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STRESS AND STRAIN DEFINITION OF AN OPEN PROFILE THIN-WALLED BEAM AT CONSTRAINED TORSION BY BOUNDARY ELEMENT METHOD

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Abstract. Thin-walled beams with open profile at constrained torsion are investigated in this paper. A thin-walled beam loaded by an external bi-moment at constrained torsion is investigated in this paper. An analytical variant of the boundary element method (BEM) is presented, which is based on a new scheme of the integral ratios transformation of the initial parameters method in a system of linear algebraic equations. Only one dimensional integrals are used defining the one dimensional continuum.
Key words: thin-walled beam, constrained torsion, bi-moment, boundary element method (BEM), matrix equation, boundary parameters.

1. Introduction

Thin-walled rods with an open profile are widely used in practice. The theory of their investigation is developed by V. Z. Vlasov [8]. The terms *free* and *constrained* torsion are differentiated at the thin-walled rod. Free torsion is realized when all cross sections possess one and the same deplanation. This state can be observed very seldom. In most cases, deplanation varies according to a certain law and as a result, the longitudinal fibres are subjected to elongations or shortenings and in the cross sections of the thin-walled rod normal stresses appear, which create additional tangential stresses. In this case, we have a situation of *constrained* torsion.

The term bending-torsion bi-moment (B_ω) is defined for the additional normal stress definition as a result of deplanation, which is a self-balanced section value. Besides the internal moment – the bi-moment, we have an external loading bi-moment (analogously to an internal loading equivalent to bending

moment and loading with a concentrated moment). The term bending-torsion moment (M_ω) [2] is implemented for the calculation of the additional tangential stresses in a thin-walled rod. These stresses appear only at constrained torsion and are uniformly distributed along the profile thickness.

The boundary element method (BEM) is a principally new one for calculation and solution of the differential equations for the problems of mechanics. Only the object boundary is subjected to discretization. The necessary boundary parameters of the object are defined from the linear system of algebraic equations and the integral equations are used for the internal point values calculation [9]. References [4], [5], [6] and [7] are used as a general knowledge.

Basic advantages of the method are: reduction of the geometrical dimension of the problem by one, [10–11], easy investigation of infinite domains [11] and usage of fundamental solutions for integral equations [1], [12]. Method of Boundary Elements describes with enough precision the behavior of the model at discretization, as well as for two dimensional, so as for three dimensional linear elastic problems and for problems related with the theory of the potential and also for investigating dynamic phenomenon in continuous media [9].

The objective of the work presented in this paper is investigation on constrained torsion of a thin-walled beam by using the Boundary Element Method (BEM).

2. Analytical variant of Boundary Element Method (BEM)

The analytical variant of the Method of Boundary Elements is applicable in the area of civil engineering mechanics, strength of materials and theory of elasticity for calculation of rods and plates constructions for static and dynamic loading and for stability. An advantage appears the same approach to the problem algorithm for problems from statics, dynamics and stability. The solution of these types of problems differs only by the fundamental functions, and the matrix form of the equations allows compatibility of different problems. A characteristic feature of the variant is: "The suggested approach of BEM does not possess an equivalent in the scientific literature, it is based on very well known solutions of differential equations in the form of the Method of the Initial Parameters."[1].

The analytical variant compared with the classical BEM is constrained in the field of application. The developed variant of BEM according to the authors with very small variations can be adapted to solving problems from electronics, thermal technics, physics, hydrogasdynamics and other sciences, where the corresponding processes can be defined by means of differential equations.

Only one dimensional integrals are used in analytical variant of Boundary Element Method (BEM) [1], which define one dimensional continuum. In mechanics, such types of objects are the rod and the rod systems. There is a possibility for two and three dimensional objects to be reduced to one dimensional continuum – a generalized rod for slabs, shells and spatial constructions. The presented variant is based on a new transformation scheme of the integral relations of the method of the initial parameters in a system of linear algebraic equations. The essence of the presented method consists of an initial discretization of the linear system of modules of rods and slabs. We understand as a module to be a rod or a generalized rod to which are adduces the plates and the shells. In the rod being one dimensional there are only two boundary points $x = 0$ and $x = \ell$. The solution of the Koshi problem for an elastic rod is presented below:

$$\mathbf{Y}(x) = \mathbf{A}(x)\mathbf{X}(0) + \mathbf{B}(x), \tag{1}$$

where: $\mathbf{Y}(x)$ – vector-column of the parameters of the rod strain in point X;
$\mathbf{A}(x)$ – quadratic matrix of the fundamental functions;
$\mathbf{X}(0)$ – vector-column of the initial parameters;
$\mathbf{B}(x)$ – vector-column of the elements of the given load.

For a system of rods, if several rods are connected in one construction, a matrix equation from type (1) can be derived.

$$\mathbf{A} = \begin{pmatrix} A_1(x) & & & \\ & \cdots & & \\ & & A_i(x) & \\ & & & \cdots \\ & & & A_m(x) \end{pmatrix}, \ \mathbf{Y} = \begin{pmatrix} Y_1(x) \\ \cdots \\ Y_i(x) \\ \cdots \\ Y_m(x) \end{pmatrix}, \ \mathbf{X} = \begin{pmatrix} X_1(x) \\ \cdots \\ X_i(x) \\ \cdots \\ Y_m(x) \end{pmatrix} \tag{2}$$

Vectors $\mathbf{Y}$ and $\mathbf{X}$ will contain parameters for the stress and the strain state of all rods in the current and in the initial points and $\boldsymbol{m}$ is the rod system number. The matrix diagonal blocks $\mathbf{A}$ represent the fundamental functions describing the rod condition. The matrix equation (1) for boundary value of the variable $x = \ell$ for each rod is transformed in the form presented below:

$$\mathbf{A}^*(\ell)\mathbf{X}^*(0, \ell) = -\mathbf{B}(\ell), \tag{3}$$

where vectors $\mathbf{Y}$ and $\mathbf{X}$ contain rod parameters in the boundary points $x = \ell$ and $x = 0$; vector $\mathbf{B}$ consists of load elements of all rods at $x = \ell$; matrix $\mathbf{A}$ contains the boundary values of the fundamental functions at $x = \ell_i$.

The essence of the transformation is a transfer of the finite parameters of vector **Y** to the place of the zero parameters of vector **X**, at which vector **Y** becomes a zero one and is cancelled from investigation. Vector $\mathbf{X}^*$ contains the unknown initial and finite boundary parameters of all the system rods. Matrix A^* is zeroed in definite columns (A_0) and elements are inserted in it that compensate the parameter transfer. The compensating elements connected with the transfer of the finite parameters from **Y** in **X** are presented in an auxiliary matrix **C** for descriptiveness of the boundary element method algorithm.

Matrix $\mathbf{A}^*$ is derived by the following summation:

$$\mathbf{A}^* = \mathbf{A}_0 + \mathbf{C}. \tag{4}$$

Vector of the loading **B** in equation (3) consists elements on the basis of the theory of the generalized functions and the spline-functions [3].

3. Constrained torsion of thin-walled beam with Boundary Element Method

The solution of the Koshi problem for constrained torsion of thin wall rod in a matrix form is presented below:

$$\begin{aligned} GI_{yc}\theta(x) = {} & GI_{yc}\theta(0) + GI_{yc}\theta'(0).x + [-B_\omega(0)]\,(chkx - 1) \\ & + [-M_\omega(0)]\,\frac{shkx - kx}{k} + \int_0^x \frac{shk\,(x-\xi) - k\,(x-\xi)}{k} m\,(\xi)\,d\xi \end{aligned} \tag{5}$$

where: k – bending-torsion characteristics of the cross section

$$k^2 = GI_{yc}/EI_\omega \tag{6}$$

The equation appears to be another form of the well known solution of V. Z. Vlasov [1]. The other parameters of the constrained torsion are defined after differentiation. The equation possesses the following form in a matrix definition:

$$(7)\quad \begin{pmatrix} GJ_{yc}\theta_{(x)} \\ GJ_{yc}\theta'_{(x)} \\ B_\omega(x) \\ M_\omega(x) \end{pmatrix} = \begin{pmatrix} 1 & x & -A_{13} & -A_{14} \\ & 1 & -A_{23} & -A_{13} \\ & & A_{33} & A_{34} \\ & & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} GJ_{yc}\theta_{(0)} \\ GJ_{yc}\theta'_{(0)} \\ B_\omega(0) \\ M_\omega(0) \end{pmatrix} + \int_0^x \begin{pmatrix} A_{14}(x-\xi) \\ A_{13}(x-\xi) \\ -A_{34}(x-\xi) \\ -A_{33}(x-\xi) \end{pmatrix} m(\xi)d\xi$$

where: $A_{13} = chkx - 1$, $A_{14} = \dfrac{shkx - kx}{k}$, $A_{23} = k.shkx$, $A_{33} = chkx$, $A_{34} = \dfrac{shkx}{k}$, k – bending-torsion characteristics of the cross section area.

Numerical example 1.

A thon-walled beam with a cross section [No 18 , $k = 2.150$ is clamped at the one end and loaded by means of a concentrated torque at the other (Fig. 1). The task is to define the stress and deformation state of the beam.

It is given: $M = -400$ Nm, $J_\omega = 51.99 \times 10^{-10}$ m^6, $J_{yc} = 598.3 \times 10^{-10}$ m^4, $\omega = 38.42 \times 10^{-4}$ m^2, $E = 2.1 \times 10^{11}$ N/m^2, $G = 8.4 \times 10^{10}$ N/m^2, $L = 3.5$ m.

The beam is build up of one module. We numerate the nods. The arrow shows the beginning and the end of the module.

We form the vectors $\mathbf{X}^*$, $\mathbf{Y}$, $\mathbf{B}$.

$$\mathbf{X}^* = \left(\begin{array}{c|l} 1 & GJ_{yc}\theta^{0-1}_{(0)} = 0; GJ_{yc}\theta'^{0-1}_{(\ell)} \\ 2 & GJ_{yc}\theta'^{0-1}_{(0)} = 0; GJ_{yc}\theta^{0-1}_{(\ell)} \\ 3 & B^{0-1}_\omega(0) \\ 4 & M^{0-1}_\omega(0) \end{array}\right), \quad \mathbf{Y} = \left(\begin{array}{c|l} 1 & GJ_{yc}\theta^{0-1}_{(0)} = 0 \\ 2 & GJ_{yc}\theta'^{0-1}_{(\ell)} \\ 3 & B^{0-1}_\omega(\ell) = 0 \\ 4 & M^{0-1}_\omega(\ell) = -GJ_{yc}\theta'^{0-1}_{(\ell)} \end{array}\right),$$

$$\mathbf{B} = \left(\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 400 \end{array}\right)$$

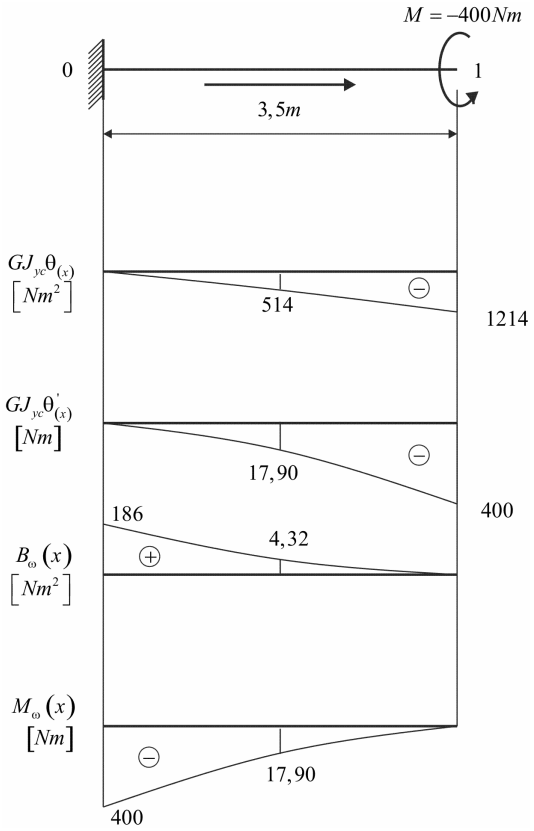

Fig. 1. Diagrams of the internal forces

We define the matrix equation (3).

$$\left(\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & -1 & -925.9069 & -427.6192 \\ 2 & -1 & & -1992.8485 & -925.9069 \\ 3 & & & 926.9069 & .431.1192 \\ 4 & 1 & & 1992.8485 & 926.9069 \end{array}\right) \left(\begin{array}{c} GJ_{yc}\theta'^{0-1}_{(\ell)} \\ GJ_{yc}\theta^{0-1}_{(\ell)} \\ B^{0-1}_{\omega}(0) \\ M^{0-1}_{\omega}(0) \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ -400 \end{array}\right)$$

The matrix equation solution using MATLAB gives the values of the beam boundary parameters.

$GJ_{yc}\theta'^{0-1}_{(\ell)} = -399.5$ Nm; $GJ_{yc}\theta^{0-1}_{(\ell)} = -1214$ Nm^2;

$$\left[\theta^{0-1}_{(\ell)} = \frac{-1214}{GJ_{yc}} = -0.2415 \text{ rad} = -13.84^{\circ}\right];$$

$B_\omega^{0-1}(0) = 186$ Nm^2; $M_\omega^{0-1}(0) = -400$ Nm.

The results achieved are compared with the results of the analytical solution in Table 1 and the diagrams are shown in Fig. 1, respectively.

Table 1

	$\theta_{(\ell)}^{0-1}$	$GJ_{yc}\theta_{(\ell)}'^{0-1}$ Nm	$B_\omega^{0-1}(0)$ Nm^2	$M_\omega^{0-1}(0)$ Nm
Analytical	$-13.84°$	-400	186.05	-400
BEM	$-13.84°$	-399.5	186	-400

In Eurocode 3 "Design of Steel Constructions" (DSC) for a random cross section the general torsion torque T appears to be a sum of two torques:

$$T = T_t + T_w \tag{8}$$

where: T_t is the moment created by the free torsion (torsion according to St. Venan)

T_w is the moment created by the constrained torsion. It causes shear stresses τ_w, and normal stresses σ_w appear after the existing bi-moment B_ω.

The effect caused by the free torsion ($T = T_w$) can be neglected at elements with open cross section.

In DSC there are no instructions for defining the internal forces, as well as for defining the stresses caused by the free and the constrained torsion [13].

The torque caused by constrained torsion is complicated to be defined. Constructive activities are recommended for its avoiding [14].

BEM possesses an effective routine for calculation the internal forces at constrained torsion.

Numerical example 2.

A thin-walled beam with cross section area I No 50 with $k = 0.71$ is subjected to a torsion moment $m = 1.2$ kNm and an external bio-moment $B_\omega = 0.8$ kNm^2 (Fig. 2). To be defined the stress and deformation state of the beam.

It is given: $J_{yc} = 72.20 \times 10^{-8}$ m^4, $E = 2.1 \times 10^{11}$ N/m^2, $G = 8.4 \times 10^{10}$ N/m^2.

We divide the beam into two modules and we numerate the nodes. The arrows point out the beginning and the end of each module.

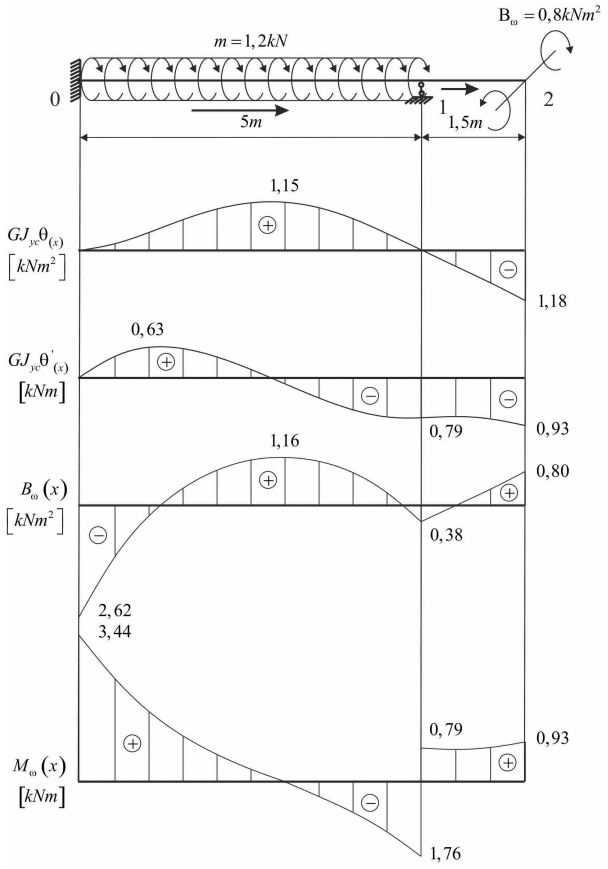

Fig. 2. Diagrams of the internal forces

We form vectors $\mathbf{X}^*$, $\mathbf{Y}$, $\mathbf{B}$.

$$\mathbf{X}^* = \begin{pmatrix} 1 & GJ_{yc}\theta^{0-1}_{(0)} = 0; M^{0-1}_{\omega}(\ell) \\ 2 & GJ_{yc}\theta'^{0-1}_{(0)} = 0; GJ_{yc}\theta^{1-2}_{(\ell)} \\ 3 & B^{0-1}_{\omega}(0) \\ 4 & M^{0-1}_{\omega}(0) \\ 5 & GJ_{yc}\theta^{1-2}_{(0)} = 0; GJ_{yc}\theta'^{1-2}_{(\ell)} \\ 6 & GJ_{yc}\theta'^{1-2}_{(0)} \\ 7 & B^{1-2}_{\omega}(0) \\ 8 & M^{1-2}_{\omega}(0) \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 1 & GJ_{yc}\theta^{0-1}_{(\ell)} = 0 \\ 2 & GJ_{yc}\theta'^{0-1}_{(\ell)} = GJ_{yc}\theta'^{1-2}_{(0)} \\ 3 & B^{0-1}_{\omega}(\ell) = B^{1-2}_{\omega}(0) \\ 4 & M^{0-1}_{\omega}(\ell) \\ 5 & GJ_{yc}\theta^{1-2}_{(\ell)} \\ 6 & GJ_{yc}\theta'^{1-2}_{(\ell)} \\ 7 & B^{1-2}_{\omega}(\ell) \\ 8 & M^{1-2}_{\omega}(\ell) = -GJ_{yc}\theta'^{1-2}_{(\ell)} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 24.0899 \\ 2 & 23.3954 \\ 3 & -39.0899 \\ 4 & -29.3954 \\ 5 & \\ 6 & \\ 7 & -0.8 \\ 8 & \end{pmatrix}$$

We derive the matrix equation (3).

$$\left(\begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & & & -A_{13} & -A_{14} & & & & \\ 2 & & & -A_{23} & -A_{13} & & -1 & & \\ 3 & & & A_{33} & A_{34} & & & -1 & \\ 4 & -1 & & A_{23} & A_{33} & & & & \\ 5 & & -1 & & & & 1.5 & -A_{13} & -A_{14} \\ 6 & & & & & -1 & 1 & -A_{23} & -A_{13} \\ 7 & & & & & & & A_{33} & A_{34} \\ 8 & & & & & 1 & & A_{23} & A_{33} \end{array}\right) \begin{pmatrix} \\ M_{\omega}^{0-1}(\ell) \\ GJ_{yc}\theta_{(\ell)}^{1-2} \\ B_{\omega}^{0-1}(0) \\ M_{\omega}^{0-1}(0) \\ GJ_{yc}\theta_{(\ell)}'^{1-2} \\ GJ_{yc}\theta_{(0)}'^{1-2} \\ B_{\omega}^{1-2}(0) \\ M_{\omega}^{1-2}(0) \end{pmatrix}$$

$$= \begin{pmatrix} -24.0899 \\ -23.3954 \\ 39.0899 \\ 29.3954 \\ \\ \\ 0.8 \\ \end{pmatrix}.$$

The matrix equation solution using MATLAB calculates the values of the beam boundary parameters.

$$M_{\omega}^{0-1}(\ell) = -1.7611 \text{ kNm}; \qquad GJ_{yc}\theta_{(\ell)}^{1-2} = -1.1833 \text{ kNm}^2;$$

$$B_{\omega}^{0-1}(0) = -2.6277 \text{ kNm}^2; \qquad M_{\omega}^{0-1}(0) = 3.4489 \text{ kNm};$$

$$GJ_{yc}\theta'^{1-2}_{(\ell)} = -0.9342 \text{ kNm}; \qquad GJ_{yc}\theta'^{1-2}_{(0)} = -0.79 \text{ kNm};$$

$$B_{\omega}^{1-2}(0) = -0.3833 \text{ kNm}^2; \qquad M_{\omega}^{1-2}(0) = 0.79 \text{ kNm}.$$

We use equations (5) for defining the stress and the strain condition of the beam substituting in them the calculated boundary parameters.
rod 0-1

$$\begin{pmatrix} GJ_{yc}\theta_{(x)}^{0-1} \\ GJ_{yc}\theta'^{0-1}_{(x)} \\ B_{\omega}^{0-1}(x) \\ M_{\omega}^{0-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & -A_{13} & -A_{14} \\ & 1 & -A_{23} & -A_{13} \\ & & A_{33} & A_{34} \\ & & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} 0.0000 \\ 0.0000 \\ -2.6277 \\ 3.4489 \end{pmatrix}$$

$$+1.2 \begin{pmatrix} \left(2\,\mathrm{ch}\,kx - 2 - k^2x^2\right)/2k^2 \\ \left(\mathrm{sh}\,kx - kx\right)/k \\ -\left(\mathrm{ch}\,kx - 1\right)/k^2 \\ -\,\mathrm{sh}\,kx/k \end{pmatrix}$$

rod 1-2

$$\begin{pmatrix} GJ_{yc}\theta_{(x)}^{1-2} \\ GJ_{yc}\theta'^{1-2}_{(x)} \\ B_{\omega}^{1-2}(x) \\ M_{\omega}^{1-2}(x) \end{pmatrix} = \begin{pmatrix} 1 & x & -A_{13} & -A_{14} \\ & 1 & -A_{23} & -A_{13} \\ & & A_{33} & A_{34} \\ & & A_{23} & A_{33} \end{pmatrix} \begin{pmatrix} 0.0000 \\ -0.7900 \\ -0.3833 \\ 0.7900 \end{pmatrix}$$

$$+0.8 \begin{pmatrix} \mathrm{ch}\,k\,(x-1.5)_+ - H\,(x-1.5) \\ k.\,\mathrm{sh}\,k\,(x-1.5)_+ \\ -\,\mathrm{ch}\,k\,(x-1.5)_+ \\ -k.\,\mathrm{sh}\,k\,(x-1.5)_+ \end{pmatrix}$$

Table 2. Results of the calculations

Coordinates $x(m)$	Parameters			
	$GJ_{yc}\theta_{(x)}$ kNm^2	$GJ_{yc}\theta'_{(x)}$ kNm	$B_\omega(x)$ kNm^2	$M_\omega(x)$ kNm
0.00	0.00	0.00	−2.6277	3.4489
0.50	0.1325	0.4694	−1.1857	2.3795
1.00	0.4191	0.6357	−0.1979	1.6132
1.50	0.7341	0.5966	0.4616	1.0523
2.00	0.9934	0.4235	0.8766	0.6254
2.50	1.1443	0.1708	1.1002	0.2782
3.00	1.1582	−0.1174	1.1606	−0.0336
3.50	1.0274	−0.4014	1.0658	−0.3496
4.00	0.7640	−0.6408	0.8035	−0.7102
4.50	0.4013	−0.7897	0.3403	−1.1612
5.00	0.00	−0.79	−0.3826	−1.7602
0.00	0.00	−0.79	−0.3833	0.79
0.50	−0.3789	−0.7417	−0.0043	0.7417
1.00	−0.7573	−0.7878	0.3740	0.7878
1.50	−1.1834	−0.9342	0.80	0.9342

4. Conclusion

In this paper is shown one application of the analytical variant of the Boundary Element Method, presented in [1], for investigation of thin-walled beams with an open profile at constrained torsion. The numerical results confirm a very good precision of the solution. The method gives a new engineering approach to solving problems in the deformable solid mechanics.

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