

A Method for Reducing the Number of Arguments in Systems of Partial Differential Equations with Variable Coefficients

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Many problems in physics, continuum mechanics and other natural sciences demand the solution of systems of partial differential equations of different types. When solving specific problems of natural origin, the solution existence is usually not to be discussed, but follows from the problem itself. Difficulties consist in obtaining a formal solution of the system of partial differential equations which must involve a sufficient number of parameters, integration constants or functions, aiming at the satisfaction of mixed boundary conditions on complicated boundary surfaces. However, sufficiently general and powerful analytical methods do not exist to provide always such a formal solution. In practice, various special methods, valid for specified sets of partial differential equations, are applied. Often numerical methods are used, having well-known advantages and disadvantages.

The basic idea when designing analytical methods for solution of partial differential equations, consists in the repeated reducing the number of arguments. In such a way the system given is transformed into an equivalent system of partial differential equations containing a smaller number of arguments and parameters as well as undetermined integration functions or constants. So, in many cases, a system of ordinary differential equations may be obtained at last.

In this paper a rather general method for reducing the number of arguments in systems of linear partial differential equations with variable coefficients is presented.

Let us consider, without losing generality and for clarity, the two argument case. Let a system of p partial homogeneous differential equations with variable coefficients be given, containing q functions $u^\beta(x, y)$ of two arguments x and y . Let K and L be the partial derivative highest order with respect to (w.r.t.) x and y , correspondingly. Then the system, in a most general form, reads:

$$(1) \quad \sum_{\beta=1}^q \sum_{k=0}^K \sum_{l=0}^L D_{\beta kl}^\alpha(x, y) \frac{\partial^{k+l} u^\beta(x, y)}{\partial x^k \partial y^l} = 0,$$

where $D_{\beta kl}^\alpha = D_{\beta lk}^\alpha$ and $(\alpha = 1, 2, \dots, p)$.

The coefficients $D_{\beta kl}^a$ must satisfy some natural conditions such as differentiability, etc., that will be made clear in the text. The number q of the unknown functions may be different from the number p of the equations, i. e. the system (1) may be underdetermined, determined or overdetermined.

We try to find a solution of the system (1) in the form:

$$(2) \quad u^\beta(x, y) = \sum_{j=0}^{m-1} Y_j^\beta(y) \omega^{(j)}(x, y), \quad (\beta = 1, 2, \dots, q),$$

where the following notation is used:

$$\omega^{(j)}(x, y) = \frac{\partial^j \omega(x, y)}{\partial x^j}, \quad \omega^{(0)}(x, y) = \omega(x, y),$$

and m is a chosen number so that a richer set of integration functions could be obtained. The greater m , the richer set of integration functions. For $m=1$ the simplest case is obtained, i.e. the Fourier case of variables separation. Actually the functions sought in (2) are given as a sum of some basic functions $\omega^{(j)}(x, y)$ that are partial derivatives w.r.t. the chosen argument x of an yet unknown function $\omega(x, y)$. The arguments x and y are not necessarily cartesian coordinates.

In some specific problems, with regard to the boundary conditions and other reasons it is advisable and useful, before using the method, a change of arguments to be done. We are not going to study this problem so that we consider the system (1) is given.

If arguments x and y change places in (2), then depending on the type of (1), two different solutions may be obtained.

We demand the basic functions $\omega^{(j)}(x, y)$ to satisfy the following conditions:

$$(3) \quad \omega^{(m)}(x, y) = \sum_{j=0}^{m-1} c_j(x, y) \omega^{(j)}(x, y),$$

$$(4) \quad \frac{\partial \omega(x, y)}{\partial y} = \sum_{j=0}^{m-1} d_j(x, y) \omega^{(j)}(x, y).$$

If $c_j(x, y)$ and $d_j(x, y)$ in (3) and (4) are supposed to be known, it is possible to integrate this system and obtain $\omega(x, y)$.

Differentiating (3) w.r.t. x and performing some substitutions, the partial derivatives w.r.t. x of an arbitrary order including orders higher than m , may be expressed by $\omega^{(j)}(x, y)$ ($j=0, 1, \dots, m-1$). The following formula is obtained:

$$(5) \quad \omega^{(i+k)}(x, y) = \sum_{j=0}^{m-1} c_j^{ki}(x, y) \omega^{(j)}(x, y),$$

$$(i=0, 1, \dots, m-1), (k \geq 0),$$

where

$$c_j^{ki}(x, y) = [C^{(k)}(x, y)]_j^i, \quad C^{(0)} = I,$$

$$(6) \quad C^{(1)}(x, y) = C(x, y), \quad C^{(k)}(x, y) = \frac{\partial C^{(k-1)}(x, y)}{\partial x} + C^{(k-1)}(x, y) \cdot C(x, y).$$

$$(7) \quad \mathbf{C}(x, y) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ c_0(x, y) & c_1(x, y) & c_2(x, y) & \dots & c_{m-1}(x, y) \end{pmatrix}$$

and $\mathbf{I}$ — the unit matrix.

Differentiating (4) k -times w.r.t. x and using the Leibniz formula, the following expression is obtained:

$$(8) \quad \frac{\partial \omega^{(k)}(x, y)}{\partial y} = \sum_{j=0}^{m-1} \sum_{j'=0}^{m-1} \sum_{k'=0}^k \binom{k}{k'} \frac{\partial^{k-k'} d_{j'}(x, y)}{\partial x^{k-k'}} c_j^{k'j'}(x, y) \omega^{(j)}(x, y), \quad (k \geq 0)$$

and indicating

$$(9) \quad d_j^k(x, y) = \sum_{k'=0}^k \sum_{j'=0}^{m-1} \binom{k}{k'} \frac{\partial^{k-k'} d_{j'}(x, y)}{\partial x^{k-k'}} c_j^{k'j'}(x, y),$$

we obtain for (8) the form

$$(10) \quad \frac{\partial \omega^{(k)}(x, y)}{\partial y} = \sum_{j=0}^{m-1} d_j^k(x, y) \omega^{(j)}(x, y).$$

Differentiating (10) w.r.t. y and performing the corresponding substitutions we obtain

$$(11) \quad \frac{\partial^l \omega^{(k)}(x, y)}{\partial y^l} = \sum_{j=0}^{m-1} d_j^{lk}(x, y) \omega^{(j)}(x, y), \quad (l \geq 0),$$

where

$$d_j^{lk}(x, y) = [\mathbf{D}^{(l)}(x, y)]_j^k, \quad \mathbf{D}^{(0)} = \mathbf{I},$$

$$(12) \quad \mathbf{D}^{(1)}(x, y) = \mathbf{D}(x, y), \quad \mathbf{D}^{(l)}(x, y) = \frac{\partial \mathbf{D}^{(l-1)}(x, y)}{\partial y} + \mathbf{D}^{(l-1)}(x, y) \cdot \mathbf{D}(x, y)$$

and $\mathbf{D}(x, y)$ denotes the matrix with components $d_j^k(x, y)$.

Through equation (5) and equation (12) any partial derivative of $\omega(x, y)$ of an arbitrary order w.r.t. x and y may be expressed.

If we differentiate (3) w.r.t. y and (4) w.r.t. x m times, and equalize, we obtain

$$(13) \quad \sum_{j=0}^{m-1} \sum_{j'=0}^{m-1} \sum_{i=0}^m \binom{m}{i} \frac{\partial^{m-i} d_{j'}(x, y)}{\partial x^{m-i}} c_j^{ij'}(x, y) \omega^{(j)}(x, y) = \sum_{i=0}^{m-1} \sum_{j'=0}^{m-1} c_{j'}^i(x, y) d_j^{ij'}(x, y) \omega^{(j)}(x, y) + \sum_{j=0}^{m-1} \frac{\partial c_j^i(x, y)}{\partial y} \omega^{(j)}(x, y).$$

In order to satisfy (13) a sufficient condition is the coefficients before the basic functions $\omega^{(j)}(x, y)$ to be equalized to zero:

$$(14) \quad d_f^m(x, y) - \sum_{j'=0}^{m-1} c_{j'}(x, y) d_{j'}^{l'}(x, y) - \frac{\partial c_j(x, y)}{\partial y} = 0, \quad (j=0, 1, \dots, m-1).$$

The equations (14) are integrability conditions for (3) and (4).

Using formulae (5) and (11) as well as the Leibniz formula we obtain:

$$(15) \quad \begin{aligned} & \frac{\partial^{k+l}}{\partial x^k \partial y^l} \sum_{j=0}^{m-1} Y_j^\beta(y) \omega^{(j)}(x, y) \\ &= \frac{\partial^k}{\partial x^k} \sum_{j=0}^{m-1} \sum_{l'=0}^l \binom{l}{l'} Y_j^{\beta(l')}(y) \frac{\partial^{l-l'} \omega^{(j)}(x, y)}{\partial y^{l-l'}} \\ &= \frac{\partial^k}{\partial x^k} \sum_{j=0}^{m-1} \sum_{l'=0}^l \binom{l}{l'} Y_j^{\beta(l')}(y) \sum_{j'=0}^{m-1} d_{j'}^{(l-l')j}(x, y) \omega^{(j')}(x, y) \\ &= \sum_{j=0}^{m-1} \sum_{l'=0}^l \binom{l}{l'} Y_j^{\beta(l')}(y) \sum_{j'=0}^{m-1} \sum_{k'=0}^k \binom{k}{k'} \frac{\partial^{k-k'} d_{j'}^{(l-l')j}(x, y)}{\partial x^{k-k'}} \omega^{(j'+k')}(x, y) \\ &= \sum_{j=0}^{m-1} \sum_{l'=0}^l \binom{l}{l'} Y_j^{\beta(l')}(y) \sum_{j''=0}^{m-1} \sum_{j'=0}^{m-1} \sum_{k'=0}^k \binom{k}{k'} \frac{\partial^{k-k'} d_{j'}^{(l-l')j}(x, y)}{\partial x^{k-k'}} c_{j''}^{k'j'}(x, y) \omega^{(j'')}(x, y), \end{aligned}$$

($\beta=1, 2, \dots, q$).

Now we may return to the system (1). Using (15) we obtain

$$(16) \quad \sum_{\beta=1}^q \sum_{j''=0}^{m-1} \sum_{k=0}^K \sum_{l=0}^L D_{\beta kl}^a(x, y) \sum_{j=0}^{m-1} \sum_{l'=0}^l \binom{l}{l'} Y_j^{\beta(l')}(y) \sum_{j'=0}^{m-1} \sum_{k'=0}^k \binom{k}{k'} \frac{\partial^{k-k'} d_{j'}^{(l-l')j}(x, y)}{\partial x^{k-k'}} c_{j''}^{k'j'}(x, y) \omega^{(j'')}(x, y) = 0, \quad (a=1, 2, \dots, p).$$

By reducing the double sums w.r.t. l and l' so that the expressions before the derivatives of $Y_j^\beta(y)$ in (16) to be formed, we obtain

$$(17) \quad \sum_{\beta=1}^q \sum_{j=0}^{m-1} \sum_{l=0}^L \sum_{i=0}^{m-1} G_{\beta il}^{aj}(x, y) Y_j^{\beta(l)}(y) \omega^{(i)}(x, y) = 0, \quad (a=1, 2, \dots, p),$$

where

$$(18) \quad \begin{aligned} G_{\beta il}^{aj} &= \sum_{k=0}^K \sum_{k'=0}^k \sum_{l'=0}^L \sum_{j'=0}^{m-1} \binom{l+l'}{l'} \binom{k+k'}{k'} D_{\beta(k+k')(l+l')}^a(x, y) \\ & \quad \frac{\partial^{k'} d_{j'}^{l'}(x, y)}{\partial x^{k'}} c_{i'}^{k'j'}(x, y) \end{aligned}$$

($a=1, 2, \dots, p$),
($\beta=1, 2, \dots, q$).

A sufficient condition to satisfy (17) is

$$(19) \quad \sum_{\beta=1}^q \sum_{j=0}^{m-1} \sum_{l=0}^L G_{\beta il}^{aj}(x, y) Y_j^{\beta(l)}(y) = 0,$$

$$\begin{aligned} & (a = 1, 2, \dots, p), \\ & (\beta = 1, 2, \dots, q), \end{aligned}$$

i.e. again equalizing to zero the expressions before the basic functions $\omega^{(l)}(x, y)$.
If the following representation is possible:

$$(20) \quad G_{\beta il}^{aj}(x, y) = \psi_i^a(x, y) B_{\beta il}^{aj}(y),$$

$$\begin{aligned} & (a = 1, 2, \dots, p), \\ & (\beta = 1, 2, \dots, q), \end{aligned}$$

then, after substituting (20) in (19), $\psi_i^a(x, y)$ are eliminated and (19) becomes a system of ordinary differential equations for $Y_j^{\beta}(y)$, i.e.

$$(21) \quad \sum_{\beta=1}^q \sum_{j=0}^{m-1} \sum_{l=0}^L B_{\beta il}^{aj}(y) Y_j^{\beta(l)}(y) = 0,$$

$$\begin{aligned} & (a = 1, 2, \dots, p), \\ & (i = 0, 1, \dots, m-1). \end{aligned}$$

Putting (20) in (18) we obtain

$$(22) \quad \sum_{k=0}^K \sum_{k'=0}^k \sum_{l'=0}^L \sum_{j'=0}^{m-1} \binom{k+k'}{k'} \binom{l+l'}{l'} D_{\beta(k+k')(l+l')}^a(x, y) c_i^{kj'}(x, y) \frac{d^{k'} d_{j'}^{l'}(x, y)}{\partial x^{k'}}$$

$$= \psi_i^a(x, y) B_{\beta il}^{aj}(y),$$

$$\begin{aligned} & (a = 1, 2, \dots, p), \quad (\beta = 1, 2, \dots, q), \\ & (i = 0, 1, \dots, m-1), \quad (l = 0, 1, \dots, L). \end{aligned}$$

The simple analysis reveals that it is very easy to exclude $\psi_i^a(x, y)$, $d_j(x, y)$ and $c_j(x, y)$ from the system (22) so that, together with the treated in the same way (14), to become a system of conditions containing only $D_{\beta kl}^a(x, y)$ and $B_{\beta il}^{aj}(y)$. If it is possible, from this last system we determine $B_{\beta il}^{aj}(y)$ as a function of $D_{\beta kl}^a(x, y)$. Thus, we obtain the system (21) of ordinary differential equations for the unknown function $Y_j^{\beta}(y)$. It may be solved by a corresponding method, including numerical.

The function $\omega(x, y)$ is obtained from the conditions (3) and (4) where $c_j(x, y)$ and $d_j(x, y)$ are determined as functions of $D_{\beta kl}^a(x, y)$ and $B_{\beta il}^{aj}(y)$.

The obtained in this way $Y_j^{\beta}(y)$ and $\omega^{(l)}(x, y)$ we put in (2) and so we get the solution of the given system (1). It is obvious that this solution will contain some free parameters as well as integration functions and constants resulting from the solutions of the systems (3), (4) and (21). Their determination is done by satisfying the boundary conditions where (2) has been substituted.

If it is not possible to determine $B_{\beta il}^{aj}(y)$ as a function of $D_{\beta kl}^a(x, y)$, this means the given system (1) has no analytical solution from the type (2) at

the chosen value of m . Then a greater value of m may be tested to solve the problem.

As a special case the method exposed is very convenient to integrate systems of partial differential equations with constant coefficients [1], [2], [3].

Obviously there are no principal difficulties for the application of the method exposed to systems of partial differential equations with variable coefficients with more than two arguments. Following the scheme given above, one can obtain all the necessary formulae. In principle these formulae are quite analogous to those obtained above. But in the many argument case there are much more indices in the general expressions and their writing out is not perspicuous. But it is not difficult to use the method for solving a given system of partial differential equations.

In practice two ways are possible. At the first one the number of arguments may be reduced consequently; at the second the decomposition may be carried out directly to a system of ordinary differential equations w.r.t. one of the arguments.

The proposed here method concerns linear systems of partial differential equations but the idea used may be applied obviously to the case of some special sets of non-linear partial differential equations. The method is applicable in the case if having done the necessary analogous assumptions, operations, etc., it is possible to come to some system, naturally non-linear, similar to (21), not containing expressions with the basic functions $\omega^{(j)}(x, y)$.

For better understanding the method we shall consider its application in the case of systems of usual equations of the mathematical physics. Let us be given a second order system of partial differential equations:

$$(23) \quad \sum_{\beta=1}^q \left(D_{\beta 20}^a \frac{\partial^2 u^\beta}{\partial x^2} + D_{\beta 11}^a \frac{\partial^2 u^\beta}{\partial x \partial y} + D_{\beta 02}^a \frac{\partial^2 u^\beta}{\partial y^2} + D_{\beta 10}^a \frac{\partial u^\beta}{\partial x} + D_{\beta 01}^a \frac{\partial u^\beta}{\partial y} + D_{\beta 00}^a u^\beta \right) = 0, \\ (a = 1, 2, \dots, p)$$

where the coefficients are functions of x and y .

According to the method, choosing $m=2$, we look for a solution in the form:

$$(24) \quad u^\beta(x, y) = Y_0^\beta(y) \omega(x, y) + Y_1^\beta(y) \omega'(x, y), \\ (\beta = 1, 2, \dots, q),$$

where the function $\omega(x, y)$ satisfies

$$(25) \quad \omega''(x, y) = c_0(x, y) \omega(x, y) + c_1(x, y) \omega'(x, y), \\ \frac{\partial \omega(x, y)}{\partial y} = d_0(x, y) \omega(x, y) + d_1(x, y) \omega'(x, y).$$

Then

$$\frac{\partial \omega'(x, y)}{\partial y} = \left(\frac{\partial d_0}{\partial x} + d_1 c_0 \right) \omega + \left(d_0 + \frac{\partial d_1}{\partial x} + d_1 c_1 \right) \omega', \\ \frac{\partial \omega''(x, y)}{\partial y} = \left(\frac{\partial^2 d_0}{\partial x^2} + d_0 c_0 + 2 \frac{\partial d_1}{\partial x} c_0 + d_1 \frac{\partial c_0}{\partial x} + d_1 c_0 c_1 \right) \omega \\ + \left(2 \frac{\partial d_0}{\partial x} + d_0 c_1 + \frac{\partial^2 d_1}{\partial x^2} + 2 \frac{\partial d_1}{\partial x} c_1 + d_1 c_0 + d_1 c_1^2 + d_1 \frac{\partial c_0}{\partial x} \right) \omega',$$

$$\begin{aligned}
\frac{\partial^2 \omega(x, y)}{\partial y^2} &= \left(d_0^2 + \frac{\partial d_0}{\partial y} + d_1 \frac{\partial d_0}{\partial x} + d_1^2 c_0 \right) \omega + \left(2d_0 d_1 + \frac{\partial d_1}{\partial y} + d_1 \frac{\partial d_1}{\partial x} + d_1^2 c_1 \right) \omega', \\
\frac{\partial^2 \omega'(x, y)}{\partial y^2} &= \left(\frac{\partial^2 d_0}{\partial x \partial y} + 2d_0 \frac{\partial d_0}{\partial x} + \frac{\partial d_1}{\partial x} \cdot \frac{\partial d_0}{\partial x} + d_1 \frac{\partial^2 d_0}{\partial x^2} + 3d_1 c_0 \frac{\partial d_1}{\partial x} + d_1^2 \frac{\partial c_0}{\partial x} \right. \\
&\quad \left. + 2d_0 d_1 c_0 + c_0 \frac{\partial d_1}{\partial y} + d_1^2 c_0 c_1 \right) \omega + \left[\frac{\partial d_0}{\partial y} + d_0^2 + 3d_1 \frac{\partial d_0}{\partial x} + d_1^2 c_0 + 2d_0 d_1 c_1 \right. \\
&\quad \left. + c_1 \frac{\partial d_1}{\partial y} + 3d_1 c_1 \frac{\partial d_1}{\partial x} + c_1^2 d_1^2 + 2d_0 \frac{\partial d_1}{\partial x} + \frac{\partial^2 d_1}{\partial x \partial y} + \left(\frac{\partial d_1}{\partial x} \right)^2 + d_1 \frac{\partial^2 d_1}{\partial x^2} + d_1^2 \frac{\partial c_1}{\partial x} \right] \omega', \\
(26) \quad \omega'''(x, y) &= \left(c_0 c_1 + \frac{\partial c_0}{\partial x} \right) \omega + \left(c_0 + c_1^2 + \frac{\partial c_1}{\partial x} \right) \omega'.
\end{aligned}$$

For the sake of convenience, here and further on, the arguments x and y are omitted.

The integrability conditions for (25), after corresponding transformations, are

$$\begin{aligned}
(27) \quad \frac{\partial^2 d_0}{\partial x^2} - c_1 \frac{\partial d_0}{\partial x} + 2c_0 \frac{\partial d_1}{\partial x} + d_1 \frac{\partial c_0}{\partial x} - \frac{\partial c_0}{\partial y} &= 0, \\
\frac{\partial^2 d_1}{\partial x^2} + c_1 \frac{\partial d_1}{\partial x} + 2 \frac{\partial d_0}{\partial x} + d_1 \frac{\partial c_1}{\partial x} - \frac{\partial c_1}{\partial y} &= 0.
\end{aligned}$$

After performing the necessary operations according to the method given above we get the corresponding to (22) conditions in the form:

$$\begin{aligned}
D_{\beta 20}^a c_0 + (D_{\beta 11}^a + d_1 D_{\beta 02}^a) \left(\frac{\partial d_0}{\partial x} + c_0 d_1 \right) + (d_0 D_{\beta 02}^a + D_{\beta 01}^a) d_0 + D_{\beta 02}^a \frac{\partial d_0}{\partial y} + D_{\beta 00}^a &= B_{\beta 00}^{a0} \psi_0^a, \\
D_{\beta 20}^a c_1 + (D_{\beta 11}^a + d_1 D_{\beta 02}^a) \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + (d_0 D_{\beta 02}^a + D_{\beta 01}^a) d_1 + D_{\beta 02}^a \frac{\partial d_1}{\partial y} \\
+ D_{\beta 10}^a &= B_{\beta 10}^{a0} \psi_1^a, \\
2D_{\beta 02}^a d_0 + D_{\beta 01}^a &= B_{\beta 01}^{a0} \psi_0^a, \\
2D_{\beta 02}^a d_1 + D_{\beta 11}^a &= B_{\beta 11}^{a0} \psi_1^a, \\
D_{\beta 02}^a &= B_{\beta 02}^{a0} \psi_0^a, \\
D_{\beta 20}^a \left(c_0 c_1 + \frac{\partial c_0}{\partial x} \right) + (D_{\beta 11}^a + d_1 D_{\beta 02}^a) \left[c_0 \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + \frac{\partial}{\partial x} \left(\frac{\partial d_0}{\partial x} + c_0 d_1 \right) \right] + D_{\beta 10}^a c_0 \\
+ (d_0 D_{\beta 02}^a + D_{\beta 01}^a) \left(\frac{\partial d_0}{\partial x} + c_0 d_1 \right) + D_{\beta 02}^a \left[\frac{\partial d_1}{\partial x} \left(\frac{\partial d_0}{\partial x} + c_0 d_1 \right) + d_0 \frac{\partial d_0}{\partial x} \right. \\
+ \left. \frac{\partial}{\partial y} \left(\frac{\partial d_0}{\partial x} + c_0 d_1 - c_0 \right) \right] &= B_{\beta 00}^{a1} \psi_1^a, \\
D_{\beta 20}^a \left(c_0 + c_1^2 + \frac{\partial c_1}{\partial x} \right) + \left[c_1 D_{\beta 11}^a + D_{\beta 02}^a \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + D_{\beta 01}^a \right] \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + D_{\beta 10}^a c_1 \\
+ D_{\beta 02}^a \frac{\partial}{\partial y} \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + D_{\beta 11}^a \frac{\partial}{\partial x} \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) &= B_{\beta 10}^{a1} \psi_1^a, \\
D_{\beta 11}^a c_0 + 2D_{\beta 02}^a \left(\frac{\partial d_0}{\partial x} + c_0 d_1 \right) &= B_{\beta 01}^{a1} \psi_0^a,
\end{aligned}$$

$$\begin{aligned}
(28) \quad & D_{\beta 11}^{\alpha} c_1 + 2D_{\beta 02}^{\alpha} \left(d_0 + \frac{\partial d_1}{\partial x} + c_1 d_1 \right) + D_{\beta 11}^{\alpha} = B_{\beta 11}^{\alpha 1} \psi_1^{\alpha}, \\
& D_{\beta 02}^{\alpha} = B_{\beta 12}^{\alpha 1} \psi_1^{\alpha}, \\
& (\alpha = 1, 2, \dots, p), \\
& (\beta = 1, 2, \dots, q).
\end{aligned}$$

It is immediately seen that $\psi_0^{\alpha}(x, y)$ and $\psi_1^{\alpha}(x, y)$, as well as $d_0(x, y)$, $d_1(x, y)$, $c_0(x, y)$, $c_1(x, y)$ are determined directly from some relations in (28) and they can be substituted in the rest of the relations in (28) and (27). For example we may eliminate ψ_0^{α} , ψ_1^{α} , d_0 , d_1 , c_0 , c_1 in the following way:

$$\begin{aligned}
(29) \quad & \psi_0^{\alpha} = \frac{D_{\beta 02}^{\alpha}}{B_{\beta 02}^{\alpha 0}}, \quad \psi_1^{\alpha} = \frac{D_{\beta 02}^{\alpha}}{B_{\beta 12}^{\alpha 1}}, \\
& d_0 = \frac{1}{2} \left(\frac{B_{\beta 01}^{\alpha 0}}{B_{\beta 02}^{\alpha 0}} - \frac{D_{\beta 01}^{\alpha}}{D_{\beta 02}^{\alpha}} \right), \quad d_1 = \frac{1}{2} \left(\frac{B_{\beta 11}^{\alpha 0}}{B_{\beta 12}^{\alpha 1}} - \frac{D_{\beta 11}^{\alpha}}{D_{\beta 02}^{\alpha}} \right), \\
& c_0 = - \frac{D_{\beta 02}^{\alpha 0} \left(\frac{B_{\beta 01}^{\alpha 1}}{B_{\beta 02}^{\alpha 0}} - 2 \frac{\partial d_0}{\partial x} \right)}{D_{\beta 11}^{\alpha} + 2d_1 D_{\beta 02}^{\alpha}}, \\
& c_1 = - \frac{D_{\beta 02}^{\alpha} \left(\frac{B_{\beta 11}^{\alpha 1}}{B_{\beta 12}^{\alpha 1}} - 2 \frac{\partial d_1}{\partial x} - 2d_0 \right) - D_{\beta 01}^{\alpha}}{D_{\beta 11}^{\alpha} + 2d_1 D_{\beta 02}^{\alpha}},
\end{aligned}$$

where $D_{\beta 02}^{\alpha} \neq 0$ at some value of β . Substituting (29) in (27) and (28) we obtain a system of conditions only between $B_{\beta i l}^{\alpha j}(y)$ and $D_{\beta k l}^{\alpha}(x, y)$. If possible we determine $B_{\beta i l}^{\alpha j}(y)$ as functions of $D_{\beta k l}^{\alpha}(x, y)$, then we substitute them in (25) and (30):

$$\begin{aligned}
(30) \quad & \sum_{\beta=1}^q \sum_{l=0}^2 [B_{\beta i l}^{\alpha 0}(y) Y_0^{\beta(i)}(y) + B_{\beta i l}^{\alpha 1}(y) Y_1^{\beta(i)}(y)] = 0, \\
& (\alpha = 1, 2, \dots, p), \\
& (i = 0, 1, \dots, m-1).
\end{aligned}$$

Thus, we obtain subsequently the solutions for $Y_0^{\beta}(y)$, $Y_1^{\beta}(y)$ and $\omega(x, y)$ as functions of the known coefficients $D_{\beta k l}^{\alpha}(x, y)$, some parameters and integration functions (coefficients). The system (23) solution is given by (24).

The necessary transformations may be done using computer analytical transformation systems like REDUCE-2. The matrices used make the method's algorithm very convenient.

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