

Stiffness Matrix of a Bar Supported on Elastic Springs*

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A straight bar having constant moment of inertia, to which translational and rotational springs are attached is considered (Fig. 1). On its ends the bar is supported arbitrarily.

By means of such a model it is possible to represent a structure from the Fig. 2. a, if deformations due to shear and normal forces are neglected. Regular semi-frame (Fig. 2. b) will have its own interpretation in the model having no translational springs.

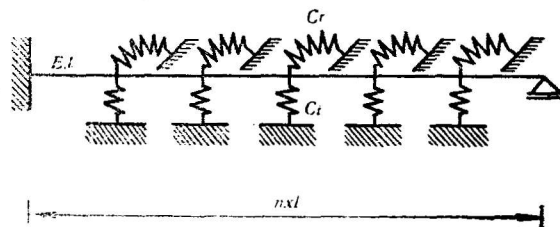


Fig. 1

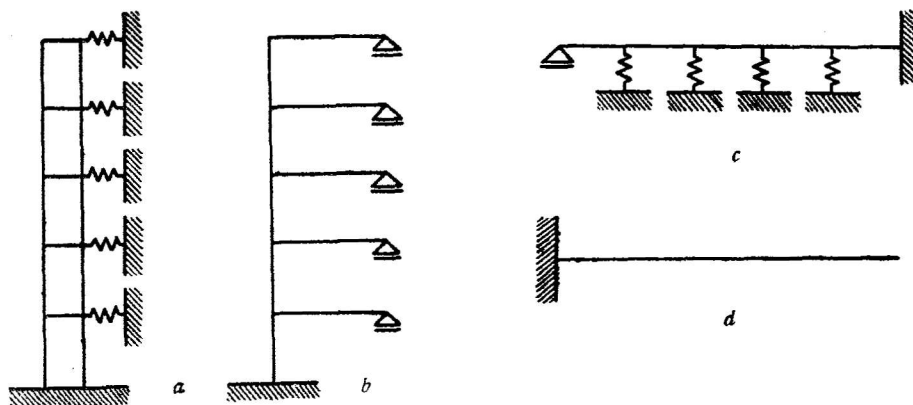


Fig. 2 a, b, c, d

* Статията е докладвана на Четвърти конгрес по теоретична и приложна механика, Варна, 1981 год.

For the beam which is supported on discretely distributed elastic springs (Fig. 2.c), the bar from the Fig. 1. will be without rotational springs.

Similar consideration can be applied to the bar from Fig. 2.d to which springs are not connected.

1. Recursion formula of node displacements

The bar from Fig. 1 is loaded by the lateral discret load at the nodes (Fig. 3).

Positive sign for displacement u_j is in downward direction, and clockwise for angle φ_j (Fig. 3.b).

The equations of equilibrium for node j according to Fig. 4. are:

$$(1) \quad M_{j,j+1} + M_{j,j-1} + M_j = \Phi$$

$$(2) \quad T_{j,j-1} - T_{j,j+1} - P_j + R_j = \Phi.$$

Bending moments and shear forces, as well as forces due to translational and rotational springs are expressed by displacements and angles, and by geometric and elastic properties in the sense of simple method of deformation. Therefore expressions (1) and (2) yield:

$$\varphi_{j+1} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) \varphi_j + \varphi_{j-1} = 3 \frac{M_{j+1} - M_{j-1}}{l}$$

$$\varphi_{j+1} - \varphi_{j-1} = \frac{2}{l} \left[M_{j+1} - \left(2 + \frac{C_r l^3}{12EI}\right) M_j + M_{j-1} \right] + \frac{1}{6} \frac{l^2}{EI} P_j.$$

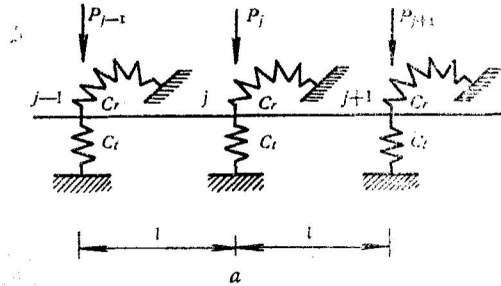


Fig. 3 a, b

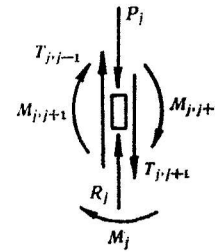


Fig. 4

After addition of these two equations, and subtraction lower from upper, and at the same time incrementing index of all members by 1, two equations with φ_{j+1} and φ_j result. From these equations φ_j can be expressed once di-

rectly and another time from φ_{j+1} so, that indexes of all the numbers are decremented by 1.

$$\begin{aligned} \varphi_j = & \frac{1}{\left(4 + \frac{1}{2} \frac{C_r l}{EI}\right)^2 - 4} \frac{1}{l} \left[-2M_{j+2} + \left(12 + \frac{5}{2} \frac{C_r l}{EI} - 4 \frac{C_r l^3}{12EI}\right) M_{j+1} \right. \\ & - \left(6 + 2 \frac{C_r l}{EI} + 8 \frac{C_e l^3}{12EI} + \frac{C_r l}{EI} \frac{C_r l^3}{12EI}\right) M_j - \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) M_{j-1} \\ & \left. + 2 \frac{1}{6} \frac{l^3}{EI} P_{j+1} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) \frac{1}{6} \frac{l^3}{EI} P_j \right] \end{aligned} \quad (3)$$

$$\begin{aligned} \varphi_j = & \frac{1}{\left(4 + \frac{1}{2} \frac{C_r l}{EI}\right)^2 - 4} \frac{1}{l} \left[\left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) M_{j+1} + \left(6 + 2 \frac{C_r l}{EI} \right. \right. \\ & \left. + 8 \frac{C_r l^3}{12EI} + \frac{C_r l}{EI} \frac{C_e l^3}{12EI}\right) M_j - \left(12 + \frac{5}{2} \frac{C_r l}{EI} - 4 \frac{C_e l^3}{12EI}\right) M_{j-1} \\ & \left. + 2M_{j-2} - \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) \frac{1}{6} \frac{l^3}{EI} P_j - 2 \frac{1}{6} \frac{C_r l^3}{EI} P_{j-1} \right]. \end{aligned} \quad (4)$$

After equating (3) and (4) the recursion formula for node displacements of bar supported by elastic springs is obtained

$$\begin{aligned} & M_{j+2} - \left(4 + \frac{C_r l}{EI} - 2 \frac{C_e l^3}{12EI}\right) M_{j+1} + \left(6 + 2 \frac{C_r l}{EI} \right. \\ & \left. + 8 \frac{C_e l^3}{12EI} + \frac{C_r l}{EI} \frac{C_e l^3}{12EI}\right) M_j - \left(4 + \frac{C_r l}{EI} - 2 \frac{C_e l^3}{12EI}\right) M_{j-1} \\ & + M_{j-2} = \frac{1}{6} \frac{l^3}{EI} \left[P_{j+1} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) P_j + P_{j-1} \right]. \end{aligned} \quad (5)$$

Recursion formula (5) thus relates displacements of nodes to the forces at the node j and its neighbourhood. Therefore, its form correspond to diffe-

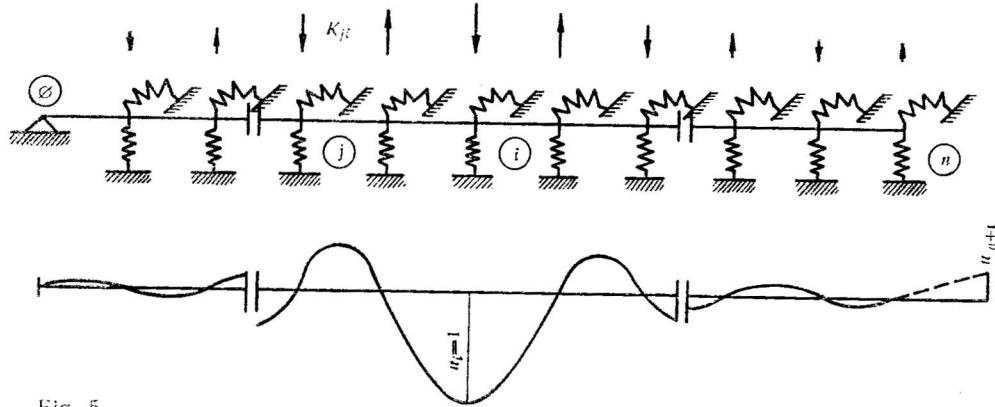


Fig. 5

rencial equation of elastic curve for the bar, which is loaded laterally by continuous load. Recursion formula can have two special forms: All forces except $P_i=1$ are equal to zero. Right hand side is equal to zero, displacements u_j

form the matrix of flexibility, and (5) becomes recursion formula for flexibility, and its order is 4.

Bar is loaded with such lateral forces, that all displacements except $u_i=1$ are equal to zero (Fig. 5), and forces k_{ji} form the stiffness matrix.

2. Stiffnesses of a bar

Recursion formula for node displacements (5) becomes recursion formula of stiffnesses of II order

$$(6) \quad k_{j+1,i} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{ji} + k_{j-1,i} = \Phi.$$

Introducing

$$(7) \quad \text{Ch } b = 2 + \frac{1}{4} \frac{C_r l}{EI}.$$

Solution of recursion formula (6) is

$$(8) \quad k_{ji} = (-1)^j (C_1 \text{Ch } jb + C_2 \text{Sh } jb) \quad 1 \leq j \leq i-2$$

$$(9) \quad k_{ji} = (-1)^j (\bar{C}_1 \text{Ch } jb + \bar{C}_2 \text{Sh } jb) \quad i+2 \leq j \leq n-1.$$

Constants of integration are determined from the boundary conditions and the conditions of continuity.

The boundary conditions for the ends of the bar, if its displacements are equal to zero, as left end of bar on Fig. 5. are

$$f(u)_0 = \Phi \quad f(k)_n = \Phi.$$

If the end of the bar is free or supported on elastic spring (right end on Fig. 5.), then the boundary conditions are

$$f(k)_1 = \Phi \quad f(k)_{n-1} = \Phi$$

$$(M_{n+1} \neq \Phi, \quad M_{-1} \neq \Phi).$$

Stiffnesses on the end of the bar are not determined with (8) and (9) therefore it is necessary for the ends of the bar to use the conditions of equilibrium

$$\Sigma M = \Phi \quad \Sigma F = \Phi.$$

Conditions of continuity are

$$k_{i-1,i} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{i-2,i} + k_{i-3,i} = 6 \frac{EI}{l^3}$$

$$k_{i+3,i} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{i+2,i} + k_{i+1,i} = 6 \frac{EI}{l^3}.$$

Stiffnesses $k_{i+1,i}$, k_{ii} and $k_{i,i-1}$ are not determined by expressions (8) and (9). In recursion formula for node displacements there is $u_i=1$. These stiffnesses are determined from the equations

$$k_{ii} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{i-1,i} + k_{i-2,i} = -6 \frac{EI}{l^3} \left(4 + \frac{C_r l}{EI} - 2 \frac{C_e l^3}{12EI}\right)$$

$$k_{i+1,i} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{ii} + k_{i-1,i} = 6 \frac{EI}{l^3} \left(6 + 2 \frac{C_r l}{EI} + 8 \frac{C_e l^3}{12EI} + \frac{C_r l}{EI} \frac{C_e l^3}{12EI}\right)$$

$$k_{i+2,i} + \left(4 + \frac{1}{2} \frac{C_r l}{EI}\right) k_{i+1,i} + k_{ii} = -6 \frac{EI}{l^3} \left(4 + \frac{C_r l}{EI} - 2 \frac{C_e l^3}{12EI}\right).$$

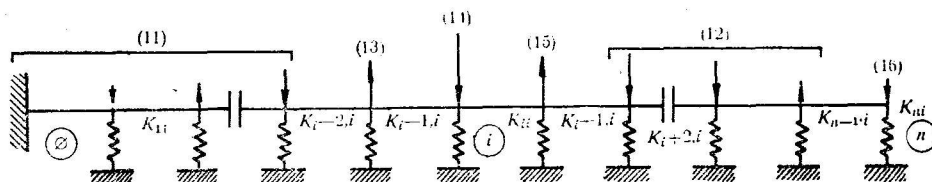


Fig. 6

So for the bar from the Fig. 6. is

$$(10) \quad b = \ln(2 + \sqrt{3})$$

$$(11) \quad k_{ji} = 72\sqrt{3} \frac{EI}{l^3} (-1)^{i+j} \frac{\text{Ch}(n+1-i)b - 2 \text{Ch}(n-i)b}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}jb \quad 1 \leq j \leq i-2$$

$$(12) \quad k_{ji} = 72\sqrt{3} \frac{EI}{l^3} (-1)^{i+j} \frac{\text{Ch}(n+1-j)b - 2 \text{Ch}(n-j)b}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}ib \quad i+2 \leq j \leq n-1$$

$$(13) \quad k_{i-1,i} = -\frac{EI}{l^3} \left[72\sqrt{3} \frac{\text{Ch}(n+1-i)b - 2 \text{Ch}(n-i)b}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}(i-1)b - 6 \right]$$

$$(14) \quad k_{ii} = -\frac{EI}{l^3} \left[72\sqrt{3} \frac{\text{Ch}(n+1-i)b - 2 \text{Ch}(n-i)b}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}ib - 48 \right] + C_e$$

$$(15) \quad k_{i+1,i} = -\frac{EI}{l^3} \left[72\sqrt{3} \frac{\text{Ch}(n-i)b - 2 \text{Ch}(n-i-1)b}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}ib - 6 \right].$$

For stiffnesses in node n are valid expressions:

$$(16) \quad k_{ni} = 36\sqrt{3} \frac{EI}{l^3} (-1)^{n+1} \frac{1}{\text{Sh}(n+1)b - 2 \text{Sh}nb} \text{Ch}ib$$

$$(17) \quad k_{n,n-1} = -\frac{EI}{l^3} \left[36\sqrt{3} \frac{1}{\text{Sh}(n+1)b - 2 \text{Sh}nb} - 6 \right]$$

$$(18) \quad k_{nn} = \frac{EI}{l^3} \left[36\sqrt{3} \frac{\text{Ch}nb - \frac{1}{2\sqrt{3}} \text{Sh}nb}{\text{Sh}(n+1)b - 2 \text{Sh}nb} - 24 \right] + C_e.$$

The stiffness matrix is symmetric, i. e. $k_{ji} = k_{ij}$. It comes from the fact, that work of stiffnesses k_{ij} and k_{ji} on the unity displacements u_i and u_j are equal.

Stiffnesses of a bar supported on the translational springs are equal to the stiffnesses of a bar without springs. Difference exists only for the stiffnesses k_{ii} and k_{nn} , and it is equal to the magnitude of the specific resistance of spring.

Literature

1. Korošec, V. Stiffness matrix of a regular semi-frame, 15. — Yugoslav congress for theoretical and applied mechanics, Kupari 1981, 75-81