

A Transversely Isotropic Plate Indented by a Flat Annular Rigid Stamp Overlying an Elastic Foundation

M. Lal

1. Introduction

One of the simplest three-part mixed boundary value problems in elasticity is the contact problem for a flat annular rigid stamp. At present, there seems to be no method available for obtaining the solution in a closed form. Williams [1], Cooke [2], Noble [3] and Jain and Kanwal [4] showed that such a problem can be reduced to the solution of a Fredholm integral equation, and tried to solve the equation by an iterative procedure. Gubenko and Mossakovsku [5] have pointed out that the problem can be reduced to the solution of a pair of simultaneous Fredholm integral equations. They also pointed out that the iterative solution of these equations is very complicated.

The present paper proposes a simple technique for the solution of an elastic contact problem for a transversely-isotropic plate indented by a flat annular rigid stamp. The normal pressure on the contact region is continuous at all points except the inner and outer edges of the stamp. In a transversely-isotropic material, the physical properties are symmetrical about the axis. The exact method of analysis for such three dimensional problems in transversely isotropic materials usually rests on the solution of homogeneous displacement equations of equilibrium in terms of two potential functions, which are harmonic in two distinct spaces as shown by Elliot [6] and Shield [7]. By involving harmonic functions the mathematical calculations become easy as compared to the isotropic case where a biharmonic function is involved.

The above mixed boundary value problem is reduced to the solution of a set of triple integral equations involving Bessel functions. The displacement on the contact region can, however, be expressed by Fourier series. This assumption helps us in the reduction of the triple integral equations to a set of infinite linear algebraic equations.

The set of infinite algebraic equations is solved numerically in section 5 of this paper. The displacement and stresses are calculated for cadmium crystal.

2. Basic equations and their solution

In the case of axial symmetry, the displacement vector $\mathbf{U}$ assumes the form $(u_r, 0, u_z)$ in a cylindrical coordinate system (r, θ, z) . The equations

of equilibrium and the required stress displacement relations for a homogeneous, isotropic, elastic medium are given by

$$2.1 \quad \mu \nabla^2 \mathbf{U} + (\lambda + \mu) \nabla (\Delta \cdot \mathbf{U}) = 0,$$

$$2.2 \quad \sigma_{zz}(r, z) = (\lambda + 2\mu) \frac{\partial u_z}{\partial z} + \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right),$$

$$2.3 \quad \sigma_{rs}(r, z) = \mu \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right),$$

where λ, μ are Lamé's constants.

The solution of the system of partial differential equations (2.1) is given by

$$2.4 \quad 2\mu u_r(r, z) = \frac{\partial \phi_1}{\partial r} + z \frac{\partial \phi_2}{\partial r},$$

$$2.5 \quad 2\mu u_z(r, z) = \frac{\partial \phi_1}{\partial z} + z \frac{\partial \phi_2}{\partial z} - (3 - 4\sigma) \phi_2,$$

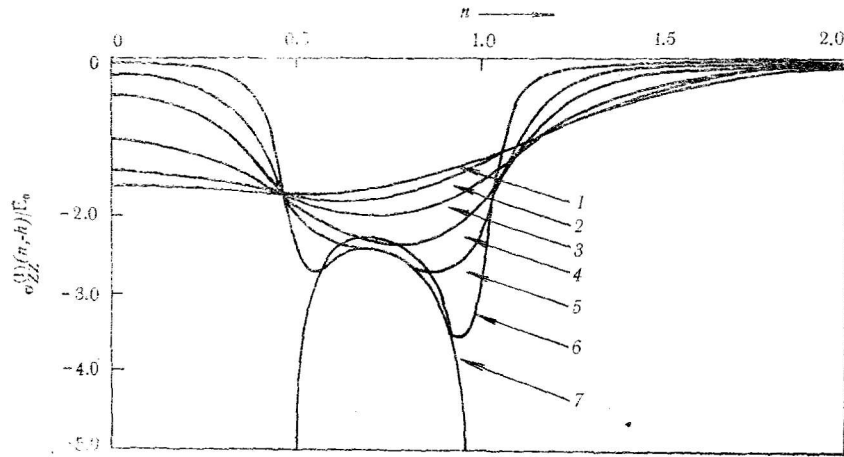


Fig. 1. Radial distribution of $\sigma_{zz}^{(1)}(r, -h)$ for $a=0.5$, $b=1.0$ and $h=1.0$
1 — $z=0$; 2 — $z=0.3$; 3 — $z=0.5$; 4 — $z=0.7$; 5 — $z=0.8$; 6 — $z=1.0, 9$; 7 — $z=1.0$

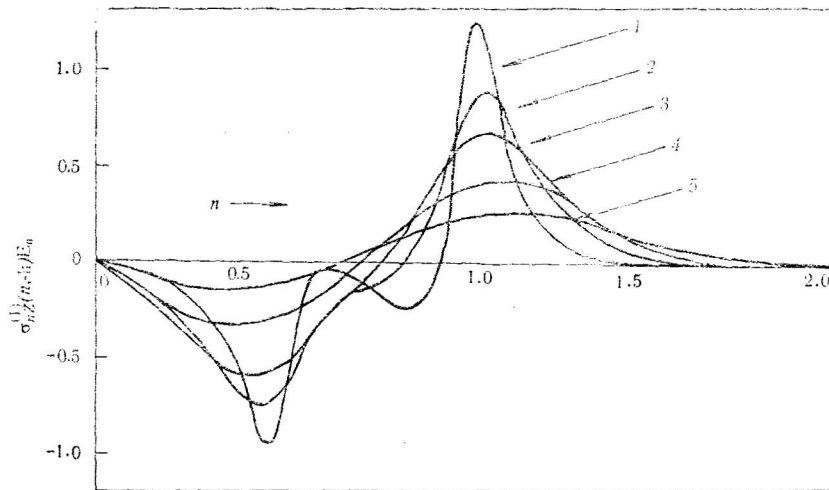


Fig. 2. Radial distribution of $\sigma_{zz}^{(1)}(r, -h)$ for $a=0.5$; $b=1.0$ and $h=1.0$
1 — $z=0.9$; 2 — $z=0.8$; 3 — $z=0.7$; 4 — $z=0.5$; 5 — $z=0.3$

where $\varphi_1(r, z)$ and $\varphi_2(r, z)$ are harmonic functions and σ is Poisson's ratio.

We take the following representation for these harmonic functions:

$$2.6 \quad \begin{aligned} \varphi_1(r, z) &= \int_0^\infty \xi [A(\xi) e^{-\xi z} + B(\xi) e^{\xi z}] J_0(\xi r) d\xi, \\ \varphi_2(r, z) &= \int_0^\infty \xi [C(\xi) e^{-\xi z} + D(\xi) e^{\xi z}] J_0(\xi r) d\xi, \end{aligned}$$

where A, B, C and D are arbitrary functions of ξ only.

Substituting the expressions for φ_1 and φ_2 obtained from equation (2.6) into equations (2.4) and (2.5) and using the values so obtained for $2\mu u_r(r, z)$ and $2\mu u_z(r, z)$ and substituted in equations (2.2) and (2.3) following expressions are obtained for the displacement and stress components:

$$2.7 \quad 2\mu u_r(r, z) = \int_0^\infty \xi [(A + zC)\xi e^{-\xi z} + (B + zD)\xi e^{\xi z}] J_1(\xi r) d\xi,$$

$$2.8 \quad \begin{aligned} 2\mu u_z(r, z) &= \int_0^\infty \xi [-\{\xi(A + zC) + (3 - 4\sigma)C\} e^{-\xi z} \\ &\quad + \{\xi(B + zD) - (3 - 4\sigma)D\} e^{\xi z}] J_0(\xi r) d\xi, \end{aligned}$$

$$2.9 \quad \begin{aligned} \sigma_{zz}(r, z) &= \int_0^\infty \xi [\{\xi(A + zC) + 2(1 - 4\sigma)C\} \xi e^{-\xi z} \\ &\quad + \{\xi(B + zD) - 2(1 - 4\sigma)D\} \xi e^{\xi z}] J_0(\xi r) d\xi, \end{aligned}$$

$$2.10 \quad \begin{aligned} \sigma_{rz}(r, z) &= \int_0^\infty \xi [-\{\xi(A + zC) + (1 - 2\sigma)C\} \xi e^{-\xi z} \\ &\quad + \{\xi(B + zD) - (1 - 2\sigma)D\} \xi e^{\xi z}] J_1(\xi r) d\xi. \end{aligned}$$

3. Formulation of the problem

Consider an infinite transversely-isotropic elastic plate bounded by the planes $z=0$ and $z=-h$ and in perfect contact with the semi-infinite isotropic, homogeneous elastic foundation ($z \geq 0$) having different elastic properties. The free surface $z=-h$ is indented over an annular area by a rigid punch of arbitrary profile. The stress and displacement fields in the elastic plate and foundation satisfy the equations (2.1) to (2.3). The problem, therefore, is to determine the solution which must satisfy the boundary conditions:

$$3.1 \quad u_z^{(1)}(r, -h) = \varepsilon_0; \quad a \leq r \leq b,$$

$$3.2 \quad \sigma_{zz}^{(1)}(r, -h) = 0; \quad 0 \leq r \leq a; \quad r > b,$$

$$3.3 \quad \sigma_{rz}^{(1)}(r, -h) = 0, \quad r \geq 0,$$

Also the elastic plate and the foundation being in perfect contact, the following continuity conditions must be satisfied at the interface $z=0$:

$$3.4 \quad \begin{aligned} [u_z^{(1)}(r, 0)]_1^2 &= [u_r^{(i)}(r, 0)]_1^2 = 0 \\ [\sigma_{zz}^{(1)}(r, 0)]_1^2 &= [\sigma_{rz}^{(i)}(r, 0)]_1^2 = 0 \end{aligned} \quad r \geq 0,$$

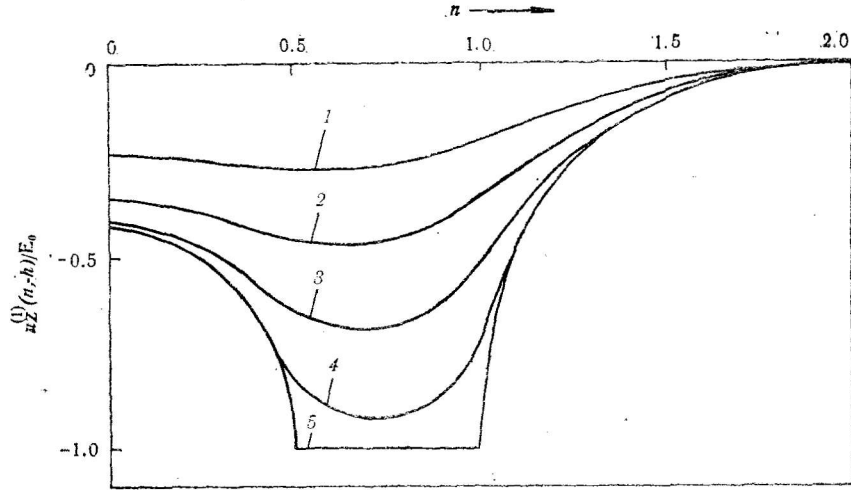


Fig. 3. Radial distribution of $u_z^{(1)}(r, -h)$ for $a=0.5$, $b=1.0$ and $h=1.0$
1 — $z=0.3$; 2 — $z=0.5$; 3 — $z=0.7$; 4 — $z=0.9$; 5 — $z=1.0$

the superscripts 1 and 2 refer respectively to the regions 1 ($-h \leq z \leq 0$) and 2 ($z \geq 0$).

If the elastic constants in regions 1 and 2 are denoted by μ_1, σ_1 and μ_2, σ_2 respectively, the solution for region 1 is obtained from equations (2.7)–(2.10) by replacing μ, σ by μ_1, σ_1 respectively and the solution for region 2 may be obtained from the same equations by replacing A, C, μ, σ by E, F, μ_2, σ_2 respectively and setting $B=D=0$, where A, B, C, D, E and F are arbitrary functions of ξ , to be determined from the boundary and continuity condition (3.1) to (3.4).

The continuity conditions (3.4) will be satisfied if we take

$$\begin{aligned} 4\sigma'A &= K_3 E = K_1 \xi^{-1} F, \quad 4\sigma'B = \mu'(3-4\sigma')E - K_1 \xi^{-1} F, \\ 4\sigma'C &= K_2 F, \quad 4\sigma'D = 2\xi\mu'E + \mu'(3-4\sigma_2)F, \end{aligned}$$

where

$$3.6 \quad \begin{aligned} \mu &= \mu_1/\mu_2, \quad u' = \mu - 1, \quad \sigma' = 1 - \sigma_1, \\ k_1 &= \mu(3-4\sigma_2)(1-2\sigma_1) - (1-2\sigma_2)(3-4\sigma_1), \\ k_3 &= 1 + \mu(3-4\sigma_2), \quad k_3 = \mu + 3 - 4\sigma_1. \end{aligned}$$

The boundary condition (3.3) and the use of equation (3.5) lead to the following relation between E and F :

$$3.7 \quad \begin{aligned} E &= -\xi^{-1} F [6(\sigma_1 + \sigma_2) - 4(1 + 2\sigma_1\sigma_2) + \xi h k_2 \\ &+ \{2(\sigma_2 - \sigma_1) - \xi h \mu'(3-4\sigma_2)\} e^{-2\xi h}] [-k_3 + \mu'(1 - 2\xi h) e^{-2\xi h}]^{-1}. \end{aligned}$$

Equations (3.5)-(3.7) express the unknown functions $A(\xi), \dots, E(\xi)$ in terms of a single unknown function $F(\xi)$. Now if we define the new unknown function $G(\xi)$ by the relation

$$3.8 \quad G(\xi) = \frac{k_2 k_3 \xi e^{\xi h} F(\xi) X(2\xi h)}{4\sigma'[-k_3 + \mu'(1 - 2\xi h)e^{-2\xi h}]} \cdot \frac{1 - \sigma_1}{\mu_1},$$

we find that the boundary conditions (3.1), (3.2) are satisfied if $G(\xi)$ is the solution of the triple integral equations.

$$3.9 \quad \int_0^\infty \xi G(\xi) J_0(\xi r) d\xi = 0, \quad 0 \leq r \leq a, \quad r > b,$$

$$3.10 \quad \int_0^\infty [1 + H(2\xi h)] G(\xi) J_0(\xi r) d\xi = \varepsilon_0, \quad a \leq r \leq b,$$

where the functions $X(x)$ and $H(x)$ are defined by the equations

$$3.11 \quad \begin{aligned} X(x) &= 1 + [d + c(1 + x^2)]e^{-x} + dce^{-2x}, \\ H(x) &= -e^{-x}[d + c(1 + x^2) + 2dce^{-x}][X(x)]^{-1} \end{aligned}$$

with

$$3.12 \quad d = k_4/k_2, \quad c = -\mu'/k_3, \quad k_4 = (3 - 4\sigma_1) - \mu(3 - 4\sigma_2).$$

4. Solution of the integral equations

The present problem is equivalent to the cases:

(a) the circular stamp of radius b is indented on the elastic plate, and then the circular portion of the radius $a (< b)$ is removed.

(b) the surface of the infinite plate is pressed with a circular hole of the radius a , and then the infinite portion with the hole of b is released from pressing.

In case (a), the singularity of $(\sigma_{zz}^{(1)})_{z=-h}$ at $r=a$ takes together the form $(b^2 - r^2)^{-\frac{1}{2}}$ before and after removing the circular portion. Similarly, in the case (b) the singularity of $(\sigma_{zz}^{(1)})_{z=-h}$ at $r=a$ takes together the form $(r^2 - a^2)^{-\frac{1}{2}}$ before and after releasing the infinite portion with the hole. Therefore, the singularities of $(\sigma_{zz}^{(1)})_{z=h}$ in this problem are considered to have the form $(b^2 - r^2)^{-\frac{1}{2}}$ at $r=b$ and $(r^2 - a^2)^{-\frac{1}{2}}$ at $r=a$. Then, we assume $(\sigma_{zz}^{(1)})_{z=-h}$ in the region on the annular stamp as follows:

$$4.1 \quad (\sigma_{zz}^{(1)})_{z=-h} = \frac{f(r)}{[(b^2 - r^2)(r^2 - a^2)]^{-\frac{1}{2}}}, \quad a < r < b,$$

where $f(r)$ is an unknown function which is assumed to be continuous in $a < r < b$ and non-zero at $r=a$ and $r=b$.

We now change the variable to

$$4.2 \quad \begin{aligned} a &= c - d, \quad b = c + d, \\ r^2 &= c^2 + d^2 - 2cd \cos(\alpha), \end{aligned}$$

where c and $2d$ stands for the mean radius of the punch and the punch width. The variable r in $a \leq r \leq b$ will be replaced by the new one α in $0 \leq \alpha \leq \pi$, where $r=a$ corresponds to $\alpha=0$, and $r=b$ to $\alpha=\pi$. Then, the function $f(r)$ in

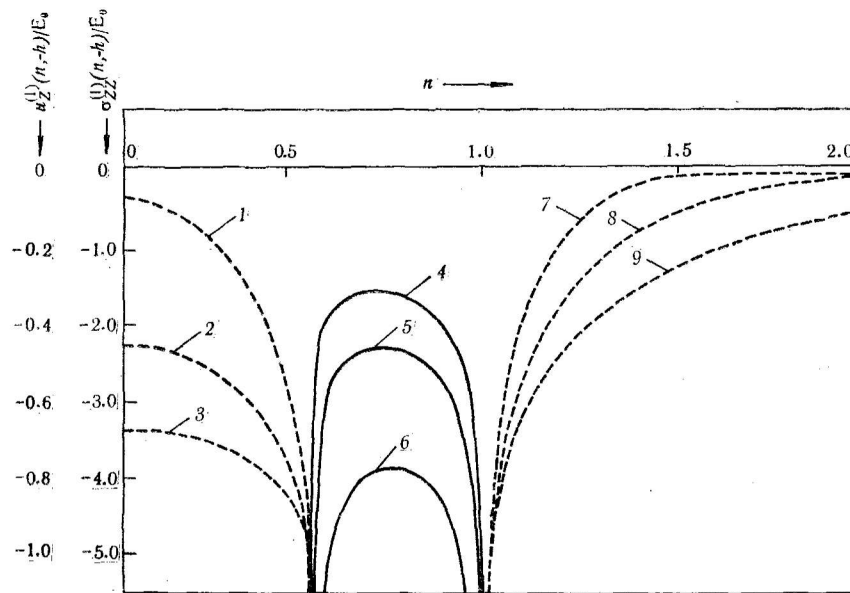


Fig. 4. The distributions of $\sigma_{zz}^{(1)}(r, -h)$ and $u_z^{(1)}(r, h)$ for various values of $h(a=0.5; b=0)$
 1 — $h=0.5$; 2 — $h=1.0$; 3 — $h=2.0$; 4 — $h=2.0$; 5 — $h=1.0$; 6 — $h=1.5$; 7 — $h=0.5$; 8 — $h=1.0$; 9 — $h=2.0$

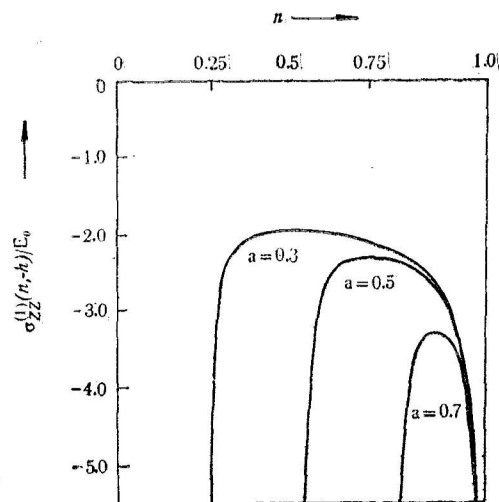


Fig. 5. The distribution of $\sigma_{zz}(r, -h)$ for various values of $a(b=1.0$ and $h=1.0)$

$a \leq r \leq b$ can be expressed as Fourier series with respect to a in $0 \leq a \leq \pi$. Considering $f(a) \neq 0$ and $f(b) \neq 0$, we may take the representation

$$4.3 \quad f(r) = 2 \sum_{n=0}^{\infty} a_n \cos(na), \quad a \leq r \leq b,$$

where the a_n are unknown coefficients. Using the relations

$$\begin{aligned} b^2 - r^2 &= 2cd(1 + \cos(a)), \\ r^2 - a^2 &= 2cd(1 - \cos(a)). \end{aligned}$$

We can rewrite equation (4.1) as

$$4.4 \quad (\sigma_{zz}^{(1)})_{z=-h} = (\varepsilon_0 / \pi cd) \sum_{n=0}^{\infty} [a_n \cos(na)] / \sin(a), \quad a < r < b.$$

Using the fact that $(\sigma_{zz}^{(1)})_{z=-h} = 0$ in $0 \leq r \leq a$, $r > b$, together with equation (4.4) we find from Hankel's inversion theorem that

$$4.5 \quad G(\xi) = \varepsilon_0 \sum_{n=0}^{\infty} a_n \int_0^{\infty} \cos(na) J_0(\xi(c^2 + d^2 - 2cd \cos a)^{\frac{1}{2}}) da,$$

which, on using the formula

$$4.6 \quad J_n(\xi c) J_n(\xi d) = (1/\pi) \int_0^{\infty} \cos(na) J_n(\xi(c^2 + d^2 - 2cd \cos a)^{\frac{1}{2}}) da,$$

gives

$$4.7 \quad G(\xi) = \varepsilon_0 \sum_{n=0}^{\infty} a_n(\xi c) J_n(\xi b).$$

Substituting (4.7) into $(u_z^{(1)})_{z=-h}$ and using the condition (3.10), we get

$$4.8 \quad \sum_{n=0}^{\infty} a_n \int_0^{\infty} [1 + H(2\xi h)] J_0(\xi r) Z_n(\xi) d\xi = 1, \quad a \leq r \leq b,$$

where

$$Z_n(\xi) = J_n(\xi c) J_n(\xi d).$$

Now using the relation

$$4.9 \quad J_0(\xi r) = J_0(\xi c) J_0(\xi d) + 2 \sum_{m=1}^{\infty} J_m(\xi c) J_m(\xi d) \cos(ma), \quad a \leq r \leq b.$$

Substituting (4.9) into (4.8) and equating the coefficients of $\cos(ma)$ on both sides of (4.8), we get an infinite set of simultaneous equations for the unknown coefficients a_m ,

$$4.10 \quad \sum_{n=0}^{\infty} a_n \int_0^{\infty} [1 + H(2\xi h)] Z_m(\xi) Z_n(\xi) d\xi = \delta_{0m}, \quad m = 0, 1, 2, \dots$$

where δ_{0m} is the Kronecker delta.

In particular, the displacement and stress on the surface of the plate are

$$u_z^{(1)}(r, -h) = \varepsilon_0 \sum_{n=0}^{\infty} a_n [I_0 + \int_0^{\infty} H(2\xi h) J_0(\xi r) Z_n(\xi) d\xi],$$

$$\sigma_{zz}^{(1)}(r, -h) = -\frac{\mu_1 \varepsilon_0}{1 - \sigma_1} \sum_{n=0}^{\infty} a_n \begin{cases} 0, & 0 \leq r \leq a, \quad r > b \\ \frac{1}{\pi c d} \cdot \frac{\cos(n\alpha)}{\sin(\alpha)}, & a < r < b \end{cases}$$

where

$$I_0 = \begin{cases} \frac{\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)} \cdot \frac{1}{c} (d/c)^n F(1/2, n+1/2; n+1; \sin^2 \beta) \\ F(1/2, n+1/2; \sin^2 \gamma), & 0 \leq r \leq a \\ \frac{(-1)^n}{\pi} \left\{ \frac{\Gamma(n+1/2)}{(n+1)} \right\}^2 \frac{1}{r} (cd/r^2)^n F(n+1/2, n+1/2; n+1; \sin^2 \beta) \\ F(n+1/2, n+1/2; n+1; \sin^2 \gamma), & b \leq r \end{cases}$$

where $\Gamma(x)$ and $F(\alpha, \beta; \gamma; x)$ are the gamma function and the Gauss hypergeometric series. Also β and γ are:

$$(\beta) = [\sin^{-1}\{(r+d)/c\} \pm \sin^{-1}\{(r-d)/c\}]/2$$

when $0 \leq r \leq a$, and

$$(\beta) = [\sin^{-1}(b/r) \pm \sin^{-1}(a/r)] \text{ when } r \geq b.$$

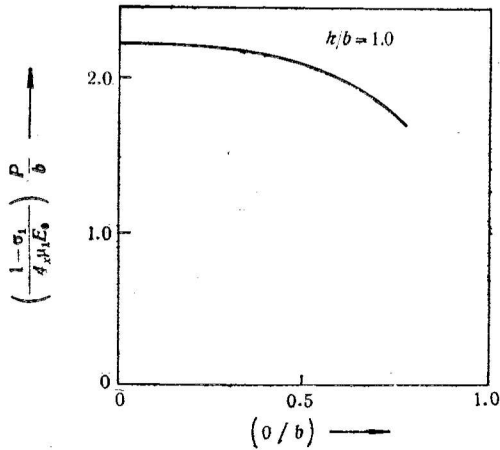


Fig. 6. The variation of the total load P with a/b ($h=1.0$)

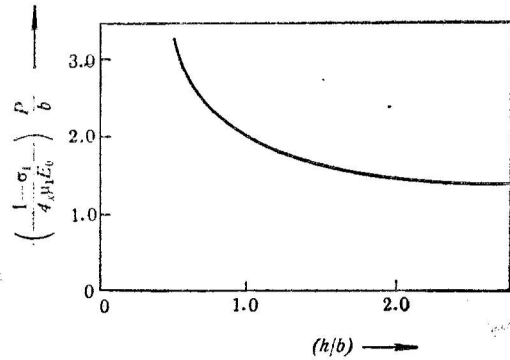


Fig. 7. The variation of the total load P with h ($a=0.5$ and $b=1.0$)

The total compressive load P on the punch is

$$P = - \int_a^b 2\pi r \sigma_{zz}^{(1)}(r, -h) dr = \frac{2\pi\mu_1\varepsilon_0}{1-\sigma_1} a_0.$$

5. Numerical results and discussion

To determine the unknown coefficient a_n , we must solve the infinite set of simultaneous equation (4.10). Let $A_{m,n}$ be the element of the m -th row and n -th column in equation (4.10)

$$A_{m,n} = A_{n,m} = \int_0^\infty [1 + H(2\xi h)] Z_m(\xi) Z_n(\xi) d\xi.$$

Since the magnitude of $H(2\xi h)$ converges rapidly to zero with the increase of ξ , this integral is evaluated directly by numerical integration of the integral

$$5.1 \quad A_{m,n} = A_{n,m} = \int_0^\infty Z_m(\xi) Z_n(\xi) d\xi = \int_0^k J_m(\xi c) J_n(\xi d) J_m(\xi c) J_n(\xi d) d\xi + A'_{m,n}$$

where

$$5.2 \quad A'_{m,n} = A'_{n,m} = \int_k^\infty J_m(\xi c) J_m(\xi d) J_n(\xi c) J_n(\xi d) d\xi$$

and k is a large value. The first integral of equation (5.1) can be evaluated by Simpson's rule. By Hankel's result, the second term in equation (5.1) is calculated by

$$5.3 \quad A'_{m,n} = \frac{1}{\pi^2 c d} [(\cos^2(ka))/k + a \text{Si}(2ka) + \{(-1)^m + (-1)^n\} \{(\sin(kb))/k \cos(ka) - c \text{Ci}(2kc) - d \text{Ci}(2kd)\} + (-1)^{m+n} \{(\sin^2(kb))/k - b \text{Si}(2kb)\}]$$

where $\text{Si}(x)$ and $\text{Ci}(x)$ are the integral sine and cosine function, respectively.

The set of simultaneous equation (4.10) is solved and the first ten roots are considered which is found to give the desired accuracy in the problem. Fig. 1-3 show the radial distributions of the displacement and the stresses for the case of $a=0.5$, $b=1.0$ and $h=1.0$. Fig. 4 shows the distributions of $(u_z^{(1)})_{z=-h}$ (the dashed lines) and $(\sigma_{zz}^{(1)})_{z=-h}$ (the full ones) for various thickness of the plate under the same dimension about the punch ($a=0.5$, $b=1.0$). These results indicate that in the cadmium crystal $(u_z^{(1)})_{z=-h}$ increase as the thickness of the plate becomes large and that $(\sigma_{zz}^{(1)})_{z=-h}$ decreases as it does small. Fig. 5 shows the radial distributions of $(\sigma_{zz}^{(1)})_{z=-h}$ for various values of a under the same dimensions $b=1.0$ and $h=1.0$. Fig. 6 and 7 show, respectively, the relations between the total compressive load and the ratio a/b or the total load P , taking a very large value in a thin plate, decreases as the thickness becomes large.

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References

1. Williams, W. E., Integral equation formulation of some three part boundary value problems. — Proc. Edin. Math. Soc., **13**, (1963), p. 317-323.
2. Cooke, J. C., Triple integral equations. — Quart. Jour. Mech. Appl. Math., **16**, (1963), p. 193-204.
3. Noble, B., The solution of Bessel function dual integral equations by a multiplying-factor method. — Proc. Camb. Phil. Soc., **59**, (1963), p. 351-371.
4. Jain, D. L., R. P. Kanani, An integral equation method for solving mixed boundary value problems. SIAM. — Jour. Appl. Math. **20**, (1971), p. 642-658.
5. Gubenko, U. S., V. I. Mossakovskii, Pressure of an axially symmetric circular die on an elastic half-space. — P. M. M., **24** (1960), p. 334-340.
6. Elliot, H. A., Three dimensional stress distribution in hexagonal aeolotropic crystals. — Proc. Camb. Phil. Soc., **44** (1948), p. 522-533.
7. Shield, R. T., Note on problems in hexagonal aeolotropic material. Proc. Camb. Phil. Soc., **47**, (1951), p. 401-409.
8. Erdelyi, A., Tables of Integral Transforms. Vol. 2 (1954). McGraw Hill, New York.