

Parametric oscillations of dynamical systems under the influence of nonlinear frictions

Nguyen van Dao

Among the parameters having a considerable influence on parametric oscillations it is worth mentioning the nonlinear frictions [1, 2]. In most dynamical systems different kinds of frictions often appear simultaneously and it is very difficult to evaluate which has the primary meaning. Therefore, the investigation of the influence of the individual forms of the frictions and the determination of the limits of the parameters, in which that influence is considerable, can be the foundation for the assertion by the observation which of the frictions appears in the system considered [3].

For the free and forced oscillations this problem was resolved by Pisarenko, G. S., Osinski, Z. and others [3, 4]. The influence of frictions on the parametric oscillations in some systems has been dealt with in the previous publication by Schmidt G. and Heinrich [1, 2]. The parametric oscillation of a linear system under the action of dry friction was discussed by Tondl, A. [5]. The investigation of the parametric oscillation in the system with quadratic friction was made by Hagedorn, P. [6].

This paper is concerned with the study of the influence of some kinds of the nonlinear frictions (Coulomb and turbulent and their combination) on the parametric oscillations of the nonlinear system (1.1). It turned out that the development and the stability of the parametric oscillations essentially depend on the nonlinear frictions and by suitable choice of the parameters of oscillating system one can control the parametric oscillations.

General equation of motion and its solution

Let us consider the nonlinear oscillating system governed by the following differential equation [10]

$$(1.1) \quad \ddot{x} + (\omega^2 + \varepsilon c \cdot \cos \gamma t)x + \varepsilon a x^3 + \varepsilon R(x, \dot{x}) = 0,$$

where ω , c , a are constants, ε is a small parameter, and the function $R(x, \dot{x})$ characterizes the nonlinear frictions. It is supposed that the system (1.1) when $c=0$ is not self-excited and that there is a resonant relation of type

$$(1.2) \quad \omega^2 = \frac{\gamma^2}{4} + \varepsilon \cdot 1.$$

By means of the formulae of substitution

$$(1.3) \quad \begin{aligned} x &= a \sin \theta, \\ \dot{x} &= \frac{\gamma}{2} a \cos \theta, \end{aligned}$$

the equation (1.1) is reduced to the standard form [7, 9]

$$(1.4) \quad \frac{\gamma}{2} \dot{a} = -\varepsilon [1x + ax^3 + c \cdot \cos \gamma t \cdot x + R(x, \dot{x})] \cos \theta,$$

$$\frac{\gamma}{2} a \dot{\psi} = \varepsilon [1x + ax^3 + c \cdot \cos \gamma t \cdot x + R(x, \dot{x})] \sin \theta,$$

$$\psi = \theta - \frac{\gamma}{2} t.$$

Since the amplitude a and phase ψ are slowly varying functions of t the technique of slowly varying parameters [7, 9] can be used to obtain the approximate solutions to equations (1.4). In the first approximation one obtains

$$(1.5) \quad \begin{aligned} \frac{\gamma}{2} \dot{a} &= -\varepsilon \left[\frac{c}{4} a \sin 2\psi + S(a, \gamma) \right], \\ \frac{\gamma}{2} a \dot{\psi} &= \varepsilon \left[\frac{1}{2} a + \frac{3}{8} aa^3 - c \frac{a}{4} \cos 2\psi + H(a, \gamma) \right], \end{aligned}$$

$$S(a, \gamma) = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta \cdot R\left(a \sin \theta, \frac{\gamma}{2} a \cos \theta\right) d\theta,$$

$$H(a, \gamma) = \frac{1}{2\pi} \int_0^{2\pi} \sin \theta \cdot R\left(a \sin \theta, \frac{\gamma}{2} a \cos \theta\right) d\theta.$$

It should be pointed out that the variables a and ψ are now representing the average values over one cycle of oscillation.

The steady state response of the system can be obtained by setting $\dot{a}$ and $\dot{\psi}$ to zero in equations (1.5), namely

$$(1.6) \quad \begin{aligned} c \frac{a}{4} \cdot \sin 2\psi_0 &= S(a_0, \gamma), \\ c \frac{a}{4} \cdot \cos 2\psi_0 &= \frac{1}{2} a_0 + \frac{3}{8} aa_0^3 + H(a_0, \gamma), \end{aligned}$$

where a_0, ψ_0 denote the steady-state values of a, ψ respectively. Eliminating ψ_0 from (1.6) the equation describing the steady-state amplitudes is obtained

$$(17) \quad W(a_0, \gamma) = 0,$$

where

$$(1.8) \quad W(a_0, \gamma) = S^2 + \left(H + \frac{A}{2} a_0 + \frac{3}{8} a a_0^3 \right)^2 - \frac{c^2}{16} a_0^2.$$

To investigate the stability of stationary oscillations the variational equations based on the steady-state solution will be studied. Let

$$a = a_0 + \delta a, \quad \psi = \psi_0 + \delta \psi,$$

where δa , $\delta \psi$ are small quantities representing the deviation from the steady-state solution. Substituting these expressions into (1.5) and retaining terms of lowest order in δa , $\delta \psi$ only and also making use of relationships (1.6) there it is obtained

$$(1.9) \quad \begin{aligned} \frac{\gamma}{2} \frac{d\delta a}{dt} &= -\varepsilon \left[\left(\frac{c}{4} \sin 2\psi_0 + S'(a_0, \gamma) \right) \delta a + c \frac{a}{2} \cos 2\psi_0 \delta \psi \right], \\ \frac{\gamma}{2} a_0 \frac{d\delta \psi}{dt} &= \varepsilon \left[\left(\frac{A}{2} + \frac{9}{8} a a_0^2 + H'(a_0, \gamma) - \frac{c}{4} \cos 2\psi_0 \right) \delta a + \frac{c}{2} a_0 \sin 2\psi_0 \delta \psi \right], \end{aligned}$$

where

$$S' = \frac{\partial S(a_0, \gamma)}{\partial a_0}, \quad H' = \frac{\partial H(a_0, \gamma)}{\partial a_0}.$$

The characteristic equation of the system (1.9) is of the form

$$(1.10) \quad \begin{aligned} \frac{\gamma^2}{4} a_0 \lambda^2 + \frac{\varepsilon}{2} \gamma (a_0 S' + S) \lambda + 2\varepsilon^2 \left[\left(\frac{A}{2} + \frac{9}{8} a a_0^2 + H' \right) \left(\frac{A}{2} a_0 + \frac{3}{8} a a_0^3 + H \right) + S S' \right] \\ - \frac{\varepsilon^2}{8} c^2 a_0 = 0. \end{aligned}$$

The characteristic roots λ will have negative real parts only when

$$(1.11) \quad \begin{aligned} \frac{\partial}{\partial a_0} (a_0 S) > 0, \\ \frac{\partial}{\partial a_0} \left[\left(\frac{A}{2} a_0 + \frac{3}{8} a a_0^3 + H \right)^2 + S^2 - \frac{c^2}{16} a_0^2 \right] = \frac{\partial}{\partial a_0} W(a_0, \gamma) > 0. \end{aligned}$$

It can be shown that the equation of vertical tangency with resonant curve is given by

$$\frac{\partial W(a_0, \gamma)}{\partial a_0} = 0.$$

Hence, the line of vertical tangency serves as a boundary separating the stable and unstable branches of the resonant curve.

2. The influence of Coulomb friction on nonlinear oscillation

Let us consider the case

$$(2.1) \quad R(x, \dot{x}) = h_0 \operatorname{sign} \dot{x},$$

where h_0 is a positive constant and

$$\text{sign } \dot{x} = \begin{cases} 1 & \text{if } \dot{x} > 0, \\ -1 & \text{if } \dot{x} < 0, \\ 0 & \text{if } \dot{x} = 0. \end{cases}$$

We note here that the case $\alpha=0$ was investigated by Tondl, A [5].
The equations (1.5) now become

$$(2.2) \quad \begin{aligned} \frac{\gamma}{2} \dot{a} &= -\varepsilon \left(\frac{c}{4} a \sin 2\psi + \frac{2}{\pi} h_0 \right), \\ \frac{\gamma}{2} a \dot{\psi} &= \varepsilon \left(\frac{A}{2} a + \frac{3}{8} \alpha a^3 - \frac{c}{4} a \cos 2\psi \right), \end{aligned}$$

for $a > 0$ and

$$(2.3) \quad \begin{aligned} \frac{\gamma}{2} \dot{a} &= -\frac{\varepsilon}{4} a \sin 2\psi, \\ \frac{\gamma}{2} a \dot{\psi} &= \varepsilon \left(\frac{A}{2} a + \frac{3}{8} \alpha a^3 - \frac{c}{4} a \cos 2\psi \right), \end{aligned}$$

for $a=0$.

The equation (1.7) is of the form

$$\frac{4}{\pi^2} h_0^2 + \frac{a_0^2}{4} \left(A + 3 \frac{\alpha}{4} a_0^2 \right)^2 - \frac{c^2 a_0^2}{16} = 0,$$

which gives

$$(2.4) \quad \eta^2 = 1 + \frac{3\varepsilon}{4\omega^2} \alpha a_0^2 \pm \frac{2\varepsilon}{\omega^2} \sqrt{\frac{c^2}{16} - \frac{4h_0^2}{\pi^2 a_0^2}}, \quad \eta = \frac{\gamma}{2\omega}.$$

This relationship is plotted in Fig. 1 for $\varepsilon\alpha/\omega^2=1/3$, $\varepsilon c/\omega^2=0.16$ and with different values of the friction coefficient h_0 : $h_0=0$ (curve 0), $\varepsilon h_0/\omega^3=\pi \cdot 10^{-2}/2$ (curve 1), $\varepsilon h_0/\omega^2=\pi \cdot 10^{-2}$ (curve 2), $\varepsilon h_0/\omega^2=\sqrt{2} \cdot \pi \cdot 10^{-2}$ (curve 3). An analysis on the stability of the steady-state curves was carried out to identify the stable branches, on which the stability conditions $h_0 > 0$, $\partial W/\partial a_0 > 0$ are fulfilled. These branches here and throughout are shown as fat plots. The resonant curves are convex toward the frequency axis.

The zero solution $a=0$ governed by equations (2.3) is unstable in the following interval of axis η^2 :

$$(2.5) \quad 1 - \frac{\varepsilon c}{2\omega^2} < \eta^2 < 1 + \frac{\varepsilon c}{2\omega^2}.$$

The inspection of Fig. 1 reveals that:

1) In the presence of the Coulomb friction the stationary regime of oscillations is established not in all the diapason of frequencies, corresponding to the zone of parametric resonance

2) The jump phenomenon takes place when decreasing the frequency γ .

3) For a certain value of frequency γ increasing the friction coefficient leads to the decrease of the amplitude of stable stationary oscillation. With the growth of the friction coefficient the resonant curves move up.

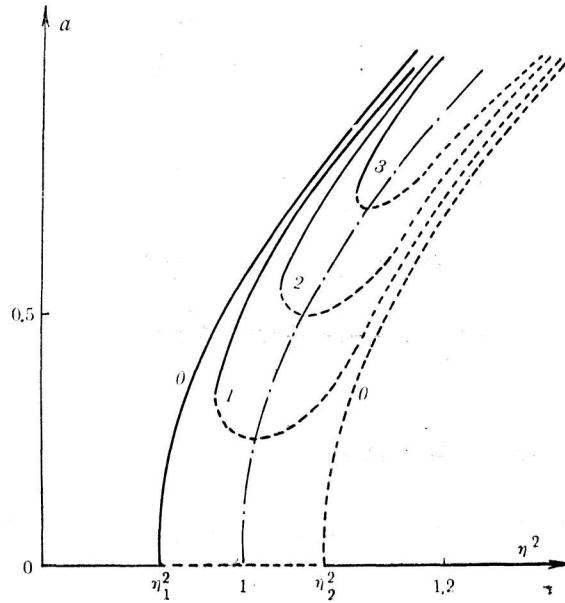


Fig. 1

We consider now the refinement of the first approximation. Replace firstly (1.4) by

$$\begin{aligned}
 \frac{da}{d\theta} = & -\frac{4\varepsilon}{\gamma^2} \left\{ \frac{2}{\pi} h_0 + \frac{c}{4} a \sin 2\psi + \frac{2}{\pi} h_0 \cos 2\theta + \frac{a}{2} \left(1 + \frac{a}{2} a^2 \right) \sin 2\theta \right. \\
 & \left. - \frac{a}{8} a^3 \sin 4\theta + \frac{c}{4} a \sin (4\theta - 2\psi) + \frac{2}{\pi} h_0 \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} [\cos (2k+1)\theta + \cos 2k\theta] \right\}, \\
 \frac{d\psi}{d\theta} = & \frac{4\varepsilon}{a\gamma^2} \left\{ \frac{1}{2} a + \frac{3}{8} a a^3 - \frac{c}{4} a \cos 2\psi + \frac{2}{\pi} h_0 \sin 2\theta - \frac{a}{2} (1 + \delta a^2) \cos 2\theta \right. \\
 (2.6) \quad & \left. + \frac{c}{2} a \cos (2\theta - 2\psi) + \frac{a}{8} a^3 \cos 4\theta - \frac{c}{4} a \cos (4\theta - 2\psi) \right. \\
 & \left. + \frac{2}{\pi} h_0 \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} [\sin 2(k+1)\theta - \sin 2k\theta] \right\}.
 \end{aligned}$$

Since a , ψ do not vary very rapidly we will integrate the right hand sides of (2.6) as if a and ψ were constant. We thus obtain

$$\begin{aligned}
 a = & a_0 - \frac{4\varepsilon}{\gamma^2} \left\{ \frac{h_0}{\pi} \sin 2\theta - \frac{a_0}{4} \left(1 + \frac{a}{2} a_0^2 \right) \cos 2\theta + \frac{a}{32} a_0^3 \cos 4\theta \right. \\
 & \left. - \frac{c}{16} a_0 \cos (4\theta - 2\psi_0) + \frac{h_0}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left[\frac{\sin 2(k+1)\theta}{k+1} + \frac{\sin 2k\theta}{k} \right] \right\},
 \end{aligned}$$

$$(2.7) \quad \psi = \psi_0 + \frac{4\varepsilon}{a_0 \gamma^2} \left\{ -\frac{h_0}{\pi} \cos 2\theta - \frac{a_0}{4} (1 + \alpha a_0^2) \sin 2\theta + \frac{c}{4} a_0 \sin (2\theta - 2\psi_0) \right. \\ \left. + \frac{\alpha}{32} a_0^3 \sin 4\theta - \frac{c}{16} a_0 \sin (4\theta - 2\psi) - \frac{h_0}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \left[\frac{\cos 2(k+1)\theta}{k+1} - \frac{\cos 2k\theta}{k} \right] \right\}.$$

The new refined first approximation will be given by

$$x = a \sin \left(\frac{\gamma}{2} t + \psi \right).$$

After some simple calculations this leads to

$$(2.8) \quad x = a_0 \sin \left(\frac{\gamma}{2} t + \psi_0 \right) + \frac{4\varepsilon}{\gamma^2} \left[-\frac{\alpha}{32} a_0^3 \sin \left(\frac{3}{2} \gamma t + 3\psi_0 \right) + \frac{c}{16} a_0 \sin \left(\frac{3}{2} \gamma t + \psi_0 \right) \right. \\ \left. + \frac{4}{\pi} h_0 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)[(2k+1)^2 - 1]} \cos (2k+1) \left(\frac{\gamma}{2} t + \psi_0 \right) \right].$$

In the relations (2.7) (2.8) a_0 and ψ_0 are the solutions of the "smoothed out" equations (2.3), (2.4).

3. The turbulent friction

If the friction is taken to be turbulent [8]

$$(3.1) \quad R(x, \dot{x}) = (h_2 + h_4 x^2) \dot{x}^2 \operatorname{sign} \dot{x}, \quad h_2 > 0,$$

then the expressions (1.5) yield

$$S(a, \gamma) = \frac{\gamma^2}{3\pi} a^2 \left(h_2 + \frac{1}{5} h_4 a^2 \right),$$

$$H(a, \gamma) = 0,$$

and the equation (1.7) becomes

$$(3.2) \quad W(a_0, \gamma) = \frac{\gamma^4}{9\pi^2} a_0^4 \left(h_2 + \frac{1}{5} h_4 a_0^2 \right)^2 + \left(\frac{4}{2} + \frac{3}{8} \alpha a_0^2 \right)^2 a_0^2 - \frac{c^2}{16} a_0^2 = 0,$$

which gives approximately

$$(3.3) \quad \eta^2 = 1 + \frac{3\varepsilon}{4\omega^2} \alpha a_0^2 \pm \varepsilon \sqrt{\frac{c^2}{4\omega^4} - \frac{64}{9\pi^2} a_0^2 \left(h_2 + \frac{1}{5} h_4 a_0^2 \right)^2}, \quad h = \frac{\gamma}{2\omega}.$$

It was found that with the given values of c , α , and h_4 , if $h_4 > 0$, then increasing the coefficient h_2 leads to the decrease of the maximum of the amplitudes of stationary oscillations. Analogously, the maximum of the amplitudes of stationary oscillations decreases when increasing h_4 (see Fig. 2), where $\varepsilon\alpha/\omega^2 = 1/3$, $\varepsilon c/\omega^2 = 0.16$ and $\varepsilon h_2 = 9\pi \cdot 10^{-2}$, $\varepsilon h_4 = 15\pi \cdot 10^{-2}$ (curve 1), $\varepsilon h_2 = 3\pi \cdot 10^{-2}$, $\varepsilon h_4 = 15\pi \cdot 10^{-2}$ (curve 2), $\varepsilon h_2 = 3\pi \cdot 10^{-2}$, $\varepsilon h_4 = 5\pi \cdot 10^{-2}$ (curve 3).

If $h_4 < 0$, then the resonant curve consists of only one branch (Fig. 3) for the case $h_2 < \frac{3\pi c}{16\omega^2} \sqrt{-\frac{27}{20} h_4}$ and two branches for $h_2 \geq \frac{3\pi c}{16\omega^2} \sqrt{-\frac{27}{20} h_4}$ (Fig. 4). The parameters of Figs. 3 and 4 are: $\varepsilon c/\omega^2 = 0.16$, $\varepsilon\alpha/\omega^2 = 1/3$, $\varepsilon h_4 = -2/3$ and $\varepsilon h_2 = 0.1$ (curve 1), $\varepsilon h_2 = 0.18$ (curve 2), $\varepsilon h_2 = 0.2$ (curve 3), $\varepsilon h_2 = 0.22$ (curve 4),

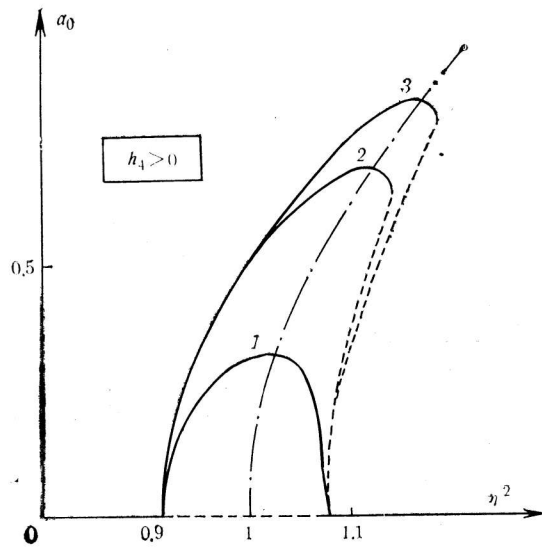


Fig. 2

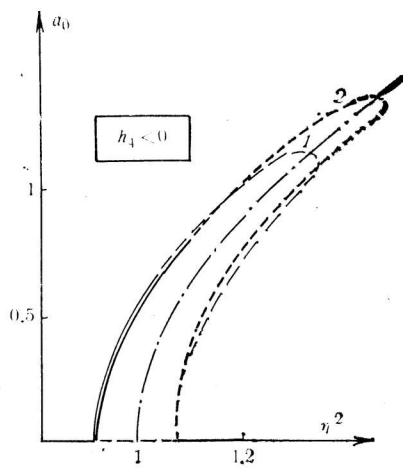


Fig. 3

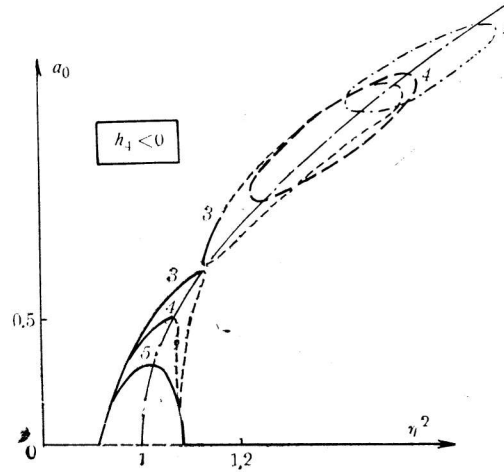


Fig. 4

$\epsilon h_2 = 0.3$ (curve 5). On these figures only the fat branches correspond to the stability of stationary oscillations, where the relations (1.11) are satisfied and the unstable branches are shown dotted to indicate that they are physically unobtainable.

4. Coulomb friction with turbulent one

In this section we shall consider the vibrating system with friction being the combination of Coulomb and turbulent frictions of the form [8]

$$(4.1) \quad R(x, \dot{x}) = (h_0 + h_2 \dot{x}^2) \operatorname{sign} \dot{x},$$

where h_0, h_2 are positive constants.

Substituting (4.1) into (1.5) and integrating it yields

$$(4.2) \quad S(a, \gamma) = \frac{2}{\pi} h_0 + \frac{h_2}{3\pi} \gamma^2 a^2,$$

and the following approximate expression for the resonant curves is found

$$(4.3) \quad \eta^2 = 1 + \frac{3\varepsilon}{4\omega^2} a a_0^2 \pm 2\varepsilon \sqrt{\frac{c^2}{16\omega^4} - \left(\frac{2h_0}{\pi\omega^2 a_0} + \frac{4h_2}{3\pi} a_0 \right)^2}, \quad \eta = \frac{\gamma}{2\omega}.$$

The conditions for real η^2 are given by

$$(4.4) \quad \frac{3\pi}{8h_2} \left(\frac{c}{4\omega^2} - \sqrt{\frac{c^2}{16\omega^4} - \frac{32h_0h_2}{3\pi^2\omega^2}} \right) \leq a_0 \leq \frac{3\pi}{8h_2} \left(\frac{c}{4\omega^2} + \sqrt{\frac{c^2}{16\omega^4} - \frac{32h_0h_2}{3\pi^2\omega^2}} \right),$$

where

$$(4.5) \quad c \geq \frac{4\omega}{\pi} \sqrt{\frac{32}{3} h_0 h_2},$$

and the equality sign in (4.4) gives the minimum and maximum amplitudes of the stationary oscillations and it corresponds to the horizontal tangents with the resonant curves.

If $c > \frac{4}{\pi} \omega \sqrt{\frac{32}{3} h_0 h_2}$, the resonant curve is closed and intersects the skeleton-curve $\eta^2 = 1 + \frac{3\varepsilon}{4\omega^2} a a_0^2$ at two points with the ordinates

$$(4.6) \quad \begin{aligned} a_{01} = a_{0\min} &= \frac{3\pi}{8h_2} \left(\frac{c}{4\omega^2} - \sqrt{\frac{c^2}{16\omega^4} - \frac{32h_0h_2}{3\pi^2\omega^2}} \right), \\ a_{02} = a_{0\max} &= \frac{3\pi}{8h_2} \left(\frac{c}{4\omega^2} + \sqrt{\frac{c^2}{16\omega^4} - \frac{32h_0h_2}{3\pi^2\omega^2}} \right). \end{aligned}$$

If

$$(4.7) \quad c = \frac{4}{\pi} \omega \sqrt{\frac{32}{3} h_0 h_2},$$

the resonant curve is tightened to the point

$$(4.8) \quad a_0 = \frac{3\pi c}{32\omega^2 h_2}, \quad \eta^2 = 1 + \frac{27\varepsilon\pi^2 c^2}{4096 h_2^2 \omega^6}.$$

If $c < \frac{4}{\pi} \omega \sqrt{\frac{32}{3} h_0 h_2}$, in the system considered there is not the nontrivial stationary oscillation.

Fig. 5 shows the resonant curves with parameters $\varepsilon a/\omega^2 = 1/3$, $\varepsilon c/\omega^2 = 0.16$, $2\varepsilon h_0/\pi\omega^2 = 10^{-2}$ for different values of h_2 . The point 1 satisfies the condition (4.7) and has coordinates (4.8), with $\varepsilon h_2 = 3\pi \cdot 10^{-2}$. The curves 2, 3, 4 and 5 correspond to $\varepsilon h_2/3\pi = 10^{-2} \cdot 23/24, 10^{-2} \cdot 5/6, 10^{-2} \cdot 3/4$ and $10^{-2} \cdot 2/3$ respectively. In Figs. 1 and 5 curves "0" correspond to the zero values of h_0 and h_2 .

It is shown that:

1) The resonant curves are closed and they are tightened with the growth of frictions.

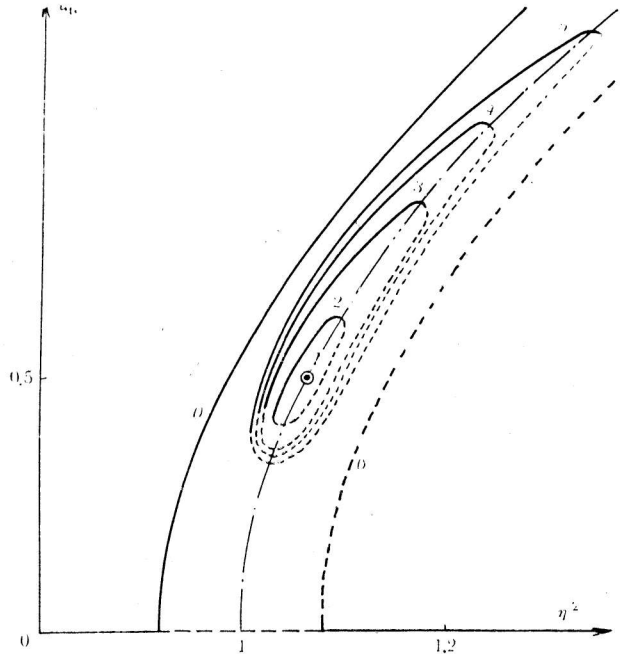


Fig. 5

2) The stationary regime of oscillations takes place not in the whole diapason of frequencies corresponding to the zone of parametric resonance. With a given modulation frequency γ increasing the coefficients of frictions leads to the decrease of the amplitude of stationary oscillation.

3) There exists a jump phenomenon when increasing as well as when decreasing the modulation frequency γ .

We prove without difficulty that the refinement of the first approximation is

$$\begin{aligned}
 a = a_0 - \frac{4\varepsilon}{\gamma^2} & \left\{ \frac{1}{2} \left(\frac{2}{\pi} h_0 + \frac{h_2}{3\pi} \gamma^2 a_0^2 \right) \sin 2\theta - \frac{a_0}{4} \left(1 + \frac{\alpha}{2} a_0^2 \right) \cos 2\theta + \frac{\alpha}{32} a_0^3 \cos 4\theta \right. \\
 & \left. - \frac{c}{16} a_0 \cos (4\theta - 2\psi_0) + \frac{1}{\pi} \sum_{k=1}^{\infty} \left[2h_0 - \frac{h_2 \gamma^2 a_0^2}{(2k+1)^2 - 4} \right] \frac{(-1)^k}{2k+1} + \left[\frac{\sin 2k\theta}{2k} + \frac{\sin 2(k+1)\theta}{2(k+1)} \right] \right\}, \\
 (4.9) \quad \psi = \psi_0 + \frac{4\varepsilon}{a_0 \gamma^2} & \left\{ -\frac{1}{2} \left(\frac{2}{\pi} h_0 + \frac{h_2}{3\pi} \gamma^2 a_0^2 \right) \cos 2\theta - \frac{a_0}{4} (1 + \alpha a_0^2) \sin 2\theta \right. \\
 & \left. + \frac{c}{4} a_0 \sin (2\theta - 2\psi_0) + \frac{\alpha}{32} a_0^3 \sin 4\theta - \frac{c}{16} a_0 \sin (4\theta - 2\psi_0) \right. \\
 & \left. - \frac{1}{\pi} \sum_{k=1}^{\infty} \left[2h_0 - \frac{h_2 \gamma^2 a_0^2}{(2k+1)^2 - 4} \right] \frac{(-1)^k}{2k+1} \left[\frac{\cos 2(k+1)\theta}{2(k+1)} - \frac{\cos 2k\theta}{2k} \right] \right\},
 \end{aligned}$$

and therefore,

$$(4.10) \quad x = a_0 \sin\left(\frac{\gamma}{2}t + \psi_0\right) + \frac{4\varepsilon}{\gamma^2} \left\{ -\frac{a}{32} a_0^3 \sin\left(\frac{3}{2}\gamma t + 3\psi_0\right) + \frac{c}{16} a_0 \sin\left(\frac{3}{2}\gamma t + \psi_0\right) \right. \\ \left. + \frac{2}{\pi} \sum_{k=1}^{\infty} \left[2h_0 - \frac{h_2 \gamma^2 a_0^2}{(2k+1)^2 - 4} \right] \frac{(-1)^k \cos(2k+1)\left(\frac{\gamma}{2}t + \psi_0\right)}{(2k+1)[(2k+1)^2 - 1]} \right\}.$$

References

1. Schmidt, G., W. Heinrich. Über nichtlineare Parameterresonanzen. Berlin, ZAMM, Nr. 3, 1972.
2. Schmidt, G. Parametererregte Schwingungen. Berlin, 1975.
3. Osinski, Z. Сравнение демпфирования колебаний разными видами трения. Труды V-ой международной конференции по нелинейным колебаниям, Киев, 1969.
4. Писаренко, Г. С. Колебания механических систем с учетом несовершенной упругости материала. Изд. Наукова думка, Киев, 1970.
5. Tondl, A. Stability for arbitrary initial conditions of a nonlinear system with periodically variable stiffness and dry friction. Proceedings of the Fourth conference on nonlinear oscillations. Prague, 1967.
6. Hagedorn, P. Zum Instabilitätsbereich erster Ordnung der Mathieugleichung mit quadratischer Dämpfung. ZAMM, 8, 1968.
7. Боголюбов, Н. Н., Ю. А. Митропольский. Асимптотические методы в теории нелинейных колебаний, М., 1963.
8. Булгаков, Б. В. Колебания, М., 1954.
9. Nguyen, van Dao. Fundamental methods of the theory of non-linear oscillations Hanoi, 1969.
10. Nguyen, van Dao. Parametric oscillations of mechanical systems with regard for the incomplete elasticity of material. Proceedings of Hanoi Polytechnical Institute, 7, 1975.

Received on 17. 2. 1976