

Dynamics of an Elastic One-link Manipulation Robot, Rotating About a Vertical Axis

S. Batchvarov, D. Tchotova

This paper considers the dynamics of an one-link anthropomorphic manipulation robot consisting of two concentrated masses m_0 and m , connected by an elastic beam (Fig. 1). The beam mass is neglected, its length is l and E is the Young's modulus, and I is the quadratic momentum of the link. The system is capable of rotating in the vertical Oxy -plane around the Oz -axis which passes through the O -mass center of m_0 and it also can rotate around the Ox -vertical axis. The setting into motion of the link in the Oxy -plane and about the Ox -axis takes in the O -joint with independent motors. The mass of these motors is included in m_0 . The C -mass center of the solid with mass m is situated along the elastic beam and lies at a d -distance from the A -point. The mechanical system has four degrees of freedom. So, let us consider as generalized co-ordinates the following: the φ -angle of the beam rotation about the Oz -axis, the η -deflection of the beam in the A -point, the ψ -deformation angle and the θ -rotation angle about the Ox -vertical axis. The ψ -angle is equal to the angle between the tangent to the link elastic line in the A -point and the undeformed state of the elastic line (Fig. 1).

Geometrically, the x and y co-ordinates of the C -mass center are:

$$(1) \quad \begin{aligned} x &= (r+l) \cos \varphi - \eta \sin \varphi + d \cdot \cos (\varphi + \psi), \\ y &= (r+l) \sin \varphi + \eta \cos \varphi + d \cdot \sin (\varphi + \psi), \end{aligned}$$

where r is the radius of the rotor actuating the link.

The kinetic energy of the so constructed model is:

$$(2) \quad \begin{aligned} T &= \frac{1}{2} J_1 \dot{\varphi}^2 + \frac{1}{2} J_2 \dot{\psi}^2 + J_2 \dot{\varphi} \dot{\psi} + \frac{1}{2} J_3 \dot{\theta}^2 + \frac{1}{2} m [(L^2 + \eta \xi) \dot{\varphi}^2 \\ &+ \xi (\xi + 2L \dot{\varphi}) + d (\dot{\varphi} + 2\dot{\psi}) \dot{\varphi} \eta \psi + \frac{1}{2} (L^2 + \xi^2) \dot{\theta}^2 + L \xi \dot{\theta}^2 \sin 2\varphi \\ &- \frac{1}{2} (L^2 - \xi^2) \dot{\theta}^2 \cos 2\varphi], \end{aligned}$$

where $L = r + l + d$, $\xi = \eta + d \psi$,

J_1 is the common inertia torque of the solid with mass m and of the rotor actuating the link in the Oxy -plane referred to the Oz -axis;

J_2 is the inertia torque of the solid with mass m referred to an axis parallel to Oz and passing through C ;

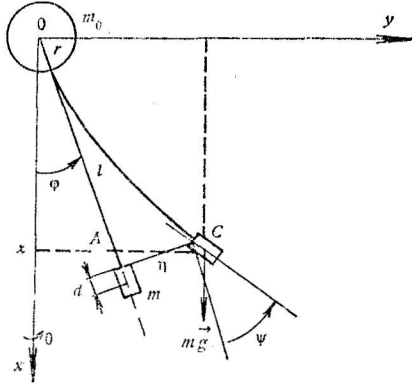


Fig. 1

J_3 is the common inertia torque of the solid with mass m and of the rotors actuating the manipulator referred to axes parallel to Ox , which pass through the respective mass centers.

The deformation energy of the mechanical system is:

$$(3) \quad \Pi = \frac{1}{2EI} (lM_0^2 + 2\frac{l^2}{2} M_0 P + \frac{l^3}{3} P^2),$$

as a result of which we get the following coefficients of influence:

$$(4) \quad \alpha_{\eta\eta} = \frac{l^3}{3EI}, \quad \alpha_{\eta\psi} = \frac{l^2}{2EI}, \quad \alpha_{\psi\psi} = \frac{l}{EI}.$$

The generalized Q_j forces ($j=1, 2, 3, 4$), which do not include the elastic forces, are defined on the basis of the virtual work expression and look as follows:

$$(5) \quad Q_\varphi = M_a^{(1)} - M_r^{(1)} - mg(L \sin \varphi + \xi \cos \varphi), \quad Q_\eta = -mg \sin \varphi, \\ Q_\psi = -mgd(\sin \varphi + \psi \cos \varphi), \quad Q_\theta = M_a^{(4)} - M_r^{(4)},$$

where $M_a^{(i)}$ are actuator torques and $M_r^{(i)}$ — resistance torques ($i=1, 4$), and they are functions of the respective generalized co-ordinates:

$$(6) \quad M_a^{(i)} = \alpha_{0i} - \alpha_{1i} \dot{q}_i + \alpha_{2i} \dot{q}_i^2, \quad M_r^{(i)} = \beta_{0i} - \beta_{1i} \dot{q}_i + \beta_{2i} \dot{q}_i^2, \\ (i=1, 4) \text{ and } q_1 = \varphi, \quad q_2 = \eta, \quad q_3 = \psi, \quad q_4 = \theta.$$

The differential motion equations are derived from Lagrange-II's equations by applying the method of the coefficients of influence as well. The following system of differential equations is obtained which describes the motion of the robot:

$$(7) \quad (J_1 + mL^2) \ddot{\varphi} + mL \ddot{\eta} + (J_2 + mLd) \ddot{\psi} = f_1(\ddot{q}_j, \dot{q}_j, q_j), \\ mL \ddot{\varphi} + m \ddot{\eta} + md \ddot{\psi} + \frac{12EI}{l^3} \eta - \frac{6EI}{l^2} \psi = f_2(\ddot{q}_j, \dot{q}_j, q_j), \\ (J_2 + mLd) \ddot{\varphi} + md \ddot{\eta} + (J_2 + md^2) \ddot{\psi} - \frac{6EI}{l^2} \eta + \frac{4EI}{l} \psi = f_3(\ddot{q}_j, \dot{q}_j, q_j), \\ \left(J_3 + \frac{1}{2} mL^2 \right) \ddot{\theta} = f_4(\ddot{q}_j, \dot{q}_j, q_j), \quad (j=1, 2, 3, 4).$$

Here by $f_i(\ddot{q}_j, \dot{q}_j, q_j)$ ($i, j=1, 2, 3, 4$) are presented the righthand sides of the equations, and they are functions of all generalized co-ordinates and of their first and second derivatives referred to time:

$$(8) \quad f_1(\ddot{q}_j, \dot{q}_j, q_j) = -m \{ \ddot{\eta} \xi_2 + \dot{\varphi} (\dot{\eta} \xi_2 + \eta \dot{\xi}_2) + d [\dot{\eta} \dot{\psi} + \eta (\dot{\psi}^2 + \psi \ddot{\psi})] \\ + g(L \sin \varphi + \xi \cos \varphi) - \left[\frac{1}{2} (L^2 - \xi^2) \sin 2\varphi + L \xi \cos 2\varphi \right] \dot{\theta}^2 + M_a^{(1)} - M_r^{(1)}, \\ f_2(\ddot{q}_j, \dot{q}_j, q_j) = m \left[(\dot{\varphi}^2 + \frac{1}{2} \dot{\theta}^2) \xi + d \dot{\varphi} \dot{\psi} - g \sin \varphi + \frac{1}{2} (L \sin 2\varphi \right. \\ \left. + \xi \cos 2\varphi) \dot{\theta}^2, \right. \\ \left. f_3(\ddot{q}_j, \dot{q}_j, q_j) = md \left[-(\ddot{\varphi} \eta + \dot{\varphi} \dot{\eta}) \psi + \dot{\varphi}^2 \eta + \frac{1}{2} \xi \dot{\theta}^2 - g(\sin \varphi + \psi \cos \varphi) \right] \right.$$

$$\begin{aligned}
& + \frac{1}{2} (L \sin 2\varphi + \xi \cos 2\varphi) \dot{\theta}^2 \Big], \\
f_4(\ddot{q}_j, \dot{q}_j, q_j) = & -m \left\{ \frac{1}{2} \xi^2 \ddot{\theta} + \xi \dot{\xi} \dot{\theta} + [L(\xi \ddot{\theta} + \dot{\xi} \dot{\theta}) + (L^2 - \xi^2) \dot{\phi} \dot{\theta}] \sin 2\varphi \right. \\
& \left. + \left[\frac{1}{2} (\xi^2 - L^2) \ddot{\theta} + (\dot{\xi} + 2L \dot{\phi}) \xi \dot{\theta} \right] \cos 2\varphi \right\} + M_a^{(4)} - M_r^{(4)},
\end{aligned}$$

where $\xi_2 = \eta + 2d\psi$.

The above system of differential equations is non-linear and is presented in the following standard form:

$$(9) \quad \sum_{i=1}^4 (a_{ij} \ddot{q}_j + c_{ij} \dot{q}_j) = \varepsilon f_{i0}(\ddot{q}_j, \dot{q}_j, q_j), \quad (i, j = 1, 2, 3, 4),$$

$$\text{where } [a_{ij}] = \begin{bmatrix} J_1 + mL^2 & mL & J_2 + mLd & 0 \\ mL & m & md & 0 \\ J_2 + mLd & md & J_2 + md^2 & 0 \\ 0 & 0 & 0 & J_3 + \frac{1}{2} mL^2 \end{bmatrix} \quad \text{and}$$

$$[c_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 \\ 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and } \varepsilon = \frac{d}{L} \ll 1 \text{ is a dimensionless value and is}$$

accepted as a small positive parameter. When $\varepsilon = 0$ the system (9) is linear and has the following characteristic equation:

$$(10) \quad a_{44} k_s^4 (b_0 k_s^4 + b_1 k_s^2 + b_2) = 0,$$

where a_{44} is the respective member of the inertia matrix, and

$$\begin{aligned}
(11) \quad b_0 = & -mJ_2(J_1 - J_2), \quad b_2 = -\frac{12E^2 l^2}{l^4} (J_1 + mL^2), \\
b_1 = & \frac{4EI}{l} m \left[\left(\frac{3d^2}{l^2} + \frac{3d}{l} + 1 \right) J_1 + \frac{3L(r-d)}{l^2} J_2 \right] + \frac{12EI}{l^3} J_2 (J_1 - J_2).
\end{aligned}$$

The roots of the characteristic equation (10) are the natural frequencies of the system:

$$\begin{aligned}
(12) \quad k_{1,2}^2 = & \frac{2EI}{l} \left\{ \frac{3}{l^2 m} + \frac{1}{J_2 (J_1 - J_2)} \left[\left(\frac{3d^2}{l^2} + \frac{3d}{l} + 1 \right) J_1 + \frac{3L(r-d)}{l^2} J_2 \pm \sqrt{K} \right] \right\}, \\
k_{3,4}^2 = & 0,
\end{aligned}$$

where

$$\begin{aligned}
K = & \left[\left(1 + \frac{3d}{l} + \frac{3d^2}{l^2} \right) J_1 + \frac{3L(r-d)}{l^2} J_2 \right]^2 + \frac{9(J_1 - J_2)^2 J_2^2}{l^4 m^2} \\
& + \frac{3(J_1 - J_2) J_2}{l^2 m} \left[\left(1 + \frac{6d}{l} + \frac{6d^2}{l^2} \right) J_1 + \frac{6L(r-d)}{l^2} J_2 - mL^2 \right].
\end{aligned}$$

The particular solutions of the non-excited system which results from (9) when $\varepsilon=0$, are as follows:

$$(13) \quad q_{j0} = a_s h_j^{(s)} \cos \gamma + \Phi_j(\gamma) h_j^{(4)}, \quad (j, s = 1, 2, 3, 4),$$

where a_s is the amplitude, $h_j^{(s)}$ are the natural form ($h_1^{(4)} = h_4^{(4)} = 1$; $h_2^{(4)} = h_3^{(4)} = 0$), $\gamma = k_s t + \alpha$, and $\Phi_j(\gamma)$ are the corrective functions. In a stationary condition:

$$(14) \quad M_{st}^{(i)} = M_{a,st}^{(i)} = M_{r,st}^{(i)}$$

from where depending on α_{ji} and β_{ji} ($j=0, 1, 2$; $i=1, 4$) the stationary angular speeds are derived:

$$(15) \quad \omega_{0i} = \frac{(\alpha_{1i} - \beta_{1i}) \pm \sqrt{(\alpha_{1i} - \beta_{1i})^2 - 4(\alpha_{2i} - \beta_{2i})(\alpha_{0i} - \beta_{0i})}}{2(\alpha_{2i} - \beta_{2i})}.$$

In (12) it can be seen that the system lacks internal resonance and further it is supposed that the external resonance is absent too, i. e. $k_s \neq \frac{p}{q} \omega_{0i}$ (p and q are natural, co-prime numbers). Under this condition the solution of the system (9) is sought according to the Kriloff-Bogolyuboff's asymptotic method [1] as follows:

$$(16) \quad q_j = a_s h_j^{(s)} \cos \gamma + \Phi_j(\gamma) h_j^{(4)} + \varepsilon u_j^{(1)}(a_s, \gamma) + \varepsilon^2 u_j^{(2)}(a_s, \gamma) + \dots, \\ (j=1, 2, 3, 4),$$

where $u_j^{(1)}(a_s, \gamma)$, $u_j^{(2)}(a_s, \gamma)$, $\dots$ ($j=1, 2, 3, 4$) are periodic functions, while the a_s amplitude and α phase, as well as $\Phi_j(\gamma)$ have to meet the differential equations:

$$(17) \quad \frac{da_s}{dt} = \varepsilon A_1(a_s) + \varepsilon^2 A_2(a_s) + \dots, \\ \frac{d\gamma}{dt} = k_s + \varepsilon B_1(a_s) + \varepsilon^2 B_2(a_s) + \dots, \\ \frac{d\Phi_j}{dt} = \omega_{0j} + \varepsilon D_1^{(j)}(\gamma) + \varepsilon^2 D_2^{(j)}(\gamma) + \dots, \quad (j=1, 4).$$

Here $A_1(a_s)$, $A_2(a_s)$, $\dots$, $B_1(a_s)$, $B_2(a_s)$, $\dots$, $D_1^{(j)}(\gamma)$, $D_2^{(j)}(\gamma)$, $\dots$ are sought functions. The $f_{0i}(\dot{q}_j, \dot{q}_j, q_j)$ functions are transformed into functions of a_s and γ , by initially adding and subtracting the respective stationary torque $M_{st}^{(i)}$ from (14) to the right-hand sides of the first and fourth equation. So $f_{0i}(a_s, \gamma)$ look as follows:

$$(18) \quad f_{0i}(a_s, \gamma) = F_0^{(i)} a_s^2 + \sum_{v=1}^2 \{ (F_v^{(i)} \cos v\gamma + F_{v+2}^{(i)} \sin v\gamma) a_s^v \\ + F_{v+4}^{(i)} \sin v\omega_{01} t + F_{v+6}^{(i)} \sin (2\gamma + v\omega_{01} t) + F_{v+8}^{(i)} \sin (2\gamma - v\omega_{01} t) \\ + F_{v+10}^{(i)} \cos (\gamma + v\omega_{01} t) + F_{v+12}^{(i)} \cos (\gamma - v\omega_{01} t) \},$$

where the coefficients $F_n^{(i)}$ ($i=1, 2, 3, 4$; $n=0, 1, 2, \dots, 14$) are shown in Table 1.

The expression (16) for q_j is differentiated twice, (17) are accounted for while the obtained results are put in (9). The coefficients in front of the first degree of ε are equalized at both sides of the equalities and the following system is obtained:

$$(19) \quad \sum_{j=1}^4 \left(a_{ij} \left(k_s^2 \frac{\partial^2 u_j^{(1)}}{\partial \gamma^2} + 2k_s \frac{\partial^2 u_j^{(1)}}{\partial \gamma \partial t} + \frac{\partial^2 u_j^{(1)}}{\partial t^2} \right) + c_{ij} u_j^{(1)} \right) = f_{0i}(a_s, \gamma)$$

$$-\sum_{j=1,4} a_{ij} k_s \frac{\partial D_1^{(j)}}{\partial \gamma} + 2a_s k_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) B_1 \cos \gamma + 2k_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) A_1 \sin \gamma,$$

($i=1, 2, 3, 4$).

In order to determine the sought unknowns, the functions $u_i^{(1)}(a_s, \gamma)$ are decomposed in a form similar to $f_{0i}(a_s, \gamma)$:

$$(20) \quad u_i^{(1)}(a_s, \gamma) = U_0^{(i)} a_s^2 + U_1^{(i)} \cos \gamma + \dots + U_{14}^{(i)} \cos(\gamma - 2\omega_{01} t).$$

The respective derivatives are found, substituted in (19) after which the coefficients in front of the same harmonic components at both sides of the obtained equalities are equalized. As a result 15 linear algebraic system equations are obtained from which the sought unknowns are determined:

$$(21) \quad F_0^{(i)} a_s^2 - \sum_{n=1,4} a_{1n} k_s \frac{\partial D_1^{(n)}}{\partial \gamma} = \sum_{j=1}^4 c_{ij} U_0^{(j)} a_s^2, \quad (i=1, 2, 3, 4).$$

From here

$$(22) \quad D_1^{(n)} = \frac{\alpha_{2n} - \beta_{2n}}{2a_{nn}} k_s h_n^{(s)2} \int_0^t a_s^2 [k_s + \varepsilon B_1(a_s)] dt, \quad (n=1, 4),$$

$$(23) \quad U_0^{(p)} = -(\alpha_{21} - \beta_{21}) \frac{l}{4EI} \lambda_p \frac{(5-p)J_2 + mL[(4-p)l + (5-p)d]}{J_1 + mL^2} k_s^2 h_1^{(s)2}, \quad (p=2, 3),$$

where $\lambda_2 = \frac{l}{3}$, $\lambda_3 = 1$, and $U_0^{(1)}, U_0^{(4)}$ are arbitrary constants.

The unknowns $A_1(a_s)$ and $B_1(a_s)$ are derived from the systems:

$$(24) \quad \sum_{j=1}^4 (-a_{ij} k_s^2 + c_{ij}) U_1^{(j)} = F_1^{(i)} a_s + 2k_s a_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) B_1,$$

$$(25) \quad \sum_{j=1}^4 (-a_{ij} k_s^2 + c_{ij}) U_3^{(j)} = F_3^{(i)} a_s + 2k_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) A_1, \quad (i=1, 2, 3, 4).$$

The basic determinants of (24) and (25) are equal to zero and in order to determine $U_n^{(j)}$ ($n=1, 3$; $j=1, 2, 3, 4$) the condition of forced vibration limitation should be maintained, i. e. the coefficients $U_n^{(j)}$ must be finite. It is supposed that

$$(26) \quad U_1^{(j)} = \sum_{r=1}^4 v_r h_j^{(r)} \quad \text{and} \quad U_3^{(j)} = \sum_{r=1}^4 w_r h_j^{(r)},$$

where v_r and w_r are unknown coefficients subject to determination. After substitution of (26) in (24) and (25) respectively and after certain transformations the following systems are obtained:

$$(27) \quad \sum_{r=1}^4 \sum_{j=1}^4 a_{ij} (k_r^2 - k_s^2) v_r h_j^{(r)} = F_1^{(i)} a_s + 2k_s a_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) B_1 \quad \text{and}$$

$$(28) \quad \sum_{r=1}^4 \sum_{j=1}^4 a_{ij} (k_r^2 - k_s^2) w_r h_j^{(r)} = F_3^{(i)} a_s + 2k_s \left(\sum_{j=1}^4 a_{ij} h_j^{(s)} \right) A_1, \quad (i=1, 2, 3, 4).$$

From the above, bearing in mind that the natural forms are orthogonal ones, the coefficients v_r and w_r are calculated:

$$(29) \quad v_r = \frac{a_s \sum_{i=1}^4 F_1^{(i)} h_i^{(r)} + 2\mu_r k_s a_s B_1}{\mu_r (k_r^2 - k_s^2)}, \quad w_r = \frac{a_s \sum_{i=1}^4 F_3^{(i)} h_i^{(r)} + 2\mu_r k_s A_1}{\mu_r (k_r^2 - k_s^2)},$$

where $\mu_r = \sum_{i=1}^4 \sum_{j=1}^4 a_{ij} h_j^{(r)} h_i^{(r)}$, ($r=1, 2, 3, 4$). In order to have finite amplitudes of the forced vibrations when $k_s = k_R$ (i. e. $r=s=R$) the following conditions should be valid:

$$(30) \quad a_R \sum_{i=1}^4 F_1^{(i)} h_i^{(R)} + 2\mu_R k_R a_R B_1 = 0,$$

$$(31) \quad a_R \sum_{i=1}^4 F_3^{(i)} h_i^{(R)} + 2\mu_R k_R A_1 = 0,$$

which results in:

$$(32) \quad B_1 = - \frac{\sum_{i=1}^4 F_1^{(i)} h_i^{(R)}}{2\mu_R k_R} = \text{const},$$

$$(33) \quad A_1 = - \frac{\sum_{i=1}^4 F_3^{(i)} h_i^{(R)} a_R}{2\mu_R k_R}.$$

When the conditions (30) and (31) are fulfilled, the amplitudes $U_1^{(j)}$ and $U_3^{(j)}$ look in the following way:

$$(34) \quad U_1^{(j)} = \sum_{\substack{r=1 \\ r \neq R}}^4 v_r h_j^{(r)} = \sum_{\substack{r=1 \\ r \neq R}}^4 h_j^{(r)} \frac{\sum_{i=1}^4 F_1^{(i)} h_i^{(r)} + 2\mu_r k_R B_1}{\mu_r (k_r^2 - k_R^2)} a_R,$$

$$U_3^{(j)} = \sum_{\substack{r=1 \\ r \neq R}}^4 w_r h_j^{(r)} = \sum_{\substack{r=1 \\ r \neq R}}^4 h_j^{(r)} \frac{\sum_{i=1}^4 a_R F_3^{(i)} h_i^{(r)} + \mu_r k_R A_1}{\mu_r (k_r^2 - k_R^2)},$$

($j=1, 2, 3, 4$).

The remaining amplitudes $U_n^{(j)}$ ($n=2, 4, 5, 6, \dots, 14$) are determined from linear algebraic systems:

$$(35) \quad \sum_{j=1}^4 (-4 k_s^2 a_{ij} + c_{ij}) U_p^{(j)} = F_p^{(i)} a_s^2, \quad (p=2, 4; i, s=1, 2, 3, 4);$$

as a result of which

$$(35)' \quad U_p^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [4k_s^2] F_p^{(i)} a_s^2}{D [4k_s^2]};$$

$$(36) \quad \sum_{j=1}^4 (-n^2 \omega_{01}^2 a_{ij} + c_{ij}) U_{n+4}^{(j)} = F_{n+4}^{(i)}, \quad (n=1, 2; \quad i=1, 2, 3, 4),$$

as a result of which

$$(36)' \quad U_{n+4}^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [(n\omega_{01})^2] F_{n+4}^{(i)}}{D [(n\omega_{01})^2]};$$

$$(37) \quad \sum_{j=1}^4 [-(2k_s + n\omega_{01})^2 a_{ij} + c_{ij}] U_{n+6}^{(j)} = F_{n+6}^{(i)}, \quad (n=1, 2; \quad i=1, 2, 3, 4),$$

as a result of which

$$(37)' \quad U_{n+6}^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [(2k_s + n\omega_{01})^2] F_{n+6}^{(i)}}{D [(2k_s + n\omega_{01})^2]};$$

$$(38) \quad \sum_{j=1}^4 [-(2k_s - n\omega_{01})^2 a_{ij} + c_{ij}] U_{n+8}^{(j)} = F_{n+8}^{(i)}, \quad (n=1, 2; \quad i=1, \dots, 4) \text{ and } \omega_{01} \neq \frac{2}{n} k_s,$$

as a result of which

$$(38)' \quad U_{n+8}^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [2k_s - n\omega_{01}]^2 F_{n+8}^{(i)}}{D [(2k_s - n\omega_{01})^2]};$$

$$(39) \quad \sum_{j=1}^4 [-(k_s + n\omega_{01})^2 a_{ij} + c_{ij}] U_{n+10}^{(j)} = F_{n+10}^{(i)}, \quad (n=1, 2; \quad i=1, \dots, 4),$$

as a result of which

$$(39)' \quad U_{n+10}^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [(k_s + n\omega_{01})^2] F_{n+10}^{(i)}}{D [(k_s + n\omega_{01})^2]};$$

$$(40) \quad \sum_{j=1}^4 [-(k_s - n\omega_{01})^2 a_{ij} + c_{ij}] U_{n+12}^{(j)} = F_{n+12}^{(i)}, \quad (n=1, 2; \quad i=1, \dots, 4), \quad \omega_{01} \neq \frac{k_s}{n},$$

as a result of which

$$(40)' \quad U_{n+12}^{(j)} = \frac{\sum_{i=1}^4 A_{ij} [(k_s - n\omega_{01})^2] F_{n+12}^{(i)}}{D [(k_s - n\omega_{01})^2]}.$$

Table 1

$i \backslash n$	1	2
0	$\frac{1}{2}(\alpha_{21} - \beta_{21}) k_s^2 h_1^{(s)2}$	0
1	0	$\frac{mL}{d}(\omega_{01}^2 + \frac{1}{2}\omega_{04}^2) H_1^{(s)}$
2	$-\frac{1}{2}(\alpha_{21} - \beta_{21}) k_s^2 h_1^{(s)2}$	0
3	$[(\alpha_{11} - \beta_{11}) - 2(\alpha_{21} - \beta_{21})\omega_{01}] k_s h_1^{(s)}$	0
4	$\frac{mL}{d}\omega_{01} h_2^{(s)}(H_1^{(s)} + d h_3^{(s)}) k_s$	$-\frac{mL}{d}[(\omega_{01} h_1^{(s)} + \frac{1}{2}\omega_{04} h_4^{(s)}) H_1^{(s)} + d\omega_{01} h_3^{(s)2}] k_s$
5	$\frac{mL}{d} g \left(\frac{1}{2} h_1^{(s)} H_2^{(s)} a_s^2 - L \right)$	$\frac{mL}{d} g \left(\frac{1}{4} a_s^2 h_1^{(s)2} - 1 \right)$
6	$\frac{mL}{2d} \omega_{04}^2 \left[L^2 - \frac{1}{2} a_s^2 (4L h_1^{(s)} H_2^{(s)} + H_1^{(s)2} - \frac{k_s^2}{\omega_{04}^2} L^2 h_4^{(s)2}) \right]$	$\frac{mL}{2d} \omega_{04}^2 \left\{ L - a_s^2 \left[L(h_1^{(s)2} - \frac{k_s^2}{2\omega_{04}^2} h_4^{(s)2}) + H_1^{(s)} h_1^{(s)} \right] \right\}$
7	$\pm \frac{mL}{4d} g a_s^2 h_1^{(s)} H_2^{(s)}$	$\pm \frac{mL}{8d} g a_s^2 h_1^{(s)2}$
8	$\mp \frac{mL}{2d} a_s^2 \omega_{04}^2 \left(\frac{1}{4} H_3^{(s)2} + \frac{1}{2} L H_2^{(s)} h_1^{(s)} + \frac{1}{4} \frac{k_s^2}{\omega_{04}^2} L^2 h_4^{(s)2} \right)$	$\mp \frac{mL}{4d} a_s^2 \omega_{04}^2 \left(\frac{k_s^2}{2\omega_{04}^2} L h_4^{(s)2} + H_3^{(s)} h_1^{(s)} \pm \frac{k_s}{\omega_{04}} H_4^{(s)} h_4^{(s)} \right)$
10	$\mp \frac{k_s}{\omega_{04}} L H_3^{(s)} h_4^{(s)}$	
11	$\frac{mL}{2d} g a_s H_3^{(s)}$	$-\frac{mL}{2d} g a_s h_1^{(s)}$
12	$\frac{mL^2}{2d} a_s \omega_{04}^2 (H_3^{(s)} \pm \frac{k_s}{\omega_{04}} L h_4^{(s)})$	$\frac{mL}{4d} a_s \omega_{04}^2 (H_4^{(s)} \pm 2L \frac{k_s}{\omega_{04}} h_4^{(s)})$
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where $H_1^{(s)} = h_2^{(s)} + d h_3^{(s)}$; $H_2^{(s)} = \frac{1}{2} L h_1^{(s)} + H_1^{(s)}$; $H_3^{(s)} = L h_1^{(s)} + H_1^{(s)}$; $H_4^{(s)} = 2L h_1^{(s)} + H_1^{(s)}$

Here with $D[pk_s \pm q\omega_{01}]^2$ are marked the values of the characteristic determinant in which k_s^2 is substituted by the respective expressions $(pk_s \pm q\omega_{01})^2$, p and q are natural numbers. With $A_{ij}[(pk_s \pm q\omega_{01})^2]$ are marked the respective minors.

The formulas derived above help to compose in first approximation the solution of the differential equations system of motion (7). (32) and (33) are substituted in the first two differential equations (17):

$$(41) \quad \frac{da_R}{dt} = \varepsilon A_1 = -\varepsilon N^{(R)} a_R, \text{ where } N^{(R)} = \frac{\sum_{i=1}^4 F_3^{(i)} h_i^{(R)}}{2\mu_R k_R},$$

$$\frac{d\gamma}{dt} = k_R + \varepsilon B_1 = \text{const} = C_R.$$

3	4
0	$\frac{1}{2}(\alpha_{24}-\beta_{24})k_s^2 h_4^{(s)2}$
$mL(\omega_{01}^2 h_2^{(s)} + \frac{1}{2} \omega_{04}^2 H_1^{(s)})$	0
0	$-\frac{1}{2}(\alpha_{24}-\beta_{24})k_s^2 h_4^{(s)2}$
0	$[(\alpha_{14}-\beta_{14})-2(\alpha_{24}-\beta_{24})\omega_{04}]k_s h_4^{(s)}$
$-mL[\omega_{01}h_2^{(s)}(h_1^{(s)}-\frac{1}{2}h_3^{(s)}+\frac{1}{2}\omega_{04}H_1^{(s)})h_4^{(s)}]k_s$	$\frac{mL}{2d}\omega_{04}H_1^{(s)2}k_s$
$mLg\left[\frac{1}{2}a_s^2 h_1^{(s)}\left(\frac{1}{2}h_1^{(s)}+h_3^{(s)}\right)-1\right]$	0
$\frac{mL}{2}\omega_{04}\{L-a_s^2[L(h_1^{(s)2}-\frac{k_s^2}{2\omega_{04}^2}h_4^{(s)2})+H_1^{(s)}h_1^{(s)}]\}$	$-\frac{mL}{d}\omega_{01}\omega_{04}\left[L^2+a_s^2\left(\frac{1}{2}H_1^{(s)2}-H_3^{(s)2}\right)\right]$
$\pm \frac{mL}{4}ga_s^2h_1^{(s)}\left(\frac{1}{2}h_1^{(s)}+h_3^{(s)}\right)$	0
$\mp \frac{mL}{4}a_s^2\omega_{04}^2\left(\frac{k_s^2}{2\omega_{04}^2}Lh_4^{(s)2}+H_3^{(s)}h_1^{(s)}\pm\frac{k_s}{\omega_{04}}H_4^{(s)}h_4^{(s)}\right)$	$\frac{mL}{2d}a_s^2k_s^2\left\{\frac{\omega_{04}}{k_s}[Lh_1^{(s)}(H_1^{(s)}+H_3^{(s)})+\frac{1}{2}H_1^{(s)2}\pm\frac{\omega_{01}}{k_s}(H_3^{(s)2}-\frac{1}{2}H_1^{(s)2})]+L\left(\frac{\omega_{01}}{k_s}\pm 1\right)H_3^{(s)}h_4^{(s)}\right\}$
$-\frac{mL}{2}ga_s(h_1^{(s)}+h_3^{(s)})$	0
$\frac{mL}{4}a_s\omega_{04}^2(H_4^{(s)}\pm 2L\frac{k_s}{\omega_{04}}h_4^{(s)})$	$-\frac{mL^2}{2d}a_s(2\omega_{01}\pm k_s)\omega_{04}(H_3^{(s)}\pm \frac{1}{2}L\frac{k_s}{\omega_{04}}h_4^{(s)})$

From here the laws of variation of the amplitude and phase in function of time are derived:

$$(42) \quad a_R = a_0 e^{-\varepsilon N^{(R)} t}, \quad \gamma = (k_R + \varepsilon B_1) t + \alpha = C_R t + \alpha,$$

where a_0 and α depend on the initial motion conditions.

The $D_1^{(n)}$ coefficients are determined from the expressions (22) bearing in mind (41) and (42). After the integration is obtained:

$$(43) \quad D_1^{(j)} = \frac{(\alpha_{2j}-\beta_{2j})k_R h_j^{(R)2} C_R}{4a_{jj} \varepsilon N^{(R)}} (1 - e^{-2\varepsilon N^{(R)} t}) a_0^2 \quad (j=1, 4).$$

After a one-time integration of (43) the expressions for the corrective functions Φ_j ($j=1, 4$) are found:

$$(44) \quad \Phi_j = \left[\omega_{0j} + \frac{(\alpha_{2j}-\beta_{2j})k_R h_j^{(R)2} C_R a_0^2}{4a_{jj} N^{(R)}} \right] t + \frac{(\alpha_{2j}-\beta_{2j})k_R h_j^{(R)2} C_R a_0^2}{8a_{jj} \varepsilon N^{(R)2}} (e^{-2\varepsilon N^{(R)} t} - 1).$$

So, all sought values are determined and the laws of the mechanical system motion can be written in the first approximation:

$$\begin{aligned}
 (45) \quad q_j^{(R)} = & a_0 e^{-\varepsilon N^{(R)} t} h_j^{(R)} \cos(C_R t + \alpha) + \left[\left(\omega_{0j} + \frac{a_{2j} - \beta_{2j}}{4a_{jj}} \cdot \frac{k_R h_j^{(R)2} C_R a_0^2}{N^{(R)}} \right) t \right. \\
 & + \frac{a_{2j} - \beta_{2j}}{8a_{jj}} \cdot \frac{k_R h_j^{(R)2} C_R a_0^2}{\varepsilon N^{(R)2}} (e^{-2\varepsilon N^{(R)} t} - 1) \left. \right] h_j^{(4)} + \varepsilon \left\{ a_0^2 e^{-2\varepsilon N^{(R)} t} \right\} U_0^{(j)} \\
 & + \sum_{v=1}^2 \frac{\sum_{i=1}^4 A_{ij} [4k_R^2] F_{2v}^{(i)}}{D [4k_R^2]} \cos \left[\frac{1 + (-1)^v}{4} \pi - 2(C_R t + \alpha) \right] + a_0 e^{-\varepsilon N^{(R)} t} \\
 & \cdot \sum_{\substack{r=1 \\ r \neq R}}^4 \frac{h_j^{(r)}}{\mu_r (k_r^2 - k_R^2)} \left[\left(\sum_{i=1}^4 F_1^{(i)} h_i^{(r)} + 2\mu_r k_R B_1 \right) \cos(C_R t + \alpha) + \left(\sum_{i=1}^4 F_3^{(i)} h_i^{(r)} \right. \right. \\
 & \left. \left. - 2\mu_r k_R N^{(R)} \right) \sin(C_R t + \alpha) \right] + \sum_{v=1}^2 \left\{ \frac{\sum_{i=1}^4 A_{ij} [(v\omega_{01})^2] F_{v+4}^{(i)}}{D [(v\omega_{01})^2]} \sin v\omega_{01} t \right. \\
 & + \frac{\sum_{i=1}^4 A_{ij} [(2k_R + v\omega_{01})^2] F_{v+6}^{(i)}}{D [(2k_R + v\omega_{01})^2]} \sin 2 \left[\left(C_R + \frac{v}{2} \omega_{01} \right) t + \alpha \right] + \frac{\sum_{i=1}^4 A_{ij} [(2k_R - v\omega_{01})^2] F_{v+8}^{(i)}}{D [(2k_R - v\omega_{01})^2]} \\
 & \cdot \sin 2 \left[\left(C_R - \frac{v}{2} \omega_{01} \right) t + \alpha \right] + \frac{\sum_{i=1}^4 A_{ij} [(k_R + v\omega_{01})^2] F_{v+10}^{(i)}}{D [(k_R + v\omega_{01})^2]} \cos [(C_R + v\omega_{01}) t + \alpha] \\
 & \left. \left. + \frac{\sum_{i=1}^4 A_{ij} [(k_R - v\omega_{01})^2] F_{v+12}^{(i)}}{D [(k_R - v\omega_{01})^2]} \cos [(C_R - v\omega_{01}) t + \alpha] \right\} \right\}.
 \end{aligned}$$

With the obtained solution can be determined the internal dynamical efforts in all of the links.

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