq. (7a), we assume that the displacement can be expressed in terms of a scalar potential ϕ_1 as

$$(8) \quad \tilde{\mathbf{u}}_1 = \nabla_{\rho} \phi_1.$$

Substitution into Eq. (7a) shows that the equation is satisfied by

$$(9) \quad \phi_1 = -M_L^2 \left(\frac{1}{32} \tilde{\omega}^2 \rho^4 + \frac{1}{8} \rho^3 \cos \theta \right).$$

Similarly, for Eq. (7b), Helmholtz's theorem allows us to represent the displacement using a scalar potential ϕ_2 and a vector potential $\Psi = (0, 0, \psi)^T$ as

$$(10) \quad \tilde{\mathbf{u}}_2 = \nabla_{\rho} \phi_2 + \nabla_{\rho} \times \Psi,$$

Substituting this form into Eq. (7b), we find that the Navier-Cauchy equation is automatically satisfied provided that

$$(11) \quad \nabla_{\rho}^2 \phi_2 = M_L^2 \frac{\partial^2 \phi_2}{\partial \tau^2}, \quad \nabla_{\rho}^2 \psi = M_T^2 \frac{\partial^2 \psi}{\partial \tau^2}.$$

These equations are solved using Laplace transforms. We define the Laplace transforms of the potentials as

$$(12) \quad \Phi_2(\rho, \theta, s) \equiv \int_0^{\infty} \phi_2(\rho, \theta, t) e^{-s\tau} d\tau, \quad \Psi(\rho, \theta, s) \equiv \int_0^{\infty} \psi(\rho, \theta, t) e^{-s\tau} d\tau.$$

Assuming $\phi_2(\tau) = \phi'_2(\tau) = \psi(\tau) = \psi'(\tau) = 0$ for $\tau < 0$, substitution into Eq. (11) yields

$$(13) \quad \nabla_{\rho}^2 \Phi_2 = M_L^2 s^2 \Phi_2, \quad \nabla_{\rho}^2 \Psi = M_T^2 s^2 \Psi.$$

The solutions to Eq. (13) can be expressed in terms of modified Bessel functions as [12]

$$(14a) \quad \Phi_2(\rho, \theta, s) = \sum_{m=0}^{\infty} \mathcal{A}_m I_m(M_L \rho s) \cos(m\theta),$$

$$(14b) \quad \Psi(\rho, \theta, s) = \sum_{m=1}^{\infty} \mathcal{B}_m I_m(M_T \rho s) \sin(m\theta),$$

where $I_m(z)$ denotes the modified Bessel function of the first kind [13], defined by

$$(15) \quad I_m(z) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+m+1)} \left(\frac{z}{2}\right)^{2n+m}.$$

For later reference, we also introduce the Laplace transform of ϕ_1 as Φ_1 . Since ϕ_1 is independent of time τ , its Laplace transform is given by

$$(16) \quad \Phi_1(\rho, \theta, s) = -\frac{M_L^2}{s} \left(\frac{1}{32} \tilde{\omega}^2 \rho^4 + \frac{1}{8} \rho^3 \cos \theta \right).$$

To summarize the above formulation, we introduce the combined scalar potential Φ as

$$(17) \quad \Phi \equiv \Phi_1 + \Phi_2.$$

Using this definition, the relations between the (Laplace-transformed) displacement and stress components ($\tilde{U}_{\alpha}$ and $\tilde{\Sigma}_{\alpha\beta}$, respectively) and the potentials are given by

$$(18a) \quad \tilde{U}_{\rho} = \frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho} \frac{\partial \Psi}{\partial \theta}, \quad \tilde{U}_{\theta} = \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} - \frac{\partial \Psi}{\partial \rho},$$

$$(18b) \quad \frac{1}{2} \tilde{\Sigma}_{\rho\rho} = -\frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{1-\nu} \nabla_{\rho}^2 \Phi,$$

$$(18c) \quad \frac{1}{2} \tilde{\Sigma}_{\rho\theta} = -\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{2} \nabla_{\rho}^2 \Psi,$$

$$(18d) \quad \frac{1}{2} \tilde{\Sigma}_{\theta\theta} = \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Psi}{\partial \theta} \right) + \frac{\nu}{1-\nu} \nabla_{\rho}^2 \Phi,$$

where

$$(19) \quad \tilde{U}_\alpha \equiv \int_0^\infty U_\alpha e^{-s\tau} d\tau, \quad \tilde{\Sigma}_{\alpha\beta} \equiv \int_0^\infty \tilde{\sigma}_{\alpha\beta} e^{-s\tau} d\tau,$$

and ∇_ρ^2 denotes the Laplacian in polar coordinates. Substituting the results of Eqs. (14) and (16) into Eq. (18), we obtain

$$(20a) \quad \rho \tilde{U}_\rho = -\frac{1}{8s} M_L^2 \tilde{\omega}^2 \rho^4 + \mathcal{I}_{0,1}(M_L \rho s) \mathcal{A}_0 - \frac{3}{8s} M_L^2 \rho^3 \cos \theta \\ + \sum_{m=1}^{\infty} [\mathcal{I}_{m,1}(M_L \rho s) \mathcal{A}_m + m I_m(M_T \rho s) \mathcal{B}_m] \cos(m\theta),$$

$$(20b) \quad \rho \tilde{U}_\theta = \frac{1}{8s} M_L^2 \rho^3 \sin \theta \\ + \sum_{m=1}^{\infty} [-m I_m(M_L \rho s) \mathcal{A}_m - \mathcal{I}_{m,1}(M_T \rho s) \mathcal{B}_m] \sin(m\theta),$$

$$(20c) \quad \frac{\rho^2}{2} \tilde{\Sigma}_{\rho\rho} = -\frac{1}{8s} \frac{3+\nu}{1-\nu} M_L^2 \tilde{\omega}^2 \rho^4 + \mathcal{I}_{0,2}(M_L \rho s) \mathcal{A}_0 - \frac{1}{4s} \frac{3+\nu}{1-\nu} M_L^2 \rho^3 \cos \theta \\ + \sum_{m=1}^{\infty} [\mathcal{I}_{m,2}(M_L \rho s) \mathcal{A}_m + \mathcal{I}_{m,3}(M_T \rho s) \mathcal{B}_m] \cos(m\theta),$$

$$(20d) \quad \frac{\rho^2}{2} \tilde{\Sigma}_{\rho\theta} = \frac{1}{4s} M_L^2 \rho^3 \sin \theta \\ + \sum_{m=1}^{\infty} [-\mathcal{I}_{m,3}(M_L \rho s) \mathcal{A}_m - \mathcal{I}_{m,4}(M_T \rho s) \mathcal{B}_m] \sin(m\theta),$$

$$(20e) \quad \frac{\rho^2}{2} \tilde{\Sigma}_{\theta\theta} = -\frac{1}{8s} \frac{1+3\nu}{1-\nu} M_L^2 \tilde{\omega}^2 \rho^4 - \mathcal{I}_{0,5}(M_L \rho s) \mathcal{A}_0 - \frac{1}{4s} \frac{1+3\nu}{1-\nu} M_L^2 \rho^3 \cos \theta \\ + \sum_{m=1}^{\infty} [-\mathcal{I}_{m,5}(M_L \rho s) \mathcal{A}_m - \mathcal{I}_{m,3}(M_T \rho s) \mathcal{B}_m] \cos(m\theta),$$

where the auxiliary functions are defined as

$$(21a) \quad \mathcal{I}_{m,1}(z) \equiv m I_m(z) + z I_{m+1}(z),$$

$$(21b) \quad \mathcal{I}_{m,2}(z) \equiv \left[m(m-1) + \frac{z^2}{1-\nu} \right] I_m(z) - z I_{m+1}(z),$$

$$(21c) \quad \mathcal{I}_{m,3}(z) \equiv m(m-1) I_m(z) + m z I_{m+1}(z),$$

$$(21d) \quad \mathcal{I}_{m,4}(z) \equiv \left[m(m-1) + \frac{z^2}{2} \right] I_m(z) - z I_{m+1}(z),$$

$$(21e) \quad \mathcal{I}_{m,5}(z) \equiv \left[m(m-1) - \frac{\nu z^2}{1-\nu} \right] I_m(z) - z I_{m+1}(z).$$

We now determine the coefficients such that Eq. (20) satisfies the boundary conditions (1). Rewriting Eq. (1) in the Laplace domain, we obtain

$$(22) \quad \tilde{\Sigma}_{\rho\rho} \Big|_{\rho=1} = -\tilde{\sigma}_0 \frac{1}{\pi} \left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos(m\theta) \right] \frac{1}{s}, \quad \tilde{\Sigma}_{\rho\theta} \Big|_{\rho=1} = 0.$$

The coefficients are determined by matching these expressions with $\tilde{\Sigma}_{\rho\rho}$ and $\tilde{\Sigma}_{\rho\theta}$ at $\rho = 1$. For the axisymmetric mode $m = 0$, we obtain

$$(23) \quad -\frac{1}{8s} \frac{3+\nu}{1-\nu} M_L^2 \tilde{\omega}^2 + \mathcal{I}_{0,2}(M_L s) \mathcal{A}_0 = -\frac{1}{4\pi s} \tilde{\sigma}_0,$$

which yields

$$(24) \quad \mathcal{A}_0 = -\frac{\mathfrak{C}}{4} \frac{1}{s \mathcal{I}_{0,2}(M_L s)},$$

where

$$(25) \quad \mathfrak{C} \equiv \frac{1}{\pi} \tilde{\sigma}_0 - \frac{1}{2} \frac{3+\nu}{1-\nu} M_L^2.$$

For the $m = 1$ mode, we obtain

$$(26a) \quad -\frac{1}{4s} \frac{3+\nu}{1-\nu} M_L^2 + \mathcal{I}_{1,2}(M_L s) \mathcal{A}_1 + \mathcal{I}_{1,3}(M_T s) \mathcal{B}_1 = -\frac{1}{2\pi s} \tilde{\sigma}_0,$$

$$(26b) \quad \frac{1}{4s} M_L^2 - \mathcal{I}_{1,3}(M_L s) \mathcal{A}_1 - \mathcal{I}_{1,4}(M_T s) \mathcal{B}_1 = 0,$$

which leads to

$$(27a) \quad \mathcal{A}_1 = -\frac{1}{s \mathcal{D}_1(s)} \left[\frac{\mathfrak{C}}{2} \mathcal{I}_{1,4}(M_T s) + \frac{1}{4} M_L^2 \mathcal{I}_{1,3}(M_T s) \right],$$

$$(27b) \quad \mathcal{B}_1 = \frac{1}{s \mathcal{D}_1(s)} \left[\frac{\mathfrak{C}}{2} \mathcal{I}_{1,3}(M_L s) + \frac{1}{4} M_L^2 \mathcal{I}_{1,2}(M_L s) \right].$$

Here,

$$(28) \quad \mathcal{D}_m(s) \equiv \mathcal{I}_{m,2}(M_L s) \mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,3}(M_L s) \mathcal{I}_{m,3}(M_T s).$$

Finally, for modes $m \geq 2$, we obtain

$$(29a) \quad \mathcal{I}_{m,2}(M_L s) \mathcal{A}_m + \mathcal{I}_{m,3}(M_T s) \mathcal{B}_m = -\frac{1}{2\pi s} \tilde{\sigma}_0 \quad (m \geq 2),$$

$$(29b) \quad -\mathcal{I}_{m,3}(M_L s)\mathcal{A}_m - \mathcal{I}_{m,4}(M_T s)\mathcal{B}_m = 0 \quad (m \geq 2),$$

from which

$$(30) \quad \mathcal{A}_m = -\frac{1}{2\pi}\tilde{\sigma}_0 \frac{\mathcal{I}_{m,4}(M_T s)}{s\mathcal{D}_m(s)}, \quad \mathcal{B}_m = \frac{1}{2\pi}\tilde{\sigma}_0 \frac{\mathcal{I}_{m,3}(M_L s)}{s\mathcal{D}_m(s)} \quad (m \geq 2).$$

Substituting these coefficients into Eq. (20), the displacement and stress fields can be expressed in Fourier series form as

$$(31) \quad \begin{Bmatrix} \tilde{U}_\rho \\ \tilde{\Sigma}_{\rho\rho} \\ \tilde{\Sigma}_{\theta\theta} \end{Bmatrix} = \sum_{m=0}^{\infty} \begin{Bmatrix} \tilde{U}_\rho^{(m)} \\ \tilde{\Sigma}_{\rho\rho}^{(m)} \\ \tilde{\Sigma}_{\theta\theta}^{(m)} \end{Bmatrix} \cos(m\theta), \quad \begin{Bmatrix} \tilde{U}_\theta \\ \tilde{\Sigma}_{\rho\theta} \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} \tilde{U}_\theta^{(m)} \\ \tilde{\Sigma}_{\rho\theta}^{(m)} \end{Bmatrix} \sin(m\theta).$$

Here, we present only the expressions for $\tilde{\Sigma}_{\rho\rho}$; the remaining components are summarized in Table 1. The explicit forms of $\tilde{\Sigma}_{\rho\rho}^{(m)}$ are given by

$$(32a) \quad \tilde{\Sigma}_{\rho\rho}^{(0)} = -\frac{1}{4s} \frac{3+\nu}{1-\nu} M_L^2 \tilde{\omega}^2 \rho^2 - \frac{\mathfrak{C}}{2} \frac{\mathcal{I}_{0,2}(M_L \rho s)}{\rho^2 s \mathcal{I}_{0,2}(M_L s)},$$

$$(32b) \quad \begin{aligned} \tilde{\Sigma}_{\rho\rho}^{(1)} = & -\frac{1}{2s} \frac{3+\nu}{1-\nu} M_L^2 \rho \\ & - \mathfrak{C} \frac{\mathcal{I}_{1,2}(M_L \rho s) \mathcal{I}_{1,4}(M_T s) - \mathcal{I}_{1,3}(M_T \rho s) \mathcal{I}_{1,3}(M_L s)}{\rho^2 s \mathcal{D}_1(s)} \\ & - \frac{1}{2} M_L^2 \frac{\mathcal{I}_{1,2}(M_L \rho s) \mathcal{I}_{1,3}(M_T s) - \mathcal{I}_{1,3}(M_T \rho s) \mathcal{I}_{1,2}(M_L s)}{\rho^2 s \mathcal{D}_1(s)}, \end{aligned}$$

$$(32c) \quad \tilde{\Sigma}_{\rho\rho}^{(m)} = -\frac{1}{\pi} \tilde{\sigma}_0 \frac{\mathcal{I}_{m,2}(M_L \rho s) \mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,3}(M_T \rho s) \mathcal{I}_{m,3}(M_L s)}{\rho^2 s \mathcal{D}_m(s)},$$

which completes the formulation.

The solution at an arbitrary time τ can be obtained by applying the inverse Laplace transform to Eq. (31), namely,

$$(33a) \quad \tilde{u}_\alpha(\rho, \theta, \tau) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{U}_\alpha(\rho, \theta, s) e^{st} ds,$$

$$(33b) \quad \tilde{\sigma}_{\alpha\beta}(\rho, \theta, \tau) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{\Sigma}_{\alpha\beta}(\rho, \theta, s) e^{st} ds,$$

where $\gamma > 0$ is a real constant chosen such that it exceeds the real parts of all poles of the integrands. In practice, the inverse Laplace transform is often evaluated numerically. However, by reducing the inverse Laplace transform to a contour integral in the complex plane and applying the residue theorem, it is also possible to compute the solution in an explicit series representation [10, 14–16].

Table 1. List of the expressions of $\tilde{U}_\alpha^{(m)}$ and $\tilde{\Sigma}_{\alpha\beta}^{(0)}$.

$\tilde{U}_\rho^{(0)}$	$-\frac{1}{8s}M_L^2\tilde{\omega}^2\rho^3 - \frac{\mathfrak{C}}{4}\frac{\mathcal{I}_{0,1}(M_L\rho s)}{\rho s\mathcal{I}_{0,2}(M_L s)}$
$\tilde{\Sigma}_{\rho\rho}^{(0)}$	$-\frac{1}{4s}\frac{3+\nu}{1-\nu}M_L^2\tilde{\omega}^2\rho^2 - \frac{\mathfrak{C}}{2}\frac{\mathcal{I}_{0,2}(M_L\rho s)}{\rho^2 s\mathcal{I}_{0,2}(M_L s)}$
$\tilde{\Sigma}_{\theta\theta}^{(0)}$	$-\frac{1}{4s}\frac{1+3\nu}{1-\nu}M_L^2\tilde{\omega}^2\rho^2 + \frac{\mathfrak{C}}{2}\frac{\mathcal{I}_{0,5}(M_L\rho s)}{\rho^2 s\mathcal{I}_{0,2}(M_L s)}$
$\tilde{U}_\rho^{(1)}$	$-\frac{3}{8s}M_L^2\rho^2 - \frac{\mathfrak{C}}{2}\frac{\mathcal{I}_{1,1}(M_L\rho s)\mathcal{I}_{1,4}(M_T s) - I_1(M_T\rho s)\mathcal{I}_{1,3}(M_L s)}{\rho s\mathcal{D}_1(s)}$ $-\frac{1}{4}M_L^2\frac{\mathcal{I}_{1,1}(M_L\rho s)\mathcal{I}_{1,3}(M_T s) - I_1(M_T\rho s)\mathcal{I}_{1,2}(M_L s)}{\rho s\mathcal{D}_1(s)}$
$\tilde{U}_\theta^{(1)}$	$\frac{1}{8s}M_L^2\rho^2 + \frac{\mathfrak{C}}{2}\frac{I_1(M_L\rho s)\mathcal{I}_{1,4}(M_T s) - \mathcal{I}_{1,1}(M_T\rho s)\mathcal{I}_{1,3}(M_L s)}{\rho s\mathcal{D}_1(s)}$ $+\frac{1}{4}M_L^2\frac{I_1(M_L\rho s)\mathcal{I}_{1,3}(M_T s) - \mathcal{I}_{1,1}(M_T\rho s)\mathcal{I}_{1,2}(M_L s)}{\rho s\mathcal{D}_1(s)}$
$\tilde{\Sigma}_{\rho\rho}^{(1)}$	$-\frac{1}{2s}\frac{3+\nu}{1-\nu}M_L^2\rho - \mathfrak{C}\frac{\mathcal{I}_{1,2}(M_L\rho s)\mathcal{I}_{1,4}(M_T s) - \mathcal{I}_{1,3}(M_T\rho s)\mathcal{I}_{1,3}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$ $-\frac{1}{2}M_L^2\frac{\mathcal{I}_{1,2}(M_L\rho s)\mathcal{I}_{1,3}(M_T s) - \mathcal{I}_{1,3}(M_T\rho s)\mathcal{I}_{1,2}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$
$\tilde{\Sigma}_{\rho\theta}^{(1)}$	$\frac{1}{2s}M_L^2\rho + \mathfrak{C}\frac{\mathcal{I}_{1,3}(M_L\rho s)\mathcal{I}_{1,4}(M_T s) - \mathcal{I}_{1,4}(M_T\rho s)\mathcal{I}_{1,3}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$ $+\frac{1}{2}M_L^2\frac{\mathcal{I}_{1,3}(M_L\rho s)\mathcal{I}_{1,3}(M_T s) - \mathcal{I}_{1,4}(M_T\rho s)\mathcal{I}_{1,2}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$
$\tilde{\Sigma}_{\theta\theta}^{(1)}$	$-\frac{1}{2s}\frac{1+3\nu}{1-\nu}M_L^2\rho + \mathfrak{C}\frac{\mathcal{I}_{1,5}(M_L\rho s)\mathcal{I}_{1,4}(M_T s) - \mathcal{I}_{1,3}(M_T\rho s)\mathcal{I}_{1,3}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$ $+\frac{1}{2}M_L^2\frac{\mathcal{I}_{1,5}(M_L\rho s)\mathcal{I}_{1,3}(M_T s) - \mathcal{I}_{1,3}(M_T\rho s)\mathcal{I}_{1,2}(M_L s)}{\rho^2 s\mathcal{D}_1(s)}$
$\tilde{U}_\rho^{(m)}$	$-\frac{1}{2\pi}\tilde{\sigma}_0\frac{\mathcal{I}_{m,1}(M_L\rho s)\mathcal{I}_{m,4}(M_T s) - mI_m(M_T\rho s)\mathcal{I}_{m,3}(M_L s)}{\rho s\mathcal{D}_m(s)} \quad (m \geq 2)$
$\tilde{U}_\theta^{(m)}$	$\frac{1}{2\pi}\tilde{\sigma}_0\frac{mI_m(M_L\rho s)\mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,1}(M_T\rho s)\mathcal{I}_{m,3}(M_L s)}{\rho s\mathcal{D}_m(s)} \quad (m \geq 2)$
$\tilde{\Sigma}_{\rho\rho}^{(m)}$	$-\frac{1}{\pi}\tilde{\sigma}_0\frac{\mathcal{I}_{m,2}(M_L\rho s)\mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,3}(M_T\rho s)\mathcal{I}_{m,3}(M_L s)}{\rho^2 s\mathcal{D}_m(s)} \quad (m \geq 2)$
$\tilde{\Sigma}_{\rho\theta}^{(m)}$	$\frac{1}{\pi}\tilde{\sigma}_0\frac{\mathcal{I}_{m,3}(M_L\rho s)\mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,4}(M_T\rho s)\mathcal{I}_{m,3}(M_L s)}{\rho^2 s\mathcal{D}_m(s)} \quad (m \geq 2)$
$\tilde{\Sigma}_{\theta\theta}^{(m)}$	$\frac{1}{\pi}\tilde{\sigma}_0\frac{\mathcal{I}_{m,5}(M_L\rho s)\mathcal{I}_{m,4}(M_T s) - \mathcal{I}_{m,3}(M_T\rho s)\mathcal{I}_{m,3}(M_L s)}{\rho^2 s\mathcal{D}_m(s)} \quad (m \geq 2)$

4 STEADY-STATE SOLUTION

The steady-state solution can be obtained from the dynamic solution by taking the limit $\tau \rightarrow \infty$. In this context, the long-time behavior of a physical quantity $f(t)$ can be evaluated by invoking the final value theorem of the Laplace transform [10, 11, 16]. Specifically, if $\tilde{f}(s)$ denotes the Laplace transform of $f(t)$, its asymptotic value is given by

$$(34) \quad \lim_{\tau \rightarrow \infty} f(\tau) = \lim_{s \rightarrow 0} [s \tilde{f}(s)].$$

This theorem can be directly applied to Eqs. (31) and (32). However, in general, the quantities $\tilde{U}_\rho^{(1)}$ and $\tilde{U}_\theta^{(1)}$ diverge in the limit $s \rightarrow 0$. For these expressions to remain finite, the following condition must be satisfied:

$$(35) \quad \frac{1}{2}\mathfrak{C} = \frac{1}{4}M_L^2 \iff \tilde{\sigma}_0 = \frac{2\pi}{1-\nu}M_L^2,$$

This condition represents the balance between gravity and the reaction force exerted by the ground.

In what follows, we focus on the stress field. Taking the limit $s \rightarrow 0$, the steady-state expressions for each stress component are obtained as

$$(36a) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\rho}}{G} = \frac{3+\nu}{4}(1-\rho^2)\tilde{\omega}^2 - \left[1 + 2 \sum_{m=1}^{\infty} \rho^m \cos(m\theta) + \frac{1-\rho^2}{\rho^2} \sum_{m=2}^{\infty} m \rho^m \cos(m\theta) \right],$$

$$(36b) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\theta}}{G} = \frac{1-\rho^2}{\rho^2} \sum_{m=2}^{\infty} m \rho^m \sin(m\theta),$$

$$(36c) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\theta\theta}}{G} = \frac{3+\nu}{4} \left(1 - \frac{1+3\nu}{3+\nu} \rho^2 \right) \tilde{\omega}^2 - \left[1 + 2 \sum_{m=1}^{\infty} \rho^m \cos(m\theta) - \frac{1-\rho^2}{\rho^2} \sum_{m=2}^{\infty} m \rho^m \cos(m\theta) \right].$$

To further simplify these expressions, we introduce the complex variable [17]

$$(37) \quad z \equiv \rho e^{i\theta},$$

and denote its real and imaginary parts by $\Re z$ and $\Im z$, respectively. Noting that

$$(38) \quad \Re z^n = \rho^n \cos(n\theta), \quad \Im z^n = \rho^n \sin(n\theta),$$

Eq. (36a) can be rewritten as

$$(39) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\rho}}{G} = \frac{3+\nu}{4} (1-\rho^2) \tilde{\omega}^2 - \Re \left(1 + 2 \sum_{m=1}^{\infty} z^m + \frac{1-\rho^2}{\rho^2} \sum_{m=2}^{\infty} m z^m \right).$$

Since $|z| < 1$ inside the disk ($r < R$), the following summation formulas hold:

$$(40) \quad \sum_{m=1}^{\infty} z^m = \frac{z}{1-z}, \quad \sum_{m=2}^{\infty} m z^m = \frac{(2-z)z^2}{(1-z)^2}.$$

Substituting these relations into Eq. (39), we obtain

$$(41a) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\rho}}{G} = \frac{3+\nu}{4} (1-\rho^2) \tilde{\omega}^2 - \Re \left[\frac{1+z}{1-z} + \frac{1-\rho^2}{\rho^2} \frac{(2-z)z^2}{(1-z)^2} \right].$$

By the same procedure, the remaining components are obtained as

$$(41b) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\theta}}{G} = \frac{1-\rho^2}{\rho^2} \Im \left[\frac{(2-z)z^2}{(1-z)^2} \right],$$

$$(41c) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\theta\theta}}{G} = \frac{3+\nu}{4} \left(1 - \frac{1+3\nu}{3+\nu} \rho^2 \right) \tilde{\omega}^2 - \Re \left[\frac{1+z}{1-z} - \frac{1-\rho^2}{\rho^2} \frac{(2-z)z^2}{(1-z)^2} \right].$$

We note that this expression exactly incorporates contributions from all modes ($m \rightarrow \infty$), and therefore no truncation is required. These expressions can equivalently be written as

$$(42a) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\rho} + \sigma_{\theta\theta}}{G} = \left[\frac{3+\nu}{2} - (1+\nu)\rho^2 \right] \tilde{\omega}^2 - \Re [\Phi(z)],$$

$$(42b) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\theta}}{G} = \frac{1-\nu}{2} \rho^2 \tilde{\omega}^2 + 2 \frac{1-\rho^2}{\rho^2} \Psi(z),$$

where

$$(43) \quad \Phi(z) \equiv \frac{1+z}{1-z}, \quad \Psi(z) \equiv \frac{(2-z)z^2}{(1-z)^2},$$

which can be regarded as Goursat stress functions [18].

Finally, we consider the case of pure free rotation, in which neither external loads nor gravity are present. Although the present nondimensionalization employs the gravitational acceleration, the parameter

$$(44) \quad M_L \tilde{\omega} = \frac{R\omega}{v_L},$$

does not involve gravity. Accordingly, the appropriate limit is

$$(45) \quad M_L \rightarrow 0, \quad M_L \tilde{\omega} = \text{const.}$$

Applying this limit to Eq. (41), we obtain

$$(46a) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\rho\rho}}{G} = \frac{3+\nu}{4} (1-\rho^2) \tilde{\omega}^2, \quad \sigma_{\rho\theta} = 0,$$

$$(46b) \quad \frac{1-\nu}{M_L^2} \frac{\sigma_{\theta\theta}}{G} = \frac{3+\nu}{4} \left(1 - \frac{1+3\nu}{3+\nu} \rho^2 \right) \tilde{\omega}^2.$$

These results coincide with the classical solutions well established in textbooks on strength of materials [3, 6].

5 RESULTS

In photoelastic experiments [19–22], the observed fringe patterns are directly related to the difference between the principal stresses [8, 22]. Motivated by this experimental background, we focus on the principal stress difference as a key quantity characterizing the stress field inside the elastic disk.

The principal stress difference is defined as

$$(47) \quad \Delta\sigma \equiv \sqrt{(\sigma_{\rho\rho} - \sigma_{\theta\theta})^2 + 4\sigma_{\rho\theta}^2},$$

which is invariant under rotations of the coordinate axes and therefore provides a physically meaningful scalar measure of the internal stress state.

Using the steady-state stress components derived in Section 4, the spatial distribution of $\Delta\sigma$ can be evaluated throughout the disk. Figure 2 shows representative

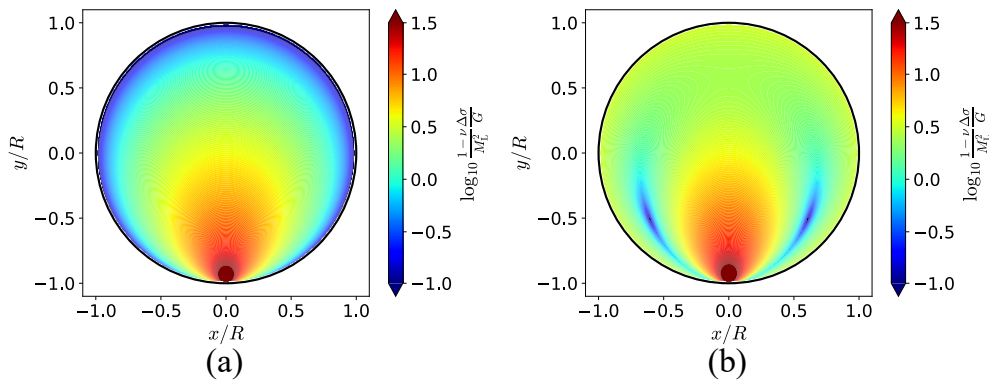


Figure 2. (Color online) Contour plots of the principal stress difference $\Delta\sigma$ for Poisson's ratio $\nu = 0.3$: (a) $\tilde{\omega} = 0.1$ and (b) 3.0.

contour plots of $\Delta\sigma$ for Poisson's ratio $\nu = 0.3$, comparing two typical cases: a small angular velocity $\tilde{\omega} = 0.1$ (a) and a large angular velocity $\tilde{\omega} = 3.0$ (b).

For $\tilde{\omega} = 0.1$, the contour pattern is dominated by the localized normal load applied at $\theta = 0$. The distribution of $\Delta\sigma$ is symmetric with respect to the loading direction, and no pronounced secondary extrema are observed away from the contact point.

In contrast, for $\tilde{\omega} = 3.0$, the stress pattern changes qualitatively. As shown in Fig. 2(b), a clear local minimum of the principal stress difference appears at a finite angular position $\theta > 0$, indicating that the rotation significantly modifies the internal stress field.

6 DISCUSSION

The emergence of a local minimum of the principal stress difference at $\theta > 0$, observed in Fig. 2(b), is a distinctive feature of the rotating case and does not appear in the non-rotating limit. This behavior can be understood from the analytical structure of the steady-state stress field.

As shown in Section 4, the combination of stress components relevant to the principal stress difference can be written in complex form as Eq. (42b). The principal stress difference $\Delta\sigma$ therefore results from the competition between two contributions: an isotropic term proportional to $\rho^2\tilde{\omega}^2$, originating from the centrifugal body force, and a spatially nonuniform term governed by the complex function $\Psi(z)$, which reflects the boundary-induced stress concentration.

For small angular velocities ($\tilde{\omega} < 1$), the rotational contribution is weak, and the stress field is dominated by the boundary term. As a result, $\Delta\sigma$ attains its extrema along the loading direction ($\theta = 0$), consistent with the symmetric pattern observed in Fig. 2(a).

When $\tilde{\omega}$ becomes sufficiently large, however, the rotational term modifies the balance between the real and imaginary parts of the complex stress combination. This leads to a stationary point of $\Delta\sigma$ at a finite angle $\theta > 0$, giving rise to the off-axis local minimum observed in Fig. 2(b). From a symmetry perspective, rotation introduces a preferred circumferential direction through the centrifugal body force, thereby breaking the mirror symmetry of the stress field with respect to the loading axis.

These results have direct implications for photoelastic experiments on rotating or driven granular disks. The appearance of off-axis minima in the principal stress difference suggests that photoelastic fringe patterns cannot be interpreted solely in terms of contact geometry when rotational effects are significant. Instead, body forces associated with rotation must be taken into account to correctly infer internal stress distributions from observed fringe patterns.

Finally, although the present analysis focuses on a solid disk with a single contact point, the same theoretical framework can be extended to more complex situations, including tangential contact forces, multiple contact points, and annular (hollow) disks. Such extensions are expected to further enrich the structure of the principal stress difference and the corresponding photoelastic patterns.

7 CONCLUSIONS

In this study, we have presented an analytical solution for the steady-state stress field of a rotating elastic disk under gravity, supported by a localized contact with a rigid ground. The problem was formulated within the framework of two-dimensional linear elastodynamics, and the steady-state solution was obtained as the limiting case of the dynamic equations. This approach allows a consistent treatment of gravity as a body force and of the singular boundary traction associated with point contact.

By employing scalar and vector potential representations together with Laplace-transform techniques, we derived explicit expressions for the displacement and stress fields under plane-stress conditions. The resulting solution admits a natural modal decomposition in the angular direction. The monopole contribution represents a pure torque mode, while the dipole contribution originates from the uniform body force associated with gravity. Higher-order modes are self-equilibrated and describe localized stress transmission emanating from the contact point. This decomposition provides a clear physical interpretation of the respective roles played by gravity and contact in shaping the non-axisymmetric stress field. In the absence of contact, the present formulation correctly recovers the classical stress distribution of a freely rotating disk, confirming the consistency of the theory.

The present analysis also highlights the usefulness of an elastodynamic formulation even for problems with a steady-state final configuration. In particular, the dipolar stress mode induced by gravity cannot be obtained within a purely steady-state Airy-stress-function framework [23] without body forces, but emerges naturally as a particular solution of the elastodynamic equations. This observation clarifies the theoretical origin of stress components that are otherwise difficult to interpret in steady-state elasticity.

Several extensions of the present work are worth pursuing. First, the analytical framework developed here can be directly applied to fully dynamic problems involving time-dependent rotation or transient contact forces. Since similar elastodynamic techniques have been successfully employed in related contexts, the extension to dynamic loading conditions is expected to be straightforward in principle. Second, the present formulation can be adapted to cases in which the localized contact force acts predominantly in the tangential direction. In such situations, global equilibrium requires the introduction of an additional torque, which can be naturally incorporated

through appropriate boundary conditions within the present theoretical framework. Third, an important extension concerns hollow disks. In practical applications such as wheels, the elastic body often consists of an annular domain, with a rigid hub surrounded by an elastic rim or tire. The analysis of hollow disks with inner and outer boundaries is therefore essential for realistic modeling and can be addressed by extending the present solution to annular geometries.

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