

On the total stability of certain systems of differential equations of the third order

Z. Zapryanov, P. Kalitzova

In 1952, proceeding from the results obtained in [1], E. A. Barbašin studied the total stability of the zero solution of the differential equation of the third order

$$(1) \quad \ddot{x} + a\ddot{x} + \varphi(\dot{x}) + f(x) = 0,$$

where $f(x)$ is a continuously differentiable function, $\varphi(\dot{x})$ — a continuous function, and a — a constant [2].

S. N. Šimanov [3] examined the total stability of the zero solution of another nonlinear differential equation of the third order

$$(2) \quad \ddot{x} + f(x, \dot{x})\ddot{x} + b\dot{x} + cx = 0,$$

where $f(x, \dot{x})$ is a continuous function which has a continuous partial derivative with respect to x .

A. I. Ogurcov [4] obtained for the equation

$$(3) \quad \ddot{x} + \psi(x, \dot{x})\ddot{x} + \varphi(\dot{x}) + f(x) = 0$$

sufficient conditions for asymptotic stability with arbitrary initial disturbances of the zero solution, from which follow, in particular, the results obtained for equations (1) and (2).

J. O. Ezeito [5], making use of Liapunoff's function as constructed by A. I. Ogurcov, specified the previous results by weakening the restrictions imposed on function ψ , which are necessary in order to demonstrate that ψ is infinitely great.

Nonlinear systems of differential equations of the third order with one nonlinearity have been studied by A. P. Tuzov [6] and V. A. Pliss [7], and with two nonlinearities by A. M. Balitinov and A. P. Effendiev [8—10].

I. V. Gaishoon [11—13] studied the stability of the zero solution of the third-order system with nonlinearity, depending on two variables

$$\frac{dx}{dt} = z,$$

$$(4) \quad \frac{dy}{dt} = x + by,$$

$$\frac{dz}{dt} = f(x, y) + az,$$

where a and b are constants, while $f(x, y)$ is a continuously differentiable function for all real values of x and y .

In this paper we examine systems from which, as a particular case (for some of them with an appropriate substitution of the variables), the systems studied in [8—11] as well as the equations studied in [2], [13] are obtained.

The proof of the theorems which we shall examine here is based on the following variant of theorem 4 in [1]:

The zero solution of a system of ordinary differential equations is totally stable, if there is a positively definite function v , so that $\frac{dv}{dt}$, taking the system into consideration, has a negative sign in the entire phase space; the set M , where $\frac{dv}{dt} = 0$, does not contain entire trajectories except the beginning $O(0, 0, 0)$; and all positive half-trajectories of the system are limited (stable according to Lagrange) ([14], ch. I of theorem 3.5).

1. Let us examine the system

$$\frac{dx}{dt} = z,$$

$$5) \quad \frac{dy}{dt} = x + by,$$

$$\frac{dz}{dt} = f(x) + \varphi(y) + \psi(z),$$

where $b = \text{const.} < 0$. Let us suppose that $f(x)$ and $\psi(z)$ are continuous, $\varphi(y)$ — continuously differentiable, and $f(x)$, $\varphi(y)$, $\psi(z)$ satisfy a certain theorem about the existence and uniqueness of the solutions of system (5). Let us further suppose that this system has as its equilibrium position only point $O(0, 0, 0)$ and $f(0) = \varphi(0) = \psi(0) = 0$.

When functions $f(x)$, $\varphi(y)$ and $\psi(z)$ are linear, i. e. $f(x) = px$, $\varphi(y) = qy$ and $\psi(z) = rz$, the conditions of Routh-Hurwitz for the total stability of the zero solution of system (5) look as follows:

$$(6) \quad r + b < 0, \quad rp + q > rb(r + b), \quad bp - q > 0.$$

As is known, these conditions are necessary and sufficient for the asymptotic stability of the zero solution of the corresponding linear system with arbitrary initial disturbances.

Let us introduce the following denominations:

$$\Phi(x) = - \int_0^x f(x) dx, \quad F(y) = \int_0^y \varphi(y) dy,$$

$$(7) \quad h(y) = a \int_0^y [\varphi(y) - bf(y)] dy,$$

$$\omega(x, y) = \Phi(x) - x\varphi(y) - bF(y).$$

Theorem 1. If there is a constant $a < 0$, so as to satisfy the conditions

1. $\frac{1}{x+by} \left[\frac{d\varphi(y)}{dy} x + b\varphi(y) + abf(y) - a^2z + af(x) + a\psi(z) \right] >$
 $> ab(a+b) \quad \text{at } x+by \neq 0;$
2. $-4bF(y)\Phi(x) - x^2\varphi^2(y) > 0 \quad \text{at } xy \neq 0,$
 $F(y) > 0 \quad \text{at } y \neq 0 \quad \text{and} \quad \Phi(x) > 0 \quad \text{at } x \neq 0;$
3. $\frac{\psi(z)}{z} \leq a \quad \text{at } z \neq 0;$
4. $h(y) > 0 \quad \text{at } y \neq 0;$
5. $\lim_{|x| \rightarrow \infty} \omega(x, y) = \infty \quad \text{at fixed } y,$

then the zero solution of system (5) is asymptotically stable with arbitrary initial disturbances.

Let us examine Liapunoff's function

$$v = - \int_0^x f(x) dx - ab \int_0^y f(y) dy + (a-b) \int_0^y \varphi(y) dy - x\varphi(y) \\ (8) \quad + \frac{ab}{2} (x+by)^2 + \frac{1}{2} [z - a(x+by)]^2.$$

In order to prove that $v(x, y, z)$ is a positively definite function of the variables x, y and z , we shall prove that $\omega(x, y)$ is a positively definite function of the variables x, y . Bearing in mind the denominations, we can write down

$$(9) \quad v(x, y, z) = \omega(x, y) + \frac{ab}{2} (x+by)^2 + \frac{1}{2} [z - a(x+by)]^2.$$

Obviously, $\omega(0, 0) = 0$, $\omega(0, y) > 0$ with $y \neq 0$ and $\omega(x, 0) > 0$ with $x \neq 0$. With $xy \neq 0$, according to condition 2 of the theorem, we obtain

$$\omega(x, y) = \Phi(x) \left[1 + \frac{x\varphi(y)}{2\Phi(x)} \right]^2 - \frac{4bF(y)\Phi(x) + x^2\varphi^2(y)}{4\Phi(x)} > 0.$$

Taking into consideration (9) and that $h(y) > 0$ with $y \neq 0$, we conclude that Liapunoff's function (8) is positively definite.

The derivative of this function, taking system (5) into consideration, can be written down as follows:

$$\frac{dv}{dt} = - \left[a - \frac{\psi(z)}{z} \right] z^2 - \left[\frac{d\varphi(y)}{dy} x + b\varphi(y) + abf(y) - a^2z \right. \\ \left. + af(x) + a\psi(z) \right] (x+by) + ab(a+b)(x+by)^2.$$

Conditions 2 and 4 of the theorem ensure a negative sign of function $\frac{dv}{dt}$.

We shall now prove that the set of points M , in which $\frac{dv}{dt} = 0$, does not contain entire trajectories of the system (5). And indeed, if we suppose that for

the trajectory $\{x(t), y(t), z(t)\}$ we have $\frac{dv}{dt}=0$, then according to conditions 1 and 3 of the theorem $x(t)+by(t)\equiv 0$. From the second equation of system (5) it follows that $\frac{dy(t)}{dt}\equiv 0$ or $y(t)=\text{const}$, and from the last one and $x(t)+by(t)\equiv 0$ it follows that $x(t)=\text{const}$. From the first equation of the system and from $x(t)=\text{const}$ it follows that $z(t)\equiv 0$. But since the system was supposed to have only one point of equilibrium (point $O(0, 0, 0)$), it follows that the examined movement can be only a zero movement.

We shall prove that all positive half-trajectories of system (5) are limited (stable according to Lagrange).

Let $P(x_0, y_0, z_0)$ be an arbitrary point of the phase space. Let us examine region D determined by the inequality $v\leq l$, where l is a positive constant so great that point P is internal for D . Let $T(P, t)$ be a trajectory of the system, starting at moment $t=0$ of point P . We proved that $v(x, y, z)$ is positively definite and $\frac{dv}{dt}\leq 0$. Consequently

$$v[T(P, t)]\leq v(P)< l,$$

which proves that $T(P, t)\subset D$ at $t\geq 0$.

All that has to be proved now is that region D is limited. From the inequality $v(x, y, z)\leq l$, owing to $a<0$, $b<0$, $\omega(x, y)\geq 0$ and $h(y)>0$ it follows that in region D we have

$$(10) \quad \frac{ab}{2}(x+by)^2\leq l,$$

$$(11) \quad \frac{1}{2}[z-a(x+by)]^2\leq l,$$

$$(12) \quad \omega(x, y)\leq l.$$

From (10) and (11) it follows that z is limited, and from $\lim_{|x|\rightarrow\infty} \omega(x, y)=\infty$, (12) and (10), it follows that x and y are limited too.

It should be noted that if in the linear case we select $a=r$, the conditions of theorem 1 coincide with those of Routh-Hurwitz.

2. Let us examine the system

$$(13) \quad \begin{aligned} \frac{dx}{dt} &= z, \\ \frac{dy}{dt} &= x+by, \\ \frac{dz}{dt} &= f(x, y, z), \end{aligned}$$

where $b=\text{const}<0$. Let us suppose that $f(x, y, z)$ is a continuous function in relation to the variables x, y and z , having a continuous particular derivative along y and satisfying the conditions of certain theorems about the existence and uniqueness of the solutions of the system (13). Let us further suppose that $O(0, 0, 0)$ is the only position of equilibrium and hence $f(0, 0, 0)=0$.

Obviously, in the linear case where $f(x, y, z)=px+qy+rz$, the conditions of Routh-Hurwitz about total stability of the zero solution are again like (6).

Let us introduce the denominations

$$\begin{aligned}\Phi(x, y) &= \int_0^x [f(0, y, 0) - f(x, y, 0)] dx, \quad F(y) = \int_0^y f(0, y, 0) dy, \\ (14) \quad h(y) &= a \int_0^y [f(0, y, 0) - bf(y, 0, 0)] dy, \\ \omega(x, y) &= \Phi(x, y) - xf(0, y, 0) - bF(y).\end{aligned}$$

Theorem 2. If there exists a negative number a , so as to fulfil the conditions

1. $\frac{1}{v} \left[\int_0^u \frac{\partial f(x, y, 0)}{\partial y} dx + (b-a)f(0, y, 0) + abf(y, 0, 0) - a^2z + af(x, y, z) \right] > y = \frac{v-u}{b}$
 $> ab(a+b) \quad \text{at } v \neq 0;$
2. $z[f(x, y, z) - f(x, y, 0)] \leq az^2;$
3. $-4bF(y)\Phi(x, y) - x^2f^2(0, y, 0) > 0 \quad \text{at } xy \neq 0,$
 $F(y) > 0 \quad \text{at } y \neq 0, \quad \Phi(x, 0) > 0 \quad \text{at } x \neq 0;$
4. $h(y) > 0 \quad \text{at } y \neq 0;$
5. $\lim_{|y| \rightarrow \infty} \int_0^y f(0, y, 0) dy = \infty;$
6. The function $g(x, y) = f(x, y, 0) - f(0, y, 0)$ is limited along y with a fixed x , then the zero solution of the system (13) is a totally stable one.

Let us examine the function of Liapunoff

$$\begin{aligned}v(x, y, z) &= - \int_0^x f(x, v, 0) dx - b \int_0^y f(0, y, 0) dy + a \int_0^y [f(0, y, 0) - bf(y, 0, 0)] dy \\ (15) \quad &+ \frac{ab}{2} (x + by)^2 + \frac{1}{2} [z - a(x + by)]^2 = \omega(x, y) + h(y) \\ &+ \frac{ab}{2} (x + by)^2 + \frac{1}{2} (z - a(x + by))^2.\end{aligned}$$

From the differentiation of (15) along t for $\frac{dv}{dt}$, taking the system (13) into consideration, we obtain

$$\frac{dv}{dt} = - \left[a - \frac{f(x, y, z) - f(x, y, 0)}{z} \right] z^2 - \left[\int_0^x \frac{\partial f(x, y, 0)}{\partial y} dx + (b-a)f(0, y, 0) \right]$$

$$+abf(y, 0, 0) - a^2z + af(x, y, z) \Big] (x + by) + ab(a + b)(x + by)^2.$$

Conditions 1 and 2 of the theorem, taken at $u=x$ and $v=x+by$, ensure a negative sign of the derivative of the Liapunoff function (15). From (16) it follows, according to conditions 1 and 2, that at $x+by \neq 0$ we have $\frac{dv}{dt} < 0$.

Consequently, the set M , where $\frac{dv}{dt} = 0$, is determined by $x+by=0$. It is easily established that it does not contain entire trajectories of the system except point $O(0,0,0)$. And indeed, let us suppose the inverse and let $\{x(t), y(t), z(t)\}$ be an entire trajectory, differing from the zero point for which $x+by \equiv 0$, then it follows from the second equation of the system that $y(t) = \text{const.}$ and hence also $x(t) = \text{const.}$ According to the first equation of (13) we get $z(t) \equiv 0$. Since by supposition point $O(0,0,0)$ is the only equilibrium position of the system, the set M cannot contain any other entire trajectories.

Function $\omega(x, y)$ at $xy \neq 0$ can be written down as follows:

$$\omega(x, y) = -bF(y) \left| 1 + \frac{xf(0, y, 0)}{2bF(y)} \right|^2 + \frac{4bF(y)\Phi(x, y) + x^2f^2(0, y, 0)}{4bF(y)}$$

From this and from the conditions of the theorem it follows that $\omega(x, y) > 0$ at $xy \neq 0$.

Bearing in mind denominations (14) and condition 3 of the theorem, we obtain $\omega(0, 0) = 0$, $\omega(0, y) > 0$ at $y \neq 0$ and $\omega(x, 0) > 0$ at $x \neq 0$. Consequently, $\omega(x, y)$ is a positively definite function of the variables x, y and $v(x, y, z)$ a positively definite function of variables x, y and z .

According to condition 3 of the theorem at $xy \neq 0$ we can write down

$$0 \leq \frac{x^2f^2(0, y, 0)}{4(-b)F(y)} < \Phi(x, y).$$

Since function $g(x, y) = f(x, y, 0) - f(0, y, 0)$ is limited along y with a fixed x the expression $\frac{xf(0, y, 0)}{2[-bF(y)]^{\frac{1}{2}}}$ is also limited with a fixed x . Since $\omega(x, y)$ at $y \neq 0$ can also be written down in the form

$$\omega(x, y) = \left\{ [-bF(y)]^{\frac{1}{2}} - \frac{xf(0, y, 0)}{2[-bF(y)]^{\frac{1}{2}}} \right\}^2 + \left[\Phi(x, y) - \frac{x^2f^2(0, y, 0)}{4(-b)F(y)} \right],$$

then, because of condition 5 of the theorem and the above-obtained results, we can conclude that $\lim_{|y| \rightarrow \infty} \omega(x, y) = \infty$ at every fixed x . Since $h(y) \geq 0$,

$\frac{ab}{2}(x+by)^2 \geq 0$ and $\frac{1}{2}[z-a(x+by)]^2 \geq 0$, it follows that $\lim_{(y) \rightarrow \infty} v(x, y, z) = \infty$ at fixed x and z .

We shall now prove that all trajectories of system (13) are limited. Let $P(x_0, y_0, z_0)$ be an arbitrary point of the phase space. Let us examine region D determined by the inequality $v \leq l$, where l is so great a positive constant that point P is internal for region D . On the basis of considerations analogous to those made when proving theorem 1, we establish that if $T(P, t)$ is a trajectory of the system starting at the moment $t=0$ from point P , then $T(P, t) \subset D$ at $t \geq 0$.

Now we shall prove that region D is limited. From $\lim_{|y| \rightarrow \infty} v(x, y, z) = \infty$ at fixed x and z and the inequality $v \leq l$ we conclude that y is limited. The limitation of x and z follows from the inequalities

$$\frac{ab}{2}(x+by)^2 \leq l$$

$$\frac{1}{2}[z-a(x+by)]^2 \leq l.$$

Consequently, region D is limited.

It is easily ascertained that if in the linear case we select $a=r$, then the conditions of theorem 2 coincide with the necessary and sufficient conditions of Routh-Hurwitz for asymptotic stability in the case of arbitrary initial disturbances.

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