

GENERAL MECHANICS

FREE VIBRATION OF CRACKED FUNCTIONALLY GRADED BEAMS WITH PIEZOELECTRIC PATCHES

RONG ZHANG

*Department of Engineering Mechanics, Shijiazhuang Tiedao University,
Shijiazhuang, 050043, P. R. China,
e-mail: rzhangv@gmail.com*

[Received 28 February 2011. Accepted 14 November 2011]

ABSTRACT. This paper presents a theoretical investigation on free vibrations of cracked functionally graded beams with piezoelectric patches by using Euler–Bernoulli beam theory and the rotational spring model. The electrical potential along the longitudinal direction is also obtained in terms of the Maxwell equation. Different graded properties and support types are discussed, and the location of the crack is found to be important to the natural frequencies. The mode shape of the electrical potential is found to be proportional to the wave number squared, and it is independent of the location and depth of the crack. The results could serve in the approximation methods for dynamic analyses of piezoelectric composite structures with cracks.

KEY WORDS: piezoelectric patch; functionally graded beam; edge crack; free vibration; mode shape of electrical potential.

1. Introduction

Dynamic characteristics of functionally graded material (FGM) beams have attracted considerable research efforts both in academic and industrial fields during recent years. Lv and Chen [1] studied free vibration of orthotropic FGM beams employing a hybrid state-space/differential quadrature method. Agarwal *et al* [2] used the existing statically exact beam finite element to study the geometric non-linear effects on static and dynamic responses in isotropic, composite and FGM beams. For thin-walled beams made of FGMs, the implications on their vibration and instability of conservative and gyroscopic forces considered in conjunction with a temperature field that yields the degradation of elastic properties were investigated [3]. Kapuria *et al* [4] proposed a third order zigzag theory based model which is validated through experiments for static and free vibration response. Chebyshev polynomials were used as basis functions while Chebyshev–Gauss–Lobatto points were used in the work of

Sridhar *et al* [5], as grid points, and free vibration analysis was carried out on FGM beams. Sina *et al* [6] developed a new beam theory different from the traditional first-order shear deformation beam theory and used it to analyze free vibration. Aydogdu and Taskin [7] used different higher order shear deformation theories and classical beam theories to investigate the free vibration of simply supported FGM beams. Ke *et al* [8] studied FGM beam vibration based on von Kármán geometric nonlinearity, since nonlinearity is sometimes encountered.

It is known, that an edge crack on a beam introduces a local flexibility, reduces the stiffness and may change the dynamic behaviour, and so the effect on the dynamic behaviour is of interest in structural safety assessment. For FGM beams, some investigations have been conducted based on the rotational spring model. Yang and Chen [9] discussed the influence of open edge cracks on the free vibration and elastic stability characteristics of an Euler-Bernoulli FGM beam, and later Yang *et al* [10] studied the forced vibration case. Kitipornchai *et al* [11] conducted the nonlinear vibration investigation by employing the Ritz method based on the same model, and Ke *et al* [12] carried out the Timoshenko beam analysis.

Beams with integrated piezoelectric layers have been widely used in smart structures for their special electromechanical properties. Piezoelectric patches could just be bonded to a portion of the beam surfaces [13–15] to save materials and electric energy (or for some other reasons). To the best of author’s knowledge, no previous work has been done on the free vibration of cracked FGM beams with piezoelectric patches.

The objective of this paper is to study the mechanical and electrical characteristics on free vibration of cracked FGM beams integrated with piezoelectric patches to gain an insight into the effects of material property distribution, boundary conditions and the location and depth of the crack. The methodology which combines the classical Euler-Bernoulli beam theory and rotational spring model in the present investigation can be used to analyze multi-crack problems also.

2. The rotational spring model

The layout of an FGM beam (length l and thickness h_b) is illustrated in Fig. 1, and an edge crack (depth a) is located at a distance of l_2 from the left end. A pair of collocated piezoelectric patches (length l_1 and thickness h_p) are mounted from the left end. The beam has a relatively wide rectangular cross section which means the plane strain simplification could be adopted in the following analysis. The Young’s modulus $E(z)$ and mass density $\rho(z)$ follow

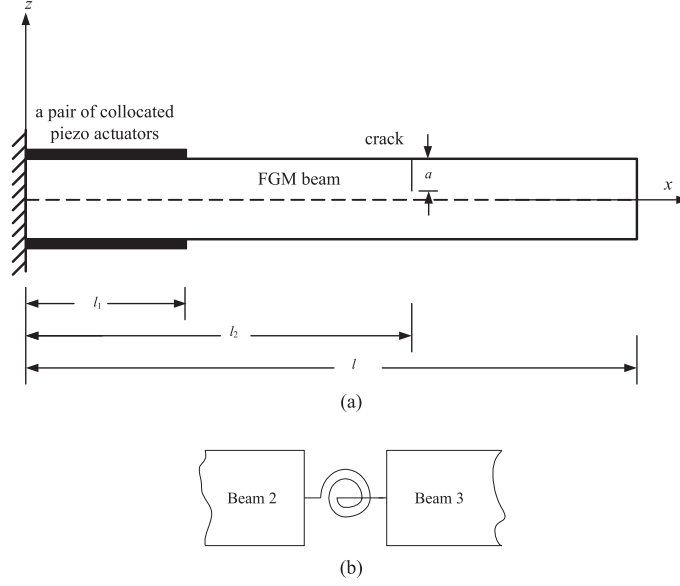


Fig. 1. (a) Geometry of the analytical model; (b) the rotational spring model

exponential variations along the transverse direction z :

$$(1) \quad E(z) = E_0 e^{\beta z}, \quad \rho(z) = \rho_0 e^{\beta z},$$

where E_0 and ρ_0 are the values of Young's modulus and mass density at the midplane ($z = 0$); β the constant characterizing the gradual variation of material properties in the transverse direction (when $\beta = 0$ the beam is homogeneous); Poisson's ratio μ is taken to be constant because it is found that the stress intensity factors (SIFs) are scarcely related to it [16].

It is assumed, that the crack is perpendicular to the beam surface and always remains open. The crack section can be treated as an elastic rotational spring which has no mass and no length based on the well-accepted rotational spring model, and the beam can be treated as three sub-beams connected by a spring and an intact intersection. The bending stiffness of the cracked section k_T is related to the compliance G by:

$$(2) \quad k_T = 1/G.$$

Kitipornchai *et al* [11] derived the expression of G on the basis of Erdogan and Wu's study [16], in terms of the crack depth ratio: (3) $\varsigma = a/h$ ($\varsigma \leq 0.7$),

$$(3) \quad G = \int_0^\varsigma \frac{72\pi(1 - \mu^2)\varsigma F(\varsigma)^2}{E(\varsigma h)h^2} d\varsigma,$$

in which:

$$(4) \quad F(\varsigma) = 1.910 - 2.752\varsigma - 4.742\varsigma^2 + 146.776\varsigma^3 - 770.750\varsigma^4 \\ + 1947.830\varsigma^5 - 2409.170\varsigma^6 + 1177.980\varsigma^7, \quad \text{when } E_2/E_1 = 0.2;$$

$$(5) \quad F(\varsigma) = 1.150 - 1.662\varsigma - 21.667\varsigma^2 - 192.451\varsigma^3 + 909.375\varsigma^4 \\ - 2124.310\varsigma^5 + 2395.830\varsigma^6 - 1031.750\varsigma^7, \quad \text{when } E_2/E_1 = 1.0;$$

$$(6) \quad F(\varsigma) = 0.650 - 0.859\varsigma - 12.511\varsigma^2 - 72.627\varsigma^3 + 267.910\varsigma^4 \\ - 535.236\varsigma^5 + 545.139\varsigma^6 - 211.706\varsigma^7, \quad \text{when } E_2/E_1 = 5.0.$$

The bending stiffness of the cross section can be determined from Eqs (2) and (3).

3. Free vibration analysis

The displacement field based on the Kirchhoff-Love hypothesis is assumed as:

$$(7) \quad w = \bar{w}(x)e^{i\omega t},$$

$$(8) \quad u = -z \frac{\partial \bar{w}(x)}{\partial x} e^{i\omega t},$$

where w and u are the displacements in the transverse z -direction and longitudinal x -direction, respectively, and ω is the natural frequency.

The poling direction of the piezoelectric material is assumed to be in the transverse direction. A differential strain is induced and it results in bending of the beam when voltages of equal magnitude, but opposite sign, are applied to the upper and lower piezoelectric patches on the beam. The strain and corresponding stresses could be deduced accordingly, as:

$$(9) \quad \bar{\varepsilon}_x = -z \frac{\partial^2 \bar{w}(x)}{\partial x^2},$$

$$(10) \quad \bar{\sigma}_x^b = -\frac{E(z)}{1 - \mu^2} z \frac{\partial^2 \bar{w}(x)}{\partial x^2},$$

$$(11) \quad \bar{\sigma}_x^p = -\tilde{C}_{11}z \frac{\partial^2 \bar{w}(x)}{\partial x^2} - \tilde{e}_{31} \bar{E}_z,$$

where the superscripts b and p represent the FGM beam and piezoelectric material respectively, the overbar denotes the spatial part of dynamic variable, the tilde denotes the transformed reduced form (Appendix A).

The electrodes at the two surfaces of the piezoelectric layer are short-circuited in the design of ultrasonic motors and resonators. The electric potential to satisfy the Maxwell equation, is assumed to follow a combined sinusoidal and linear variation [17]:

$$(12) \quad \phi = -\sin \frac{2\pi(z - h_b/2 - h_p/2)}{h_p} \bar{\phi}(x) e^{i\omega t} + \frac{2\pi(z - h_b/2 - h_p/2)}{h_p} \phi_a e^{i\omega t},$$

where $\bar{\phi}(x)$ is the spatial variation of the electric potential, and ϕ_a is the external electric voltage which is applied to the electrodes of the patches. In the present study, only the free vibration analysis is considered and thus, the second term vanishes.

Consequently, the amplitudes of the electric field $\bar{E}$ and electric displacement $\bar{D}$ are written as:

$$(13) \quad \bar{E}_x = \sin \frac{2\pi(z - h_b/2 - h_p/2)}{h_p} \frac{\partial \bar{\phi}(x)}{\partial x},$$

$$(14) \quad \bar{E}_z = \frac{2\pi}{h_p} \cos \frac{2\pi(z - h_b/2 - h_p/2)}{h_p} \bar{\phi}(x),$$

$$(15) \quad \bar{D}_x = \Xi_{11} \bar{E}_x,$$

$$(16) \quad \bar{D}_z = \Xi_{33} \bar{E}_z + e_{31} \bar{\epsilon}_x.$$

The moments in piezoelectric coupled zone (denoted by subscript 1) and uncoupled zone (denoted by subscript 2) by virtue of Eqs (10) and (11) are expressed as:

$$(17) \quad M_1 = -\int_{-\frac{h_b}{2}}^{\frac{h_b}{2}} z \bar{\sigma}_x^b dz - 2 \int_{\frac{h_b}{2}}^{\frac{h_b}{2} + h_p} z \bar{\sigma}_x^p dz = A_1 \frac{\partial^2 \bar{w}(x)}{\partial x^2} + B_1 \bar{\phi}(x),$$

$$(18) \quad M_2 = -\int_{-\frac{h_b}{2}}^{\frac{h_b}{2}} z \bar{\sigma}_x^b dz = A_2 \frac{\partial^2 \bar{w}(x)}{\partial x^2},$$

where

$$A_1 = \frac{1}{12} \left[(h_b + 2h_p)^3 - h_b^3 \right] \tilde{C}_{11} + \frac{E_0 (\beta^2 h_b^2 + 8) \sinh\left(\frac{\beta h_b}{2}\right) - 4\beta h_b \cosh\left(\frac{\beta h_b}{2}\right)}{2\beta^3 (1 - \mu^2)},$$

$$B_1 = \frac{4\pi \tilde{e}_{31}}{h_p} \int_{\frac{h_b}{2}}^{\frac{h_b}{2} + h_p} z \cos \frac{2\pi (z - h_b/2 - h_p/2)}{h_p} dz = 0,$$

$$A_2 = \frac{E_0 (\beta^2 h_b^2 + 8) \sinh\left(\frac{\beta h_b}{2}\right) - 4\beta h_b \cosh\left(\frac{\beta h_b}{2}\right)}{2\beta^3 (1 - \mu^2)}.$$

Substituting Eqs (17) and (18) into the partial differential equation of motion for beam flexure $\frac{\partial^2 M}{\partial x^2} + \int_{z \in \Omega} \rho(z) dz \frac{\partial^2 w}{\partial t^2} = 0$, yields:

$$(19) \quad A_1 \frac{\partial^4 \bar{w}(x)}{\partial x^4} + B_1 \frac{\partial^2 \bar{\phi}(x)}{\partial x^2} - C_1 \bar{w}(x) = 0,$$

$$(20) \quad A_2 \frac{\partial^4 \bar{w}(x)}{\partial x^4} - C_2 \bar{w}(x) = 0,$$

where $C_1 = \omega^2 \left(\int_{-\frac{h_b}{2}}^{\frac{h_b}{2}} \rho(z) dz + 2\rho_p h_p \right)$ and $C_2 = \omega^2 \left(\int_{-\frac{h_b}{2}}^{\frac{h_b}{2}} \rho(z) dz \right)$, Ω stands for the domain of the cross section.

The deflection solutions for the three sub-beams are:

$$(21) \quad \bar{w}_1(x) = c_{11} \sinh(\lambda_1 x) + c_{12} \cosh(\lambda_1 x) + c_{13} \sin(\lambda_1 x) + c_{14} \cos(\lambda_1 x),$$

$$(22) \quad \bar{w}_2(x) = c_{21} \sinh(\lambda_2 x) + c_{22} \cosh(\lambda_2 x) + c_{23} \sin(\lambda_2 x) + c_{24} \cos(\lambda_2 x),$$

$$(23) \quad \bar{w}_3(x) = c_{31} \sinh(\lambda_2 x) + c_{32} \cosh(\lambda_2 x) + c_{33} \sin(\lambda_2 x) + c_{34} \cos(\lambda_2 x),$$

in which $\lambda_1^4 = C_1/A_1$, $\lambda_2^4 = C_2/A_2$, the unknowns c_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3, 4$) are to be determined from the boundary conditions and compatibility conditions at the intersections of sub-beams.

The cracked FGM beams are considered with three kinds of end support, that is, clamped-free (C-F), hinged-hinged (H-H) and clamped-clamped (C-C). The associated boundary conditions require that:

$$(24) \quad \bar{w}_1(0) = 0, \quad \bar{w}_1'(0) = 0, \quad \bar{w}_3''(l) = 0, \quad \bar{w}_3'''(l) = 0 \quad \text{for C-F},$$

$$(25) \quad w_1(0) = 0, \quad w_1''(0) = 0, \quad w_3(l) = 0, \quad w_3''(l) = 0 \quad \text{for H-H},$$

$$(26) \quad w_1(0) = 0, \quad w_1'(0) = 0, \quad w_3(l) = 0, \quad w_3'(l) = 0 \quad \text{for C-F}.$$

At the intersection of piezoelectric coupled region and uncoupled region ($x = l_1$), due to the relatively thin patches, the compatibility condition enforces continuities of the transverse displacement, slope, moment and shear:

$$(27) \quad w_1(l_1) = w_2(l_1), \quad w_1'(l_1) = w_2'(l_1), \\ w_1''(l_1) = w_2''(l_1), \quad w_1'''(l_1) = w_2'''(l_1).$$

At the cracked section ($x = l_2$), the compatibility condition is the same to the preceding one except that a discontinuity in the slope exists, and so:

$$(28) \quad w_2(l_2) = w_3(l_2), \quad k_T w_2'(l_2) - A_2 w_2''(l_2) = k_T w_3'(l_2), \\ w_2''(l_2) = w_3''(l_2), \quad w_2'''(l_2) = w_3'''(l_2).$$

Substituting Eqs (21)–(23) into Eqs (24)–(28) yields a matrix equation which consists of 12 homogeneous algebraic equations:

$$(29) \quad [\mathbf{H}(\omega)] \{\mathbf{c}\} = \{\mathbf{0}\},$$

where $[\mathbf{H}]$ is a matrix nonlinearly dependent on the natural frequency ω , $\{\mathbf{c}\}$ is a vector composed of 12 unknowns in Eqs (21)–(23). To obtain a non-trivial solution, the determinant of this set of equations is set to zero:

$$(30) \quad \det [\mathbf{H}(\omega)] = |\mathbf{H}(\omega)| = 0.$$

The determinants $|\mathbf{H}_{C-F}|$, $|\mathbf{H}_{H-H}|$ and $|\mathbf{H}_{C-C}|$ for three types of end support are given in Appendix B. The associated mode shapes of displacement are easily obtained when the roots (the natural frequencies) of these transcendental equations are found.

On the other hand, note that all of the electrical variables primarily must satisfy the Maxwell equation which requires that the divergence of the electric flux density vanishes at any point within the media. This condition can be satisfied by enforcing the integration of the electric flux divergence multiplied by z across the thickness of the piezoelectric layers to be zero [17]:

$$(31) \quad \int_{\frac{h_b}{2}}^{\frac{h_b}{2} + h_p} z \nabla \bar{D} dz = 0.$$

Now, by substituting Eqs (15) and (16) we arrive at:

$$(32) \quad P_1 \frac{\partial^2 \bar{w}(x)}{\partial x^2} + Q_1 \frac{\partial^2 \bar{\phi}(x)}{\partial x^2} + R_1 \bar{\phi}(x) = 0,$$

where: $P_1 = -\tilde{e}_{31} h_p (h_b + h_p)$, $Q_1 = h_b^2 \bar{\Xi}_{11} / \pi$, $R_1 = -4 \bar{\Xi}_{33} \pi$.

Numerically solving Eq. (32) with the boundary condition $\frac{\partial \bar{\phi}(0)}{\partial x} = \frac{\partial \bar{\phi}(l_1)}{\partial x} = 0$, the mode shape of electric potential could be found also. The relation between the electrical potential and the displacement in a piezoelectric coupled sandwich beam can be expressed as [17]:

$$(33) \quad \Phi = \frac{P_1 k^2 W}{R_1 - Q_1 k^2},$$

where k is the wave number.

4. Numerical example

To illustrate the implications resulting from the proposed mechanics model and solution, unless otherwise stated, the parameters are assumed as $E_1 = 70$ GPa, $\mu = 0.33$, $\rho_1 = 2780$ kg/m³, $E_2/E_1 = 5$, $a = 0.009$ m, $l_2 = 0.6$ m, $l = 0.9$ m, $h_b = 0.03$ m, E_1 and E_2 denote Young's modulus at the top and bottom surfaces of the beam respectively, $c_{11} = 132 \times 10^9$ N · m⁻², $c_{33} = 115 \times 10^9$ N · m⁻², $c_{13} = 73 \times 10^9$ N · m⁻², $\rho_p = 7800$ kg/m³, $h_p = 0.003$ m, $l_1 = 0.3$ m, $e_{31} = -4.1$ C · m⁻², $e_{33} = 14.1$ C · m⁻², $\Xi_{11} = 7.124 \times 10^{-9}$ F · m⁻¹, $\Xi_{33} = 5.841 \times 10^{-9}$ F · m⁻¹.

Table 1 gives the fundamental frequencies for three types of boundary conditions. It is obvious, that $E_2/E_1 = 1.0$ means the beam is homogeneous, and when the value of E_2/E_1 increases the beam is strengthened (the compliance reduces). The homogeneous beams have lower frequencies than the FGM beams do whether they are strengthened or weakened, and it is because greater E_2/E_1 not only could lead to a greater beam stiffness but also could lead to a bigger bending stiffness loss at the crack section. It is clear that if the edge crack initiates from the other side the frequency change will be different. Therefore, the frequency change associated with the gradient is rather complicated. In other words, the natural frequency is more sensitive to a crack when the beam has lower mass density. The same conclusions can be drawn from the results of the second and third modes which are listed in Tables 2 and 3.

It is shown in Fig. 2, that the crack location influences the natural frequencies differently not only in modes, but also in support types. The

Table 1. Fundamental frequency (Hz)

E_2/E_1	C-F	H-H	C-C
0.2	207.159	563.964	1312.722
1.0	199.568	554.289	1271.628
5	207.573	576.451	1318.995

Table 2. Second frequency (Hz)

E_2/E_1	C-F	H-H	C-C
0.2	1256.71	2304.245	3566.177
1.0	1234.03	2253.535	3496.750
5	1286.14	2320.832	3613.190

Table 3. Third frequency (Hz)

E_2/E_1	C-F	H-H	C-C
0.2	3529.18	5385.498	7329.102
1.0	3477.66	5168.450	7038.342
5	3599.32	5298.385	7208.989

crack implies the beam is intact when it is located at the right tip, and the corresponding frequency is the maximum. It is worth to notice that when the crack is located at several certain positions, the frequencies for some mode are very close to the maximal and it means the crack almost does not contribute to the present mode, but the crack does contribute to the remaining modes. The number of the points increases when the restriction of the boundaries gets stiffer (C-C case is stiffer than H-H case while H-H case is stiffer than C-F case).

The numerical solutions for Eq. (32) are obtained by a finite difference technique with Richardson extrapolation. Figure 3 shows the first three mode shapes of the displacement and electrical potential. The distribution of electrical potential is totally different to that of displacement. Actually, the electrical potential is mainly dependent on the moment of the coupled beam. It is because the piezoelectric patches are relatively thin, and so in Eq. (33) Q_1 is little in comparison with R_1 . As a result, the electrical potential is approximately proportional to the wave number squared which is closely related to the curve or the moment.

A crack is often stochastic-distributed and may grow, and so the effects of the varying depth and location should be investigated. Figure 4 shows that

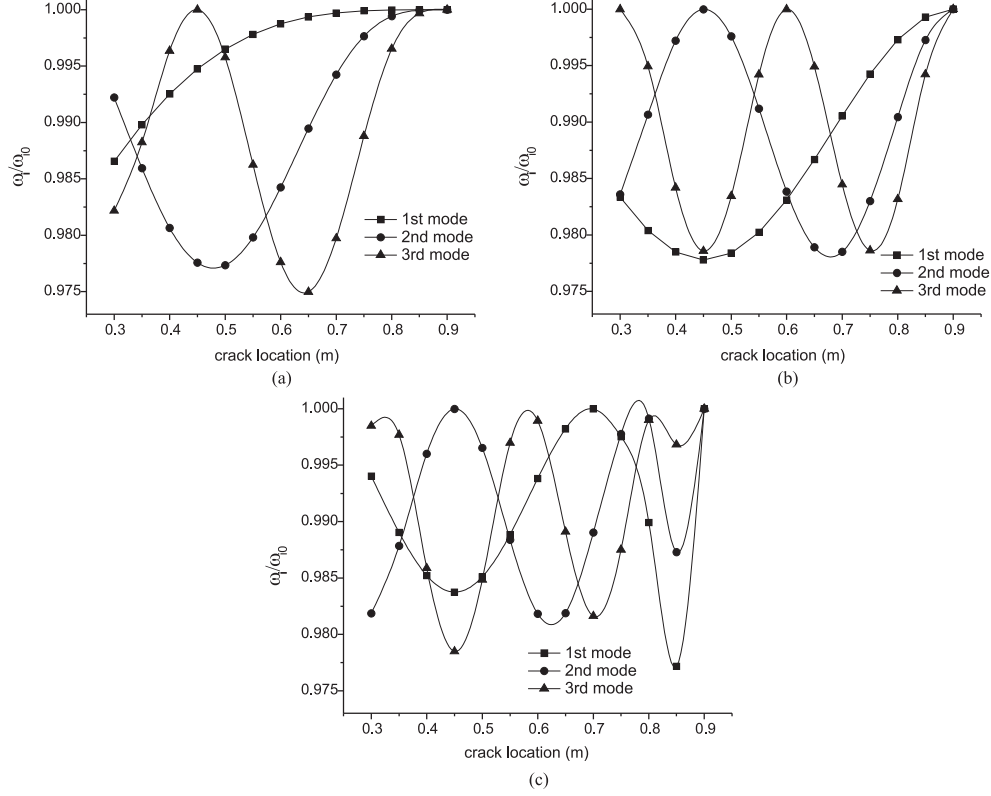


Fig. 2. Frequency ratio of the beams with a crack at varying locations (ω_i designates the i th frequency of a cracked beam, ω_{i0} designates the counterpart of an intact beam).
 (a) C-F; (b) H-H; (c) C-C

the mode shape of electrical potential almost remains which means that it is independent of the crack. This finding appears to be a new one.

5 Conclusion

Free vibration of cracked FGM beams with piezoelectric patches is theoretically investigated in this paper by using the analytical approach based on the rotational spring model. The numerical results indicate that the edge crack hardly reduces the natural frequency of some mode when it is located at several positions, but it does influence the counterparts of remaining modes. After the Maxwell equation is enforced, the distribution of the electrical potential of the patches in the longitudinal direction is obtained, and it is found that the electrical potential is mainly related to the moment. Moreover, the dynamic re-

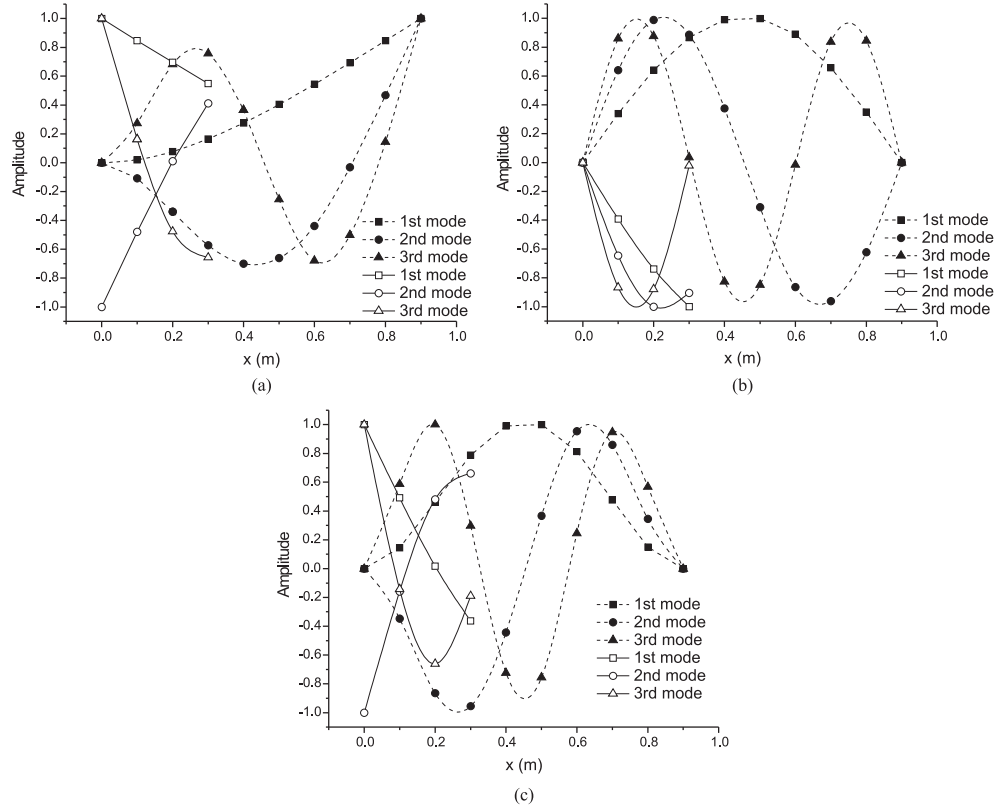


Fig. 3. First three mode shapes. Full curve, electrical potential; broken curve, displacement. (a) C-F; (b) H-H; (c) C-C.

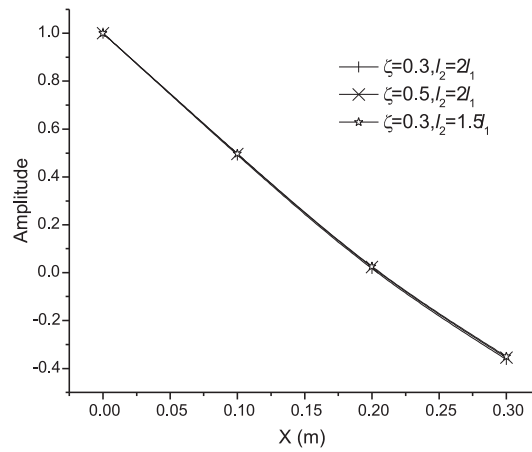


Fig. 4. First mode shapes of electrical potential (C-C) for different crack's parameters

sponse of the electric potential can be evaluated using the mode superposition method based on the formulation and findings. The present methodology can be utilized for piezoelectric sandwich beams with multiple edge cracks when all the cracks occur in un-piezoelectric-coupled region, because the effect of piezoelectric patches on the SIFs is not considered in Erdogan and Wu's solution [16] which is employed here. Further investigations for the SIFs need conducting to solve the problems with cracks occurring in piezoelectric coupled region.

REFERENCES

- [1] LV, C. F., W. Q. CHEN. Free Vibration of Orthotropic Functionally Graded Beams with Various End Conditions. *Struct. Eng. Mech.*, **20** (2005), No. 4, 465–476.
- [2] AGARWAL, S., A. CHAKRABORTY, S. GOPALAKRISHNAN. Large Deformation Analysis for Anisotropic and Inhomogeneous Beams Using Exact Linear Static Solutions. *Compos. Struct.*, **72** (2006), No. 1, 91–104.
- [3] LIBRESCU, L., S. Y. OH, O. SONG. Thin-walled Beams Made of Functionally Graded Materials and Operating in a High Temperature Environment: Vibration and Stability. *J. Therm. Stresses*, **28** (2005), No. 6–7, 649–712.
- [4] KAPURIA, S., M. BHATTACHARYYA, A. N. KUMAR. Bending and Free Vibration Response of Layered Functionally Graded Beams: A Theoretical Model and Its Experimental Validation. *Compos. Struct.*, **82** (2008), No. 3, 390–402.
- [5] SRIDHAR, R., A. CHAKRABORTY, S. GOPALAKRISHNAN. Wave Propagation Analysis in Anisotropic and Inhomogeneous Uncracked and Cracked Structures Using Pseudo Spectral Finite Element Method. *Int. J. Solids Struct.*, **43** (2006), No. 16, 4997–5031.
- [6] SINA, S. A., H. M. NAVAZI, H. HADDADPOUR. An Analytical Method for Free Vibration Analysis of Functionally Graded Beams. *Mate. Design*, **30** (2009), No. 3, 741–747.
- [7] AYDOGDU, M., V. TASKIN. Free Vibration Analysis of Functionally Graded Beams with Simply Supported Edges. *Mater. Design*, **28** (2007), No. 5, 1651–1656.
- [8] KE, L. L., J. YANG, S. KITIPORNCHAI. An Analytical Study on the Nonlinear Vibration of Functionally Graded Beams. *Meccanica*, **45** (2010), No. 6, 743–752.
- [9] YANG J., Y. CHEN. Free Vibration and Buckling Analyses of Functionally Graded Beams with Edge Cracks. *Compos. Struct.*, **83** (2008), No. 1, 48–60.
- [10] YANG, J., Y. CHEN, Y. XIANG, X. L. JIA. Free and Forced Vibration of Cracked Inhomogeneous Beams under an Axial Force and a Moving Load. *J. Sound Vib.*, **312** (2008), No. 1–2, 166–181.

- [11] KITIPORNCHAI, S., L. L. KE, J. YANG, Y. XIANG. Nonlinear Vibration of Edge Cracked Functionally Graded Timoshenko Beams. *J. Sound Vib.*, **324** (2009), No. 3–5, 962–982.
- [12] KE, L. L., J. YANG, S. KITIPORNCHAI, Y. XIANG. Flexural Vibration and Elastic Buckling of a Cracked Timoshenko Beam Made of Functionally Graded Materials. *Mech. Adv. Mater. Struct.*, **16** (2009), No. 6, 488–502.
- [13] TRINDADE, M. A., A. BENJEDDOU, R. OHAYON. Piezoelectric Active Vibration Control of Damped Sandwich Beams. *J. Sound Vib.*, **246** (2001), No. 4, 653–677.
- [14] LESIEUTRE, G. A., U. LEE. A Finite Element for Beams Having Segmented Active Constrained Layers with Frequency-Dependent Viscoelastics. *Smart Mater. Struct.*, **5** (1996), No. 5, 615–627.
- [15] VEL, S. S., R. C. BATRA. Analysis of Piezoelectric Bimorphs and Plates with Segmented Actuators. *Thin Wall. Struct.*, **39** (2001), No. 1, 23–44.
- [16] ERDOGAN, F., B. H. WU. The Surface Crack Problem for a Plate with Functionally Graded Properties. *J. Appl. Mech.*, **64** (1997), No. 3, 448–456.
- [17] WANG, Q., S. T. QUEK. Flexural Vibration Analysis of Sandwich Beam Coupled with Piezoelectric Actuator. *Smart Mater. Struct.*, **9** (2000), No. 1, 103–109.

Appendix A

The 3D constitutive equations of piezoelectric material are given by:

$$\begin{aligned}
 (A1) \quad \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} &= \begin{pmatrix} C_{11}^E & C_{12}^E & C_{13}^E & 0 & 0 & 0 \\ C_{12}^E & C_{11}^E & C_{13}^E & 0 & 0 & 0 \\ C_{13}^E & C_{13}^E & C_{33}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}^E - C_{12}^E}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55}^E \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} \\
 &\quad - \begin{pmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & 0 & 0 \\ e_{15} & 0 & 0 \\ 0 & e_{15} & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix},
 \end{aligned}$$

$$(A2) \quad \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{pmatrix} \Xi_{11} & 0 & 0 \\ 0 & \Xi_{11} & 0 \\ 0 & 0 & \Xi_{33} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{Bmatrix}.$$

In terms of the Kirchhoff-Love theory, the following assumptions are satisfied in plane strain state:

$$(A3) \quad \begin{cases} \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{23} = 0 \\ \sigma_{33} = \sigma_{13} = \sigma_{23} = 0. \end{cases}$$

Employing the above assumptions yields the 2D constitutive equations, viz.,

$$(A4) \quad \begin{cases} \sigma_{11} = \tilde{C}_{11}^E \varepsilon_{11} - \tilde{e}_{31} E_1 \\ D_1 = \tilde{\Xi}_{11} E_1 \\ D_3 = \tilde{\Xi}_{33} E_3 + \tilde{e}_{31} \varepsilon_{11} \end{cases},$$

where:

$$\tilde{C}_{11}^E = C_{11}^E - \frac{(C_{13}^E)^2}{C_{33}^E}, \quad \tilde{e}_{31} = e_{31} - \frac{C_{13}^E}{C_{33}^E} e_{33}, \quad \tilde{\Xi}_{11} = \Xi_{11}, \quad \tilde{\Xi}_{33} = \Xi_{33} + \frac{e_{33}^2}{C_{33}^E}.$$

Herein $C_{11}^E, C_{13}^E, C_{33}^E$ are the elastic modulus components of the piezoelectric material measured at constant electric field; e_{31}, e_{33} are the piezoelectric constants; Ξ_{11}, Ξ_{33} are the dielectric constants.

Appendix B

Let $J_{11} = \sinh(\lambda_1 l_1), \quad J_{12} = \cosh(\lambda_1 l_1), \quad J_{13} = \sin(\lambda_1 l_1), \quad J_{14} = \cos(\lambda_1 l_1),$

$$J_{21} = \sinh(\lambda_2 l_1), \quad J_{22} = \cosh(\lambda_2 l_1), \quad J_{23} = \sin(\lambda_2 l_1), \quad J_{24} = \cos(\lambda_2 l_1),$$

$$J_{31} = \sinh(\lambda_2 l_2), \quad J_{32} = \cosh(\lambda_2 l_2), \quad J_{33} = \sin(\lambda_2 l_2), \quad J_{34} = \cos(\lambda_2 l_2),$$

$$J_{41} = \sinh(\lambda_2 l), \quad J_{42} = \cosh(\lambda_2 l), \quad J_{43} = \sin(\lambda_2 l), \quad J_{44} = \cos(\lambda_2 l).$$

The determinants are given as

[illegible]

$$|\mathbf{H}_{C-C}| =$$

[illegible]