

A Generalized Method for Kinematic Analysis and Synthesis of Plane Mechanisms

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1. Introduction

It is well-known that the theory of mechanisms and machines is a science about the general methods for investigating the properties of mechanisms and machines and about the process of designing their schemes. There is quite a number of methods for kinematic analysis and synthesis having a smaller or a larger scope of application. The limitations of these methods are mainly in two directions—they solve problems of mechanism analysis only or of synthesis only; they can be applied only to a certain kind of mechanisms. During the last years the involvement of computers allowed to reduce synthesis to a multiple comparative analysis of a set of mechanisms existing in a bounded space of their parameters. A similar approach towards mechanism synthesis is nonradical and irrational and should be employed only when no other methods for synthesis can be applied. On the other hand, the kinematic equivalence of mechanisms is used only within the framework of the kinematic substitution of higher kinematic pairs aiming to achieve an equivalence of the kinematic analysis of mechanisms with higher kinematic pairs and that of linkages.

The present paper exposes a generalized approach and a method for analysis and synthesis of plane mechanisms based on the theorems of Willis [1], Aronhold-Kennedy [2, 3], Bobillier [4] and Freudenstein [5] and is actually a generalization and further elaboration of the contributions found in [6, 7, 8, 9, 10, 11].

2. Kinematic equivalence of mechanisms

To expose the kinematic equivalence of mechanisms is a problem related not only to the kinematic analysis of mechanisms but also to the dimensional and type synthesis of mechanisms [10]. In the design of mechanisms, the synthesis conditions are most often reduced to transfer functions expressing quantitatively the necessary transformation of the motion. The four possible cases of transformation of the motion are illustrated on Fig. 1 (a, b, c), while on the tables on Fig. 2a one can find the first transfer functions ψ_ϕ , S'_ϕ , ψ'_σ , S'_σ , determined with the help of Willis and Aronhold-Kennedy theo-

rems, as well as the second transfer functions $\psi''_\phi, S''_\phi, \psi''_\sigma, S''_\sigma$ found by means of Bobillier and Freudenstein theorems. The first and second transfer functions expressed by the abscissae x_P of the relative instantaneous rotation centre (IRC) $P \equiv P_{12}$ and by the inclination angle μ of the relative motion

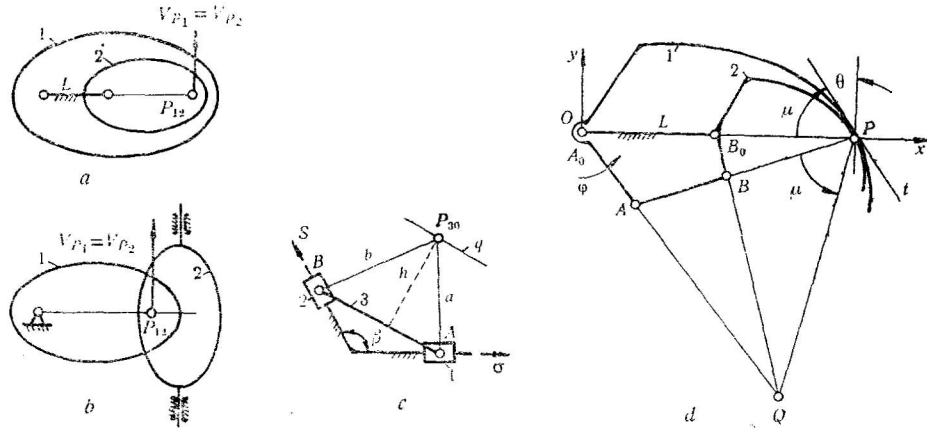


Fig. 1 a, b, c, d

centrodes at the point P , determine the velocities and accelerations of all plane mechanisms. Vice versa, the first two transfer functions serve to determine and use in synthesis the kinematic invariants x_P and μ of all plane mechanisms (Fig. 2b). The abscissae x_P of the IRC P is a polar radius-vector $(r_P)_{12}$ of the first centrod z_{12} of the relative motion between the links 1 and 2 (Fig. 1d) and is defined by the expressions $(r_P)_{12} - (r_P)_{21} = L$ and $\omega_1(r_P)_{12} = \omega_2(r_P)_{21}$.

$$(1) \quad x_P = (r_P)_{12} = L \frac{\psi'}{\psi' - 1} = f(\phi); \quad \psi' = \frac{d\psi}{d\phi} = \frac{\omega_2}{\omega_1}.$$

The angle μ is obtained as an angle added to the angle of obliquity Θ of the relative motion centrodes to reach 90°

$$(2) \quad \operatorname{tg} \mu = \operatorname{ctg} \Theta = \frac{(r_P)_{12}}{(r_P)_{21}} = \frac{\psi'(1 - \psi')}{\psi''}.$$

Let z_{12} and z_{21} be centrodes of a four-bar linkage relative motion. Then, in view of the Bobillier theorem, the straight line AB defining the position of the link with a plane displacement motion, and the collineation axis q form an angle μ as well. If the points A_0 and B_0 are curvature centres of the corresponding trajectories (roulettes) of the points A and B , and if in the expression (2) the substitution $d\phi = d\sigma/r$ and $d\psi = dS/R$ is carried out, where σ , r and S , R are the corresponding displacements of the two points and the curvature radii to their trajectories, then one obtains

$$(3) \quad \operatorname{tg} \mu = \frac{\frac{dS}{d\sigma} \left(1 - \frac{r}{R} \frac{dS}{d\sigma} \right)}{r \frac{d^2 S}{d\sigma^2}}.$$

output		input	output	
rotation (ψ)	translation (S)		rotation (ψ)	translation (S)
$\psi'_\phi = \frac{x_P}{x_P - L}$ $\psi''_\phi = \frac{\psi'_\phi(1 - \psi'_\phi)}{\text{tg } \mu}$	$S'_\phi = x_P$ $S''_\phi = \frac{S'_\phi}{\text{tg } \mu}$	rotation (ϕ)	$x_P = L \frac{\psi'_\phi}{\psi'_\phi - 1}$ $\text{tg } \mu = \frac{\psi'_\phi(1 - \psi'_\phi)}{\psi''_\phi}$	$x_P = S'_\phi$ $\text{tg } \mu = \frac{S'_\phi}{S''_\phi}$
$\psi'_\sigma = \frac{1}{x_P}$ $\psi''_\sigma = -\frac{\psi'^2_\sigma}{\text{tg } \mu}$	$S'_\sigma = -\frac{b}{a}$ $S''_\sigma = \frac{S'_\sigma}{h} (1 + S'^2_\sigma - 2S'_\sigma \cos \beta)^{-\frac{1}{2}}$	translation (σ)	$x_P = \frac{1}{\psi'_\sigma}$ $\text{tg } \mu = -\frac{\psi'^2_\sigma}{\psi''_\sigma}$	$\frac{b}{a} = -S'_\sigma$ $h = \frac{S'_\sigma}{S''_\sigma} (1 + S'^2_\sigma - 2S'_\sigma \cos \beta)^{-\frac{1}{2}}$

Fig. 2 a, b

Depending on the type of transformation of the motion, the expression (3) assumes a concrete form as shown on Fig. 2b. For the mechanisms with a translatory motion of the input and output link (double-slide mechanisms — Fig. 1c), $\mu=0$, i.e. the collineation axis q is parallel to the coupler AB . For the aims of analysis and synthesis it is necessary to determine the distance h between the straight lines AB and q by means of the first two transfer functions S'_σ and S''_σ . Setting $R=\infty$, $r \neq \infty$, in view of (3)

$$(4) \quad \text{tg } \mu = \frac{S'_\sigma}{r S''_\sigma}$$

is obtained.

The geometric construction on Fig. 3 implies

$$(5) \quad \text{tg } \mu = \frac{h_1}{r} \frac{\sin \gamma}{\sin \beta}.$$

After making the right-hand sides of (4) and (5) equal, setting $r=\infty$, and meanwhile $h_1=h$:

$$(6) \quad h = \frac{S'_\sigma}{S''_\sigma} \frac{\sin \beta}{\sin \gamma} = \frac{S'_\sigma}{S''_\sigma} \frac{\overline{AQ}}{\overline{AB}} = \frac{S'_\sigma}{S''_\sigma} \frac{\overline{AQ}}{(\overline{AQ}^2 + \overline{BQ}^2 + 2\overline{AQ} \cdot \overline{BQ} \cos \beta)^{0.5}}$$

is obtained

Since $\frac{\overline{BQ}}{\overline{AQ}} = \frac{b}{a} = -S'_\sigma$ it is finally obtained that

$$(7) \quad h = \frac{S'_\sigma}{S''_\sigma} (1 + S'^2_\sigma - 2S'_\sigma \cos \beta)^{-0.5}.$$

A generalized kinematic scheme of four types of equivalent plane mechanisms is shown on Fig. 4. The kinematic equivalence of the centrode mechanism with cams z_{12} and z_{21} , of the cam mechanism with cam k and of the

The diagram shows a lever arm AD pivoted at point A. A force Q is applied at point Q on the lever arm. A horizontal force is applied to block B, which is constrained to move vertically. The geometry is defined by angles β , γ , γ' , μ , and γ'' , and distances h , h_1 , and r .

tive and absolute IRC P and Q and the angle μ . The straight line BP defines the position of the normal towards the cam profile of the cam mechanism, to the instantaneously equivalent coupler of the linkage and to the band of the band mechanism. Obviously, the point A is the centre of curvature of the cam profile of the cam mechanism, an imaginary kinematic pair of the linkage and final contact point of the band and the cam of the band mechanism. Therefore, the evolute e_A of the cam profile of the cam mechanism is a cam of the band mechanism.

3. Kinematic analysis and synthesis

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Fig. 2a), while the position function is $S = \sigma \cos \beta + (l^2 - \sigma^2 \sin^2 \beta)^{0.5}$. The kinematic analysis of the remaining five kinds of linkages is exposed in tabular form on Fig. 6. The expressions in lines 1 to 7 define the position of the mechanisms, the expressions written in the next two lines serve to determine

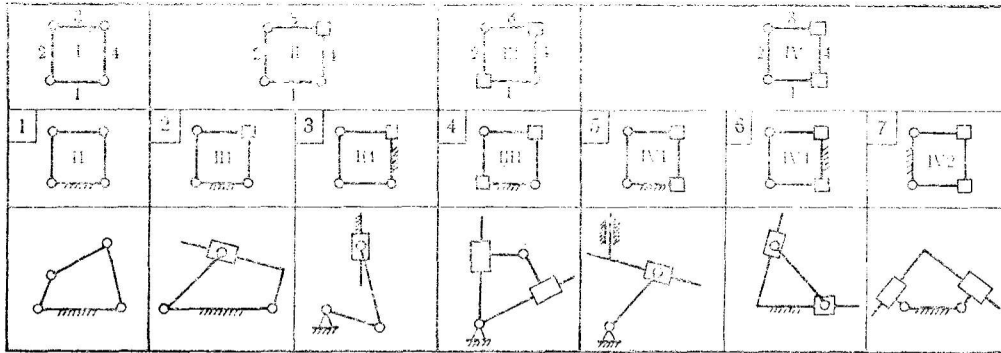


Fig. 5

x_p and the first transfer function, resp., while the expressions in the last three lines help to define $\operatorname{tg} \mu$ and the second transfer function, resp. (see Fig. 2a). x_p and $\operatorname{tg} \mu$ are determined with the help of the analytic geometry apparatus.

The kinematic analysis in graphic form is accomplished following the same order: the kinematic scheme of the mechanism in a certain position is constructed, the IRC P and Q are determined, x_p and $\operatorname{tg} \mu$ are measured and the first two transfer functions are calculated.

The problems for synthesis using three infinitely close relative and absolute positions, the problems for defining equivalent four-bar linkages to the second order of the relative and absolute motions and the problems of determining the centre of curvature to a trajectory of a link point with generalized plane motion are all solved in a similar way.

Let from a four-bar linkage (Fig. 7) be given the frame 1 with length L , the output link 4 with length R and its position defined by the angle ψ (usually these parameters are determined by the admissible angles of force transmission). By means of the first transfer function the position of the IRC $P \equiv P_{42}^{(1)}$ is found and the straight line BP is defined. The first and second transfer functions help to define $\operatorname{tg} \mu$ ($\mu \equiv \mu_{42}^{(1)}$), the collineation axis q and the absolute IRC $Q \equiv Q_{31}$. The point A is found as an intersection point of the straight lines PB and QO , and thus the mechanism scheme is obtained.

The problem of finding an equivalent four-bar linkage, at least to the second order of the relative motion, is again reduced to synthesis, given new values of R and ψ or of only one of the two parameters. Equivalence is reached if the values of x_p and $\operatorname{tg} \mu$ are preserved, i. e. values of the first and second transfer functions.

Let link 2 be a frame. The absolute and relative IRC change their places $P_{31}^{(2)} \equiv Q_{31}$ and $Q_{42} \equiv P_{42}^{(1)}$. Then the synthesis by three infinitely close absolute positions, the determination of equivalent four-bar linkages at least to the second order of the absolute motion of link 4 and finding the curvature centre (the point E) to a trajectory of a point (the point D) from a link with generalized plane motion (link 4) is carried out in the way already described. The-

refore, the mathematical model for synthesis by three infinitely close positions of the four-bar mechanisms written on Fig. 8 in lines 2 to 6 holds when determining equivalent four-bar linkages at least to the second order of the absolute motion. Concerning the synthesis of four-bar linkages by three inini-

Diagram	1	2	3	4	5
1					
2	$\alpha = \arctg \frac{r \sin \varphi}{L - r \cos \varphi}$	$\alpha = \arctg \frac{e}{\sigma}$	$\alpha = 90^\circ - \gamma$		
3	$d^2 = r^2 + L^2 - 2rL \cos \varphi$	$d = \sqrt{e^2 + \sigma^2}$	$d = \sigma \cos \alpha$		
4	$\beta = \arccos \frac{d^2 + R^2 - l^2}{2Rd}$ $d^2 - l^2 = R^2$	$\beta = \arcsin \frac{h}{d}$	$\beta = \arccos \frac{d}{R}$		
5	$\psi = 180^\circ - (\alpha + \beta)$	$\psi = 90^\circ - (\alpha + \beta)$			
6	$x_A = r \cos \varphi; y_A = r \sin \varphi$	$x_A = e; y_A = \sigma \equiv S$	$x_A = R \cos \varphi + \sqrt{l^2 - (R \cos \varphi - e)^2}$	$x_A = R \cos \psi$	$x_A = R \cos \varphi + R \cos \varphi \cdot \operatorname{ctg} \gamma$
7	$x_B = L - R \cos \psi; y_B = R \sin \psi$	$x_B = R \cos \psi; y_B = R \sin \psi$			
8	$k_2 = \frac{y_B - y_A}{x_B - x_A}$	$k_2 = \operatorname{tg} \psi$	$k_2 = \frac{y_B - y_A}{x_B - x_A}$	$k_2 = \operatorname{tg} \psi$	$k_2 = \operatorname{tg} \gamma$
9		$x_P = x_A - \frac{y_A}{k_2}$			
10	$x_Q = \frac{L \operatorname{tg} \psi}{\operatorname{tg} \psi - \operatorname{tg} \varphi}; y_Q = x_Q \operatorname{tg} \varphi$	$x_Q = \sigma \operatorname{tg} \psi; y_Q = \sigma$			
11		$k_3 = \frac{y_Q}{x_Q - x_P}$			$k_3 = \operatorname{tg} \psi$
12		$\operatorname{tg} \mu = \frac{k_3 - k_2}{1 + k_2 k_3}$			

Fig. 6

tely close absolute positions, finding equivalent four-bar linkages at least to the second order of the absolute motion and defining the curvature centre to a trajectory of a point from a link with generalized plane motion, the mathematical model remains the same and the calculations are accomplished with

The general procedure for synthesis of the four types of plane mechanisms allows to define quickly the mechanisms that are perspective in type and parameters and solve one and the same problem. If a certain variant of the synthesized mechanism is considerably superior to the others, this means the type and

parameters of the mechanism are finally accomplished. If not, the type of the mechanism should be decided in the next stage of constructive design taking in consideration the satisfaction of all additional conditions of the design [10]. Therefore, the determination of the mechanism type depends essentially on the results from the general synthesis method, which should be termed as type-dimensional synthesis.

4. Polycontour mechanisms

There is a wide variety in the structure of polycontour mechanisms. A large part of them is formed by consequently connected elementary mechanisms — real or imaginary four-bar linkages. Quite a big set of mechanisms of different classes contain n -linked contours ($n=5, 6, \dots$) in their kinematic chains. If these mechanisms include at least one four-bar contour, they can be transformed into mechanisms consisting of consecutively connected four-bar linkages introducing one or more links kinematically compatible to the second order of the motion of the given kinematic chain [9]. Every imaginary link connects a real joint point of an n -ary link ($n>2$) from a four-bar contour with the centre of curvature to its trajectory in the plane of the opposite link of the same contour. In this way, the neighbouring p -bar contour ($p>4$) is transformed into one four-bar and one $p-1$ -bar contour (Fig. 9). If the given kinematic scheme does not possess a second four-bar contour (see Fig. 9, scheme 4), then the newly obtained four-bar contour is used in order to introduce the next imaginary link, if necessary.

In the six-bar chain with one four-bar contour (Fig. 9, scheme 1) one imaginary link is introduced which connects the joint point between links 3 and 5 with a centre of a curvature to its trajectory in the plane of link 1. Concerning mechanisms of II and III class with a ternary frame, the kinematic analysis is reduced to determining the transfer functions of two four-bar linkages. If the mechanisms are of III and IV class with a binary frame, their kinematic analysis involves only analysis of a four-bar linkage. For the 5 schemes of 8-bar chains shown on Fig. 9, some characteristic variants of mechanisms are given, their kinematic analysis requiring to analyse three four-bar linkages if the frame is a quaternary link, two four-bar linkages if it is a ternary link and one four-bar linkage if it is a binary link.

The first transfer function of any polycontour mechanism, represented by k consecutively connected real or imaginary four-bar linkages, is the product of the first two transfer functions of the four-bar linkages:

$$(8) \quad \frac{dY_k}{dX_k} = \prod_{i=1}^k \frac{\dot{Y}_i}{\dot{X}_i} = \prod_{i=1}^k \frac{dY_i}{dX_i}.$$

After differentiation of the product (1) and a transformation, the second transfer function of the polycontour mechanism is represented as a sum:

$$(9) \quad \frac{d^2Y_k}{dX_k^2} = \sum_{i=1}^k \frac{d^2Y_i}{dX_i^2} \left(\frac{dY_1}{dX_1} \right)^2 \left(\frac{dY_2}{dX_2} \right)^2 \dots \left(\frac{dY_{i-1}}{dX_{i-1}} \right)^2 \left(\frac{dY_{i+1}}{dX_{i+1}} \right) \dots \frac{dY_k}{dX_k} \\ = \sum_{i=1}^k \left[\frac{d^2Y_i}{dX_i^2} \prod_{m=1}^{i-1} \left(\frac{dY_m}{dX_m} \right)^2 \prod_{n=i+1}^k \frac{dY_n}{dX_n} \right]$$

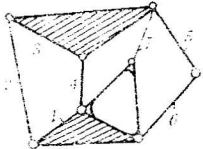
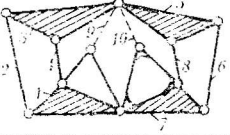
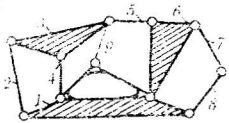
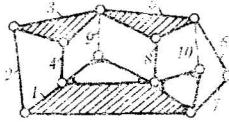
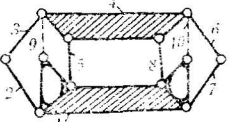
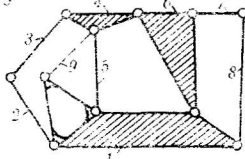
Kinematic chain	Links			Class	Four-bar linkages		
	fixed	input	output		first	second	third
	1	2	6	II	1,2,3,4	1,7,5,6	—
	1	6	2	III	1,6,5,7	1,7,3,2	—
	6	1	5	III	6,1,7,5	—	—
	6	5	1	IV	6,5,7,1	—	—
	1	7	2	III	1,7,10,9	1,9,3,2	—
	1	2	7	IV	1,2,3,9	1,9,10,7	—
	1	2	8	II	1,2,3,9	1,9,5,6	1,6,7,8
	6	7	5	III	6,7,8,1	6,1,9,5	—
	1	2	7	II	1,2,3,9	1,9,5,10	1,10,6,7
	7	1	6	III	7,1,10,6	—	—
	7	6	1	V	7,6,10,1	—	—
	1	2	7	III	1,2,3,9	1,9,4,10	1,10,6,7
	2	1	3	III	2,1,9,3	—	—
	2	3	1	IV	2,3,9,1	—	—
	1	8	2	II	1,8,7,6	1,6,4,9	1,9,3,2
	1	2	8	III	1,2,3,9	1,9,4,6	1,6,7,8
	2	1	3	III	2,1,9,3	—	—
	2	3	1	IV	2,3,9,1	—	—

Fig. 9

of k addends, each of which is a product of the second transfer function of the i th four-bar linkage, the squares of the first transfer functions of the four-bar linkages preceding the i th four-bar linkage and the first transfer functions of the four-bar linkages succeeding it. Therefore, the kinematic analysis of four-bar linkages including the determination of transfer functions and curvature radii, thus introducing imaginary links if necessary, serves to accomplish kinematic analysis of polycontour mechanisms.

5. Conclusion

The general method for analysis and synthesis of plane mechanisms is marked by its universality both regarding the various problems it solves and with respect to the type of mechanisms. Based on the kinematic equivalence of mechanisms and on the apparatus of kinematic geometry, the method is applied always in the same manner no matter in graphic or analytic form, and it reveals the kinematic schemes of four types of plane mechanisms all at a time. Thus, the design of mechanisms does not begin with the choice of mechanism type but rather the type is a result of the dimensional synthesis and the procedure that followed it for the determination of the variants perspective in type and parameters and surpassing the remaining ones. Such a type-dimensional synthesis positively yields an optimal solution without loss of time and unnecessary mental strain. If the linkages, synthesized in a finite interval, have some structural error greater than the allowed one, the obtained results hold for the centrode, cam and band mechanisms, synthesized simultaneously. The estimation of the method concerning the separate types of mechanisms yields the following:

1. In comparison with the already known methods for synthesis of linkages by using computers, the method proposed hereby is far more efficient since there are three varied parameters less, and compared to the methods of approximating functions, our method does not require to define the deviation from a given function by a single expression.

2. The method proposed is quite more efficient from the methods for synthesis of cam mechanisms applying the apparatus of differential geometry. It allows: to construct normals to the cam profile, thus, defining the cam and its equidistant curves by means of points, besides, the angles of force transmission for any position of the mechanism are controlled; to construct the evolute of the cam profile, thus, making precise the determination of the minimal radius of the cam curvatures and the choice of the roller radius of the output link; to look for optimal basic dimensions of the cam mechanisms in the admissible range with the aim of achieving technological profiles that can be easily worked out in a kinematic way when cams are realized.

3. The band mechanisms cams are not demanded as envelopes of the relative positions of the band which complicates the synthesis by the known methods, but are defined by means of points locus of joint points of equivalent four-bar linkages in the plane of the input link, i. e. the evolute of the cam profile.

4. The kinematic analysis of polycontour mechanisms is reduced to kinematic analysis of consecutively connected four-bar linkages by using a general approach for the transformation of polycontour kinematic chains into chains of consecutively connected four-bar linkages, as well as for the kinematic analysis. The method is most efficient concerning linkages of class III with binary frame and mechanisms of class IV and higher, since the work volume decreases considerably.

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