

Effect of Rotation on Rayleigh-Taylor Instability in the Presence of a General Oblique Magnetic Field

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1. Introduction

The study of the hydromagnetics of dispersion relations is of major importance for the development of Rayleigh-Taylor instability. The character of equilibrium of an inviscid, incompressible fluid of variable density in the vertical direction was first investigated by Rayleigh [1]. The combined effect of horizontal and vertical magnetic fields on the Rayleigh-Taylor instability was studied by Ariel [2]. The present paper investigates theoretically the effect of rotation, including the effect of a general oblique magnetic field on the development of Rayleigh-Taylor instability.

2. The equations of the problem

Consider a fluid layer of density ρ dependent on z -coordinate. The fluid is pervaded by an oblique magnetic field $\mathbf{H}(\parallel H_x, 0, H_z)$, where H_x and H_z are constants. It is assumed that the field is incompressible and perfectly conducting. Let the configuration rotate about the z -axis uniformly with an angular velocity Ω . The linearized perturbation equations appropriate to the problem are

$$(1) \quad \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla \delta p + \frac{k}{4\pi} [(\nabla \cdot \mathbf{h}) \cdot \mathbf{H} + (\nabla \times \mathbf{H}) \times \mathbf{h}] + \mathbf{g} \delta \rho + 2\rho(\mathbf{u} \times \boldsymbol{\Omega}),$$

$$(2) \quad \frac{\partial}{\partial t} (\delta \rho) = -(\mathbf{u} \cdot \nabla) \rho,$$

$$(3) \quad \frac{\partial \mathbf{h}}{\partial t} = (\mathbf{H} \cdot \nabla) \mathbf{u},$$

$$(4) \quad \nabla \cdot \mathbf{u} = 0$$

and

$$(5) \quad \nabla \cdot \mathbf{h} = 0,$$

where $\mathbf{h}(\parallel h_x, h_y, h_z)$, δp and $\delta \rho$ are the perturbations in magnetic field, pressure and density respectively, consequent to a small disturbance which produ-

ces the velocity field $\mathbf{u}$ ($\parallel u, v, w$). k is the coefficient of magnetic permeability assumed to be constant. Analysing the disturbance into normal modes, we seek solutions whose dependence on x, y and t given by $\exp(ik_x x + ik_y y + nt)$, where k_x, k_y are wave numbers along x and y directions respectively and n is the rate at which the equilibrium departs from its initial position, we find that the linearized hydromagnetic equations take the form

$$(6) \quad n^2 [\rho w k^2 - D(\rho D w)] + \frac{kn}{4\pi} (H_z D + ik_x H_x) (D^2 - k^2) h_z - g k^2 (D \rho) w - n D (2 \rho \Omega \zeta) = 0;$$

$$(7) \quad \rho n \zeta - \frac{K}{4\pi \rho} (H_z D \xi + ik_x H_x \xi) = 2 \rho \Omega D w;$$

$$(8) \quad n h_z = ik_x H_x w + H_z D w,$$

where $\zeta (\parallel ik_x v - ik_y u)$ and $\xi (\parallel ik_x h_y - ik_y h_x)$ are the vertical components of the vectors $\text{curl } \mathbf{u}$ and $\text{curl } \mathbf{h}$, respectively.

3. Boundary conditions

If the fluid is confined between two rigid boundaries, say $z=0$ and $z=d$, then

$$(9) \quad u = v = w = 0$$

at $z=0$ and $z=d$.

The electromagnetic conditions at the boundaries require

$$(10) \quad h_z = 0 \text{ or } D h_z = 0$$

according as the boundaries are perfectly conducting or of insulating material. The boundary condition on $\mathbf{h}$, the normal component of vorticity vector, can be deduced by making use of equations $E_x = E_y = h_z = 0$. Thus,

$$(11) \quad \zeta = 0 \text{ on a rigid surface.}$$

4. A variational principle

Multiplying equation (6) by w^* the complex conjugate and integrating across the vertical extent of the fluid ($0 < z < d$), we get the following equation on making use of equations (7), (8)

$$(12) \quad n^2 \int_0^d \rho (k^2 w w^* + D w D w^*) dz - g k^2 \int_0^d (D \rho) w w^* dz + \frac{k}{4\pi} n n^* \int_0^d (D h_z D h_z^* + k^2 h_z h_z^*) dz + \frac{k}{4\pi} n^2 \int_0^d \xi \xi^* dz + n n^* \int_0^d \rho \zeta \zeta^* dz = 0,$$

the integrated parts vanish on account of boundary conditions. The equation (12) shows that in general n^2 is purely real.

Consider a change δn in n consequent upon a first arbitrary variation $\delta w, \delta h_z, \delta \zeta$ and $\delta \xi$ in w, h_z, ζ and ξ respectively that satisfy the boundary conditions of the problem. We have to the first order

$$\begin{aligned}
& -2n\delta n \int_0^d \rho(k^2 w w^* + D w D w^*) dz - \frac{k}{4\pi} (n\delta n^* + n^*\delta n) \int_0^d (D h_z D h_z^* + k^2 h_z h_z^*) dz \\
& - \frac{k}{4\pi} 2n\delta n \int_0^d \xi \xi^* dz - (n\delta n^* + n^*\delta n) \int_0^d \rho \zeta \zeta^* dz = n^2 \int_0^d [\rho w k^2 - D(\rho D w)] \delta w^* dz \\
& + n^2 \int_0^d [\rho w^* k^2 - D(\rho D w^*)] \delta w dz - g k^2 \int_0^d D \rho w \delta w^* - g k^2 \int_0^d D \rho w^* \delta w dz \\
& - \frac{k}{4\pi} n n^* \int_0^d (D^2 - k^2) h_z \delta h_z^* dz - \frac{k}{4\pi} n n^* \int_0^d (D^2 - k^2) h_z^* \delta h_z dz \\
(13) \quad & + \frac{k}{4\pi} n^2 \int_0^d (\xi \delta \xi^* + \xi^* \delta \xi) dz + n n^* \int_0^d \rho (\zeta \delta \zeta^* + \zeta^* \delta \zeta) dz.
\end{aligned}$$

Further δw , δh_z , $\delta \zeta$ and $\delta \xi$ are connected by the relations

$$(14) \quad n \delta h_z + \delta n h_z = H_z D \delta w + i k_x H_x \delta w;$$

$$(15) \quad n^* \delta h_z^* + \delta n^* h_z^* = H_z D \delta w^* - i k_x H_x \delta w^*;$$

$$(16) \quad n \delta \zeta + \delta n \zeta = \frac{k}{4\pi \rho} (H_z D \delta \xi + H_x i k_x \delta \xi) + 2\rho \Omega D \delta w;$$

$$(17) \quad n^* \delta \zeta^* + \delta n^* \zeta^* = \frac{k}{4\pi \rho} (H_z D \delta \xi^* - i k_x H_x \delta \xi^*) + 2\rho \Omega D \delta w^*.$$

Combining equations (13)–(17), we obtain

$$\begin{aligned}
& -2n\delta n \int_0^d \rho(k^2 w w^* + D w D w^*) dz - 2n\delta n \int_0^d \xi \xi^* dz = \int_0^d \left\{ n^2 [\rho w k^2 - D(\rho D w)] \right. \\
& \left. - g k^2 (D \rho) w + \frac{k}{4\pi} n (i k_x H_x + H_z D) (D^2 - k^2) h_z - n (2 \rho \Omega \zeta) \right\} \delta w^* dz \\
(18) \quad & + \int_0^d \left\{ n^2 [\rho w^* k^2 - D(\rho D w^*)] - g k^2 (D \rho) w^* + \frac{k}{4\pi} n^* (-i k_x H_x + H_z D) (D^2 - k^2) h_z^* \right. \\
& \left. - n (2 \rho \Omega \zeta^*) \right\} \delta w dz.
\end{aligned}$$

Since the quantity inside the brackets generally does not vanish for small variations δw , δw^* , δh_z , δh_z^* , $\delta \zeta$, $\delta \zeta^*$, $\delta \xi$ and $\delta \xi^*$ in w , w^* , h_z , h_z^* , ζ , ζ^* , ξ and ξ^* respectively, compatible with the boundary conditions. Thus, we have shown that, for the case under consideration, namely for any $\rho(z)$ and a general oblique magnetic field, a variational procedure for obtaining an approximate solution is possible.

5. A continuously stratified fluid of finite depth

In this section we shall find an approximate solution for a continuously stratified fluid depth d in which the density distribution is given by

$$(19) \quad \rho(z) = \rho_0 \exp \beta z,$$

where β and ρ_0 are appropriate constants.

The boundary surfaces are to be rigid and ideally conducting. Thus in accordance with the boundary conditions we choose the following trial functions for $w(z)$:

$$(20) \quad w(z) = W(1 - \cos lz),$$

where W is constant and $l = (\pi s/d)$, s being an even integer. h_z and ζ can be directly evaluated with aid of (8) and (7). We have

$$(21) \quad h_z(z) = \frac{w}{n} [lH_z + ik_x H_x (l - \cos lz)],$$

$$(22) \quad \text{and } \zeta(z) = \frac{2\Omega nl(n^2 + k_x^2 V_0^2 + l^2 V_z^2) \sin lz + 2ik_x lV_0 V_z \cos lz}{[(n^2 + k_x^2 V_0^2 + l^2 V_z^2)^2 - 4k_x^2 l^2 V_0^2 V_z^2]^2}.$$

Evaluating the integrals in (12) after substituting for $w(z)$, $h_z(z)$ and $h(z)$ we obtain the following dispersion relation between n and k :

$$(23) \quad \begin{aligned} & n^6 + n^4 \left[2(l^2 V_z^2 + k_x^2 V_0^2) + \frac{l^2(l^2 + k^2)V_z^2}{3k^2 + l^2} + k_x^2 V_0^2 + \frac{4\Omega^2 l^2 - 3g\beta k^2}{3k^2 + l^2} \right] \\ & + n^3 \left[(l^2 V_z^2 - k_x^2 V_0^2)^2 + (l^2 V_z^2 + k_x^2 V_0^2) \left\{ \frac{l^2(l^2 + k^2)V_z^2}{3k^2 + l^2} + k_x^2 V_0^2 + \frac{4\Omega^2 l^2 - 3g\beta k^2}{3k^2 + l^2} \right\} \right] \\ & + (l^2 V_z^2 - k_x^2 V_0^2)^3 \left\{ \frac{l^2(l^2 + k^2)V_z^2}{3k^2 + l^2} + k_x^2 V_0^2 - \frac{3g\beta k^2}{3k^2 + l^2} \right\} = 0, \end{aligned}$$

where $V_z = (k H_z^2 / 4\pi \rho_0)^{\frac{1}{2}}$ and $V_0 = (k H_0^2 / 4\pi \rho_0)^{\frac{1}{2}}$ are the components of the Alfvén velocity V at $z=0$.

Equation (23) is a cubic in n^2 , therefore it must give six roots. For the stability of the configuration we require n^2 to be negative. If β is negative, i. e., the potentially stable arrangement remains stable irrespective of the direction of the ambient magnetic field. Even if the configuration is thoroughly unstable ($\beta > 0$) for $k < k_1$, $k > k_2$, where k_1 and k_2 are the roots of the equation

$$(24) \quad 3k^4 \cos^2 \theta + k^2(l^2 V_0^2 \cos^2 \theta + l^2 V_z^2 - 3g\beta) + l^4 V_z^2 = 0,$$

undamped oscillatory motion takes place in the presence of a magnetic field. θ is the inclination of the direction of magnetic field to the horizontal. Therefore, the potential unstable arrangement can be stabilized for some of the disturbances by the magnetic field. It is further noted that k given by equation (24) does not contain any term involving Ω . So we conclude that the criterion of stability is independent of the rotation.

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References

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2. Ariel, P. D. Can. J. Phys., 57, 1979, 1094.