SOME PROBLEMS RELATED TO THE MATHEMATICAL MODELING OF CONVECTION FLOWS IN WATER FLOW GLAZING

M. RASHEVSKI*, S.G. SLAVTCHEV

Institute of Mechanics, Bulgarian Academy of Sciences

[Received: 19 June 2023. Accepted: 5 October 2023]

doi: https://doi.org/10.55787/jtams.23.53.4.366

ABSTRACT: This paper is a review of recent studies of the present authors related to the modeling of hydrodynamic and heat transfer processes in water-filled glazing chambers exposed to sunlight. Due to the absorption properties of water, solar visible and near-infrared irradiance induces a volumetric heat source with exponential decay. Three different scenarios of mixed natural and forced convection are discussed: two fully-developed flows in vertical flat and rectangular channels and a developing flow in a slender channel with a rectangular cross-section (similar to a Hele-Shaw cell). Three cases of temperature boundary conditions are studied for the lighted and the opposite walls, whereas in the case of the rectangular channel, linear declination of the side wall temperatures is applied. Analytical solutions for fully-developed flows in both flat and rectangular channels are reported and discussed. Some physical characteristics of the flows such as bulk liquid temperature and Nusselt number are calculated. The problem of developing flows in the slender rectangular channel is formulated in a “stream function-vorticity” version. The main conclusion is that the presented models predict quite well the average physical parameters of the mixed convection flows in the glazing chamber under solar radiation.

KEY WORDS: Water-flow glazing, Vertical slender channel, Viscous liquid, Laminar natural and mixed convection, Light absorption, Non-uniform volumetric heat source

1 INTRODUCTION

Unlike traditional architecture typologies, newly designed high-rise buildings in Europe, and in particular in Bulgaria, are using mostly light materials to overcome structural challenges. The physical mass of the building is constantly reducing, however, the capacity to absorb thermal energy (thermal mass) is also reducing. Overheating becomes a primary concern in such greenhouse-like glazed buildings, which can be
instantly overheated once the sun shines on the facade. It is now obvious that thermal mass has to be increased and the solar energy has to be either reflected or collected and used.

Perhaps, the first attempt to purposely use the building’s thermal mass to collect and store solar energy can be dedicated to the French engineer Felix Trombe [1] who created the concept for the so-called “Trombe wall” in the 60s of the last century. Later work [2] shows that the use of inertial materials such as concrete, ceramic, or clay brickwork in the European climate zones within highly insulated exterior walls can’t fully absorb and transmit the solar energy. They can store only the amount penetrating through the glazed facades but on the cost of increased structural expenses. Moreover, this process is passive and depends on building orientation and geometry and can’t respond dynamically to current exterior conditions and interior needs.

Recently, some innovative complex facade systems have emerged. Examples of reflecting facades are double-skin air-ventilated facades, glass coatings, solar blinds, etc., and for absorbing facades among others are building integrated solar absorbers [3], semi-transparent photovoltaics, etc. A comprehensive review of different facade systems is published in the Fraunhofer ISE by Maurer et al. [4]. Light-weight transparent facades for increased and actively controlled thermal mass are needed. The potential of water as a facade medium with unique spectral and thermal properties was endorsed and experts [5–10] recognized its importance for future dynamic transparent facade systems.

Several research groups are dealing with water flow glazing (WFG) and two major research projects supported by the European Commission under the 7th framework program (FP7) and Horizon 2020 program, respectively – FluidGlass and InDeWaG. The InDeWaG consortium, established in 2014, was composed of academic and industrial partners from Bulgaria, Spain, and Germany, including the Bulgarian Academy of Science. Under those projects were developed the most advanced and mature WFG technologies constructed so far in real operational environments.

The glazing unit in such systems consists of two or more glass panes, a chamber with water or water-glycol mixture, and two spacers with porous boundaries acting as water chamber inlet and outlet [6] (see Fig. 1). Once the water enters from the spacer into the glazing chamber, it is subjected to a radiant heat source from the sun. Due to its unique spectral properties, water is almost completely transparent to the visible solar radiation and almost opaque to the infrared radiation which carries more than 40% of the total solar energy flux [11]. The absorbed energy provokes natural convection which contributes to the forced fluid flow. Solar radiation is acting in the chamber as a volumetric heat source with exponential decay, which together with the temperature boundary conditions strongly influences chamber flow and heat fluxes. Finally, the absorbed energy in the glazing unit is extracted through the upper glazing spacer and transferred to be stored, used, or dissipated.
The thermal efficiency of double-pane water flow glazing is theoretically studied by two main approaches. The first one includes different energy balance models assessing the energy power per unit surface area at the center region of the windows or at building/room scale [12–16]. Those models predict the average thermal characteristics needed for designing water-chamber windows. The second approach is based on mathematical models of mixed convection and heat transfer in vertical heated channels subjected to solar radiation. In such models, the fluid is considered viscous and incompressible and the flows are usually assumed laminar and fully-developed.

The first studies of combined free and forced convection are dated since the 50s of the last century and some earlier papers on this topic are cited in [17]. Noteworthy are works which consider flows in channels with internal heat sources, including those initiated by light absorption. Theoretical and experimental studies of so-called “photo-absorption”, i.e., free convection in fluid induced only by light, are presented in the book of Luikov and Berkovsky [18]. Mixed convection in vertical channels in the presence of heat source/sink of different types is studied in [19–21].

This review paper shows recent results of the present authors [17, 22], concerning the hydrodynamic and heat transfer processes in a water-flow glazing chamber. The study is based on the Navier-Stokes equations in Boussinesq approximation and the energy equation with a heat source described by the Beer-Lambert law for light absorption in visible and near-infrared spectra. Due to the narrowness of the glazing unit in one dimension, it could be modeled as a slender channel. Two mathematical problems for well-developed flows in vertical flat and rectangular channels are formu-
lated. Based on the analytical solutions obtained in [17, 22], some substantial results are reported and discussed. Another problem for stationary convective flows in a vertical narrow channel of finite size is formulated in the “stream function-vorticity” statement. The numerical solution of the governing equations by the finite difference method is currently in progress.

2 THE GOVERNING EQUATIONS

Due to the slow circulation flow rates in the glazing chamber, only laminar flow regimes are considered. Under solar radiation, some amount of the solar heat is absorbed in the fluid and changes the temperature field. Since the temperature differences are not large, the water viscosity is constant, but the density, $\rho$, is a linear function of the temperature $T$ (Boussinesq formula):

$$\rho = \rho_0(1 - \beta(T - T_0)),$$

where $\rho_0$ is the density at the reference temperature $T_0$ and $\beta$ is the thermal expansion coefficient. The fluid motion and thermal losses in the water chamber of the glazing are described by the Navier-Stokes equations with Boussinesq approximation and the energy equation, namely,

$$\nabla \cdot \mathbf{V} = 0,$$

$$\rho_0(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla p + \mu \Delta \mathbf{V} - \rho_0 \mathbf{g} \beta(T - T_0),$$

$$\mathbf{V} \nabla T = \kappa \Delta T + \frac{Q}{\rho_0 C_P},$$

where $\mathbf{V}$ is flow velocity, $p$ pressure, $\mu$ dynamic viscosity, $\kappa$ thermal diffusivity, $C_P$ specific heat capacity, $Q$ volumetric heat source, due to the internal heating, $\mathbf{g}$ gravity acceleration vector, and $\Delta$ Laplacian operator. The terms on the left side of eq. (3) are the inertia terms responsible for the convective motion, while the terms on the right side describe respectively: pressure drop, diffusive motion, and external body forces (such as buoyancy), and the volumetric heat source in eq. (4).

The essence of the Boussinesq approximation in the dynamic equations is that the density of otherwise incompressible fluid varies only due to temperature differences in the direction of the gravitational force. Due to the thermal expansion of the liquid, the warmer regions become “lighter” reducing its specific weight and flow against gravity, while the cooler (thus “heavier”) regions sink. That’s the mechanism of the natural convection in the chamber. At relatively small flow rates natural convection is comparable with the forced one and mixed convection exists.

The left and right walls of the chamber are transparent to visible and infrared light. The solar radiation penetrates the liquid-filled glazing and induces a volumetric heat
source with intensity decreasing exponentially according to the Beer-Lambert law
(see, for example [18, 23])

\[ Q = Q_0 e^{-\alpha x}, \quad Q_0 = \cos \gamma I_0 \alpha, \]

where \( Q \) is the heat source power per unit volume measured in W/m\(^3\), \( Q_0 \) is the
source intensity at the left wall, \( I_0 \) is the global solar irradiance, \( \gamma \) is the angle of
incidence and \( \alpha \) is the absorption coefficient of the liquid. Specific values of \( I_0, \alpha, \)
and \( Q_0 \) for the most interesting wavelengths of the light spectrum are derived in [17]
and presented here in Table 1. The values are given for the visible (VIS) [400-800
nm] and near-infrared (NIR) [800-2500 nm] spectra. Although the heat flux in the
visible spectrum is more intensive than the one in the near-infrared spectrum, due to
the specific absorption properties of the water, the volumetric heat source intensity at
the left wall is much greater in the NIR interval.

<table>
<thead>
<tr>
<th>Wavelength</th>
<th>Radiation flux ( I_0 ) [W/m(^2)]</th>
<th>Absorption coefficient ( \alpha ) [1/m]</th>
<th>Volumetric heat source ( Q_0 ) [W/m(^3)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>400:800</td>
<td>324</td>
<td>0.6</td>
<td>194</td>
</tr>
<tr>
<td>800:2500</td>
<td>220</td>
<td>2179</td>
<td>479042</td>
</tr>
</tbody>
</table>

3 FORMULATION OF THE MATHEMATICAL PROBLEMS

Depending on the geometry models of the glazing and the approximation of the flow
regimes, we formulate three mathematical problems which describe the hydrody-
namic and heat transfer processes in the glazing.

3.1 FULLY DEVELOPED CONVECTIVE FLOW IN A FLAT CHANNEL

The simplest geometrical model of the glazing is a channel between two parallel
plates placed vertically in the gravity field. (see Fig. 2a). This approximation of the
geometry follows from the fact that the height and length of the glazing are much
larger than its width. Therefore, the flow in the central area of the glazing is weakly
influenced by the lateral walls.

The distance between the plates is denoted by \( d \). They are assumed transparent
and are kept at constant temperatures \( T_1 \) and \( T_2 \). Liquid of temperature \( T_0 \) enters the
channel from bellow with a flow rate per unit length in \( y \)-direction \( q = W_0 d \), where
\( W_0 \) is the mean inlet velocity. We assume that \( T_1 > T_0 \) and \( T_2 \geq T_0 \), independently
of which of the plate temperatures is higher. The flow is assumed stationary and
fully developed and therefore, the fluid velocity has only one component directed vertically along the plates and depending on the coordinate across the channel. The liquid temperature also depends on the same coordinate.

Introducing the Cartesian coordinates $x$ and $z$ and supposing the velocity $w$ and temperature $T$ to be functions of $x$, the equations (2-4) are simplified to the following system written in non-dimensional form [17]

$$\frac{d^2 w}{dx^2} + Gr \Theta = \frac{dp}{dz} = \text{const}, \quad \frac{\partial p}{\partial x} = 0,$$

(6)

$$\frac{d^2 \Theta}{dx^2} + H \rho_0 e^{-Nz} = 0.$$

(7)

The boundary conditions and the flow rate balance are as follows:

$$w = 0, \quad \Theta = r_T, \quad \text{at} \ x = 0,$$

(8)

$$w = 0, \quad \Theta = 1, \quad \text{at} \ x = 1.$$

(9)

$$\int_0^1 w dx = Re.$$

Here are introduced the following scales: distance $d$, velocity $\nu/d$, pressure $\rho_0(\nu/d)^2$, temperature difference $\Delta T = T_1 - T_0$ and dimensionless temperature $\Theta = (T - T_0)/\Delta T$,  

Fig. 2: Schemes of flat (a); and rectangular channels (b).
where $\nu = \mu/\rho_0$ is the kinematic viscosity. Some dimensionless parameters are also defined: the Reynolds number $Re = q/\nu$, the Grashof number $Gr = (g\beta\Delta Td^3)/\nu^2$, the optical density $N = \alpha d$, the heat source parameter $H_Q = Q_0d^2/(\rho_0C_P\kappa\Delta T)$ and the parameter $r_T = (T_2 - T_0)/\Delta T$. The most interesting cases of the relation between the different temperatures are expressed by the values of the last parameter. When both plates have equal temperatures ($T_2 = T_1$), $r_T$ is equal to one. It vanishes when the temperature of the left-side plate coincides with the entering fluid one and $r_T = 2$ if the right plate temperature is the mean of the other two temperatures ($T_1 = (T_2 + T_0)/2$).

The solution of the linear ordinary differential equations is given in [17]. The temperature field is independent of the flow, while the velocity profiles depend on the volumetric source intensity and the wall temperatures. Before discussing the most substantial results of that study we estimate the dimensionless parameters for a water chamber of typical width $d = 0.024$ m (Table 2).

<table>
<thead>
<tr>
<th>$N_{VIS} = 0.0144 / N_{NIR} = 52.296$</th>
<th>$\Delta T = 1$ K</th>
<th>$\Delta T = 5$ K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_Q (VIS)$</td>
<td>0.1844</td>
<td>0.0369</td>
</tr>
<tr>
<td>$H_Q (NIR)$</td>
<td>455.327</td>
<td>91.065</td>
</tr>
<tr>
<td>$Gr$</td>
<td>33681</td>
<td>168404</td>
</tr>
</tbody>
</table>

3.1.1 Influence of the heat source

At equal wall temperatures, the heat source is the main driving force for the free convection occurring in the channel. The temperature and velocity profiles for VIS and NIR spectra at $r_T = 1$ and $\Delta T = 1$ K are presented in Fig. 3.

Since solar radiation is absorbed weakly in the visible spectrum, the temperature profile has a nearly parabolic form with a comparatively low maximum close to the center of the channel. On the other hand, the more intense NIR heat source creates a comparatively high peak, shifted towards the exterior (left in Fig. 3a) channel wall. The resulting velocity fields for $Re = 0$ follow the temperature profiles. In the case of VIS heat source, there is a flow directed upwards in the center with two reverse flows on the sides. The NIR heat source creates a flow with liquid going up near the exterior (left) wall, where the fluid temperature has its maximum and going down along the interior (right) wall. Following the flow rate eq. (9) for free convection in both situations the flow(s) going downwards equates to the upward flow.

Due to the poor absorption properties of the water in the visible spectrum, it is obvious that the more significant case presents the NIR heat source.
Fig. 3: Temperature (a) and velocity (b) profiles for VIS (1) and NIR (2) spectra at $r_T = 1, \Delta T = 1$ K.

3.1.2 Influece of the Wall Temperatures

Figure 4 shows the temperature profiles in the channel width for different values of $r_T$ and different temperature scales $\Delta T$. Since the temperature difference is present in the denominator for $\Theta$, the resulting graphics show a diminishing effect when increasing the temperature difference.

The velocity fields for $Re = 0$ in Figs. 5 and 6 show that the wall temperatures have a strong impact on the channel flow. They are an important condition for the

Fig. 4: Temperature profiles for $r_T = 0$ (1), $r_T = 1$ (2), $r_T = 2$ (3) at $\Delta T = 1$K (a) and $\Delta T = 5$ K (b).
Convection Flows in Water Flow Glazing

Fig. 5: Velocity profiles for $r_T = 0$ (1), $r_T = 1$ (2), $r_T = 2$ (3) at $\Delta T = 1K$ and $Re = 0$.

Appearance of one or two reverse flows and its/their position and intensity. The direction of the flow depends on the sign of the velocity derivative on the rigid boundaries. The limiting values for the existence of two reverse flows ($r_{T1} < r_T < r_{T2}$) are given through the corresponding formulas on both walls [17]:

Fig. 6: Velocity profiles (NIR spectra) for $r_T = 0.835$ (1), $r_T = 0.84$ (2), $r_T = 0.869$ (3) at $\Delta T = 1K$ and $Re = 0$
The very special case when $r_{T1} = r_{T2}$ can be assessed through a combination of the characteristic parameters, defining a critical Reynolds number above which no reverse flow exists:

\[
Re_0 = \frac{GrH_Q}{144} \left(F_1(N) - F_2(N)\right) = \frac{GrH_Q}{12N^3} \left[2 - \left(1 + e^{-N}\right) \left(\frac{1}{N} - \frac{6}{N^2} + \frac{12}{N^3}\right)\right].
\]

As $F_1(N)$ and $F_2(N)$ are functions of the dimensionless optical depth and the product $Gr.H_Q = \frac{g\beta d Q_0}{\nu^2 \rho_0 C_p K}$ doesn’t depend on the temperature difference, $Re_0$ is determined by the physical properties of the liquid and the heat source intensity. For VIS spectrum $Re_0 = 8.5629$ and for NIR spectrum $Re_0 = 7.9495$. Velocity profiles without reverse flow, corresponding to both spectra, are displayed in Fig. 7.
3.1.3 THERMAL CHARACTERISTICS FOR THE FLAT CHANNEL

The most important thermal characteristics of the flow are the dimensionless bulk liquid temperature, the net power outlet, and the Nusselt number on the channel walls. The dimensionless bulk liquid temperature (the mean temperature growth in the channel) is only applicable for mixed convection flows using known formulas such as [24]

\[ \Theta_{\text{bulk}} = \frac{\int_0^1 w(\Theta) \, dx}{\int_0^1 w \, dx} = \frac{\int_0^1 w(\Theta) \, dx}{Re}. \]

The net power outlet directed upwards through a section of width \(d\) and unit length \(L = 1\) in the direction perpendicular to the channel plane is defined as [25]

\[ Q_{\text{heat}}^* = \rho_0 C_p L \int_0^d w(T - T_0) \, dx, \]

or in non-dimensional form by the formula

\[ Q_{\text{heat}} = \frac{Q_{\text{heat}}^*}{Q_0 d^2 L} = \frac{Pr}{H_Q} \int_0^1 w(\Theta) \, dx, \]

where \(Pr = \nu/\kappa\) is the Prandtl number. Results for the bulk liquid temperature and power outlet can be extracted from [17].

The heat fluxes through the walls are represented by the Nusselt numbers, defined according to the formulae [26]

\[ Nu_1 = -\frac{1}{r_1} \left( \frac{d\Theta}{dx} \right)_{x=1}, \quad Nu_2 = -\frac{1}{r_2} \left( \frac{d\Theta}{dx} \right)_{x=0}, \]

where \(r_1\) and \(r_2\) are defined as

\[ r_1 = \frac{T_1 - \Delta T \Theta_{\text{bulk}}}{\Delta T}, \quad r_2 = \frac{T_2 - \Delta T \Theta_{\text{bulk}}}{\Delta T}. \]

In the case of solar radiation in the NIR spectrum, these numbers are summarized for \(Re = 100\) in Table 3, where the sign of the number shows the direction of the flux.

For equal wall temperatures \((r_T = 1)\) the heat flux is positive on the right wall and negative on the left wall, due to the influence of the heat source on the bulk liquid temperature. Hence, heat is transmitted from the channel to the exterior. Similar is the situation for \(r_T = 2\) when both fluxes are directed outwards. In the case when the left wall temperature is equal to the entrance temperature \(T_0\) and the right wall
Table 3: Nusselt numbers $N u_1$ and $N u_2$ at $Re = 100$ for the case of heat source in the NIR spectrum

<table>
<thead>
<tr>
<th>$\Delta T$</th>
<th>$r_T$</th>
<th>$N u_2$ on the left channel wall</th>
<th>$N u_1$ on the right channel wall</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>-0.4524</td>
<td>-0.0377</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-0.3919</td>
<td>0.0076</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.3462</td>
<td>0.0561</td>
</tr>
<tr>
<td>$\Delta T$ = 5K</td>
<td>0</td>
<td>-1.5981</td>
<td>-0.3587</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-0.3919</td>
<td>0.7438</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-0.2964</td>
<td></td>
</tr>
</tbody>
</table>

temperature is higher than both $T_2$ and $T_0$ ($r_T = 0$), heat enters from the right wall and is dissipated through the left wall. The Nusselt numbers for $\Delta T = 1$ and $\Delta T = 5$ when $r_T = 1$ are practically equal and differ only after the 11th decimal (on the right wall) and 10th decimal (on the left wall).

3.2 FULLY DEVELOPED CONVECTIVE FLOW IN A VERTICAL RECTANGULAR CHANNEL

The two-dimensional problem of convective flows in the channel between two plates can be extended to a three-dimensional problem of mixed convection in a channel with a rectangular cross-section.

Let us consider a fully developed flow in a vertical rectangular channel of sizes $d$ and $L$ (see Fig. 2b). The fluid enters from below with constant velocity $W_0$ and temperature $T_0$. The left and right transparent walls are kept at constant temperatures $T_2$ and $T_1$, respectively. The lateral channel walls are opaque. Their temperature varies linearly from $T_2$ to $T_1$ if $T_2 > T_1$ or all channel walls have the same temperature $T_1$ when $T_2 = T_1$. In both cases of different or equal wall temperatures, the fluid temperature $T_0$ is assumed smaller than $T_1$, and therefore, $\Delta T = T_1 - T_0 > 0$.

Applying the same scales for distance, velocity, pressure, and temperature as in the previous section, the governing equations of the fully developed flow in a channel with a rectangular cross-section are represented in dimensionless form as follows:

\[ \Delta w + Gr \Theta = \frac{dp}{dz} = \text{const}, \quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \Delta \Theta + H Q e^{-N_x} = 0, \]

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian. Let us introduce the aspect ratio $l = L/d$ which is of order one when both sizes of the cross-section are comparable. Referring to the coordinate system $x, y, z$ in Fig. 2b, the boundary conditions are
Convection Flows in Water Flow Glazing

\[ x = 0, \quad -\frac{l}{2} < y < \frac{l}{2} : \quad w = 0, \quad \Theta = r_T, \]

(19) \[ x = 1, \quad -\frac{l}{2} < y < \frac{l}{2} : \quad w = 0, \quad \Theta = 1, \]

\[ y = \pm \frac{l}{2}, \quad 0 \leq x \leq 1 : \quad w = 0, \quad \Theta = \Theta_w(x) \equiv r_T + (1 - r_T)x. \]

The flow rate balance is given by

(20) \[ \int_0^1 \int_{-\frac{l}{2}}^{\frac{l}{2}} w(x, y) \, dx \, dy = lRe. \]

The last equation is used for the determination of the pressure gradient.

It is worth mentioning that similar problems of isothermal flows in a channel with a rectangular cross-section are considered in the books of Happel and Brenner [27], and Shkadov and Zapryanov [28] as examples of fully developed flows. Following the approach applied to such problems, we have found the analytical solutions of the equations as sums of a function of \( x \) only and another one depending on both independent variables [22]. Here some results of that paper are presented and compared to the ones in the flat channel case.

As the flow temperature and velocity are symmetric functions about the plane \( y = 0 \), their profiles are displayed in the half of the channel section where \( 0 \leq y \leq l/2 \). For the most interesting NIR spectrum, the temperature field in the quadratic channel with equal wall temperatures is shown in Fig. 8a. The temperature profiles in planes \( y = y_i \) located uniformly at a distance of 0.1 between the plane \( y = 0 \), and the wall

![Fig. 8: Temperature field (a) and temperature profiles (b) 1-6 in different channel sections from \( y = 0 \) to \( y = l/2 \) respectively for \( l = 1, \, Re = 0, \, r_T = 1, \, \Delta T = 1 \, K \).](image-url)
Fig. 9: Velocity field (a) and velocity profiles (b) 1-6 in different channel sections from \( y = 0 \) to \( y = l/2 \) respectively for \( l = 1, Re = 0, r_T = 1, \Delta T = 1 \, K. \)

9a

\( y = 1/2 \) are plotted in Fig. 8b. As expected, the maximum of the curves is situated closer to the lighted wall and the highest one occurs on the center line of the channel.

In the case of natural convection (\( Re = 0 \)), the dynamic field is presented in Fig. 9a. At equal wall temperatures, the flow is only induced by the volumetric heat source and goes up near the lighted wall and falls down in the vicinity of the opposite one. The velocity profiles in the planes \( y = y_i \) are illustrated in Fig. 9b.

The more the channel is extended, the closer the temperature profiles get to the ones for the flat channel, hence, the influence of the side walls decreases above certain

Fig. 10: Temperature field (a) and temperature profiles (b) 1-6 in different channel sections from \( y = 0 \) to \( y = l/2 \) respectively for \( l = 5, Re = 0, r_T = 1, \Delta T = 1 \, K. \).
value of \( l \). For example, for \( l = 5 \), visible from Figs. 10a and 10b, in about half of the channel width, the temperature profile across the \( x \) plane approaches the profile in the flat channel (Fig. 3a) and the curves coincide.

The velocity for \( l = 5 \) is presented in Figs. 11a and 11b. It is seen that in the central region around the plane \( y = 0 \) the velocity profiles in the planes \( y = y_i \) where \( 0 \leq y_i \leq 0.5 \) are very close to each other. This shows that in this area the influence of the lateral walls is negligibly small and the flow behaves as almost two-dimensional. In such situations (for slender channels with elongated rectangular sections of \( L/d \gg 1 \)) it is convenient to use the flat channel model.

3.2.1 THERMAL CHARACTERISTICS FOR THE RECTANGULAR CHANNEL

Following the same definitions as in the previous section, but applied to a channel with a rectangular cross-section, similar expressions for the thermal parameters are formulated. The dimensionless bulk liquid temperature is

\[
\Theta_{\text{bulk}} = \frac{\int_{0}^{1} \int_{-\frac{l}{2}}^{\frac{l}{2}} w\Theta \, dx \, dy}{\int_{0}^{1} \int_{-\frac{l}{2}}^{\frac{l}{2}} w \, dx \, dy} = \frac{\int_{0}^{1} \int_{-\frac{l}{2}}^{\frac{l}{2}} w\Theta \, dx \, dy}{lRe}.
\]

The amount of heat carried by the flow is characterized by the net power outlet \( Q_{\text{heat}} \) through the channel section
\[ Q_{\text{heat}} = \frac{Q^*_{\text{heat}}}{Q_0 d^2 L} = \frac{Pr}{H_Q} \int_0^1 \int_{-\frac{L}{2}}^{\frac{L}{2}} w \Theta \, dx \, dy. \]

The average Nusselt number on the left and right walls are

\[ Nu_2 = -\frac{1}{lr_2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left( \frac{d\Theta}{dx} \right)_{x=0} dy, \quad Nu_1 = -\frac{1}{lr_1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left( \frac{d\Theta}{dx} \right)_{x=1} dy. \]

The results for \( \Theta_{\text{bulk}} \) and the Nusselt number values for \( Re = 100, \, r_T = 1, \Delta T = 1 \) and \( \Delta T = 5 \) are plotted in Figs. 12 and 13.

It is worth noting that in all thermal characteristics, the double integral of the product of the velocity \( w \) and the temperature \( \Theta \) plays an important role. Its behavior with \( l \) (divided by the Reynolds number) is seen in Fig. 12. The interval of the aspect ratio is studied up to \( l = 54 \) which corresponds to real sizes of the glazing unit equal to 1.3 m length and 0.024 m width. At the end of the interval, the mean outlet temperature differs from the one obtained through the flat channel formula (13) by less than 1%. Similar are the results for the Nusselt numbers \( Nu_1 \) and \( Nu_2 \) where the discrepancy is 2.1 and 0.2% respectively. As shown in the previous section, both Nusselt numbers for \( \Delta T = 1 \) and \( \Delta T = 5 \) practically coincide.

Finally, the influence of the lateral walls on the thermal characteristics is noticeable at low aspect ratios. At high aspect ratios, the rectangular channel model fully confirms the results obtained in the flat channel case.

![Fig. 12: \( \Theta_{\text{bulk}} \) as a function of \( L/d \) aspect ratio compared to the flat channel value.](image-url)
3.3 Mixed convection in a vertical rectangular slender channel of finite height. Hele-Shaw cell approximation

Let us consider a vertical slender channel with a rectangular cross-section and a finite height $H$. The length of the channel $L$ is assumed much larger than its width $d$ (see Fig. 2b), hence, the ratio $\delta = d/L \ll 1$, while the height-to-length aspect ratio $\epsilon = H/L > 1$. The liquid of temperature $T_0$ enters the channel from below with the mean velocity $W_0$. The large area walls (left and right on the figure) are kept at temperatures $T_2$ and $T_1$, while the lateral, small area walls are assumed isolated. In the narrow channel the velocity component $u$ in the direction $x$ is negligibly small in comparison with the other two velocity components and can be ignored, sometimes referred to as the Hele-Shaw cell approximation.

The governing equations (2-4) are reduced to the following system of equations in dimensionless form

\[
\begin{align*}
\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \frac{1}{\delta Re} \left( \frac{\partial^2 v}{\partial x^2} + \delta^2 \Delta_1 v \right), \\
v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{\delta Re} \left( \frac{\partial^2 w}{\partial x^2} + \delta^2 \Delta_1 w \right) + \frac{Gr}{\delta Re^2} \Theta, \\
v \frac{\partial \Theta}{\partial y} + w \frac{\partial \Theta}{\partial z} &= \frac{1}{\delta Pr Re} \left( \frac{\partial^2 \Theta}{\partial x^2} + \delta^2 \Delta_1 \Theta + H_Q e^{-Nx} \right),
\end{align*}
\]

where $v$, $w$, and $\Theta$ are functions of all three coordinates and $\Delta_1 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is
the two-dimensional Laplacian. The scaling of the coordinates and variables is as follows: $d$ for the $x$-coordinate, $L$ for the $y$- and $z$-coordinates, $W_0$ for the velocity, $\rho_0W_0^2$ for the pressure and the same scale for the temperature as described previously. The dimensionless parameters $Re, Gr$, and $Pr$ preserve the same definitions as in the previous sections.

The boundary conditions for the system (24) are:

\[
\begin{align*}
  x = 0 &: v = w = 0, \quad \Theta = r_T, \\
  x = 1 &: v = w = 0, \quad \Theta = 1, \\
  y = \pm \frac{1}{2} &: v = w = 0, \quad \frac{\partial \Theta}{\partial y} = 0, \\
  z = 0 &: v = 0, w = 1, \quad \Theta = 0, \\
  z = \epsilon &: v = 0, \frac{\partial w}{\partial z} = 0, \quad \frac{\partial \Theta}{\partial z} = 0.
\end{align*}
\]

(25)

The conditions at $z = \epsilon$ express the requirement that the streamlines and isotherms are straight lines at the channel end if the height of the channel is sufficient for the flow to become developed. Due to the incompressibility of the fluid, the condition for a constant flow rate at an arbitrary section of the channel reads

\[
\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} w(x, y, z) \, dx \, dy = 1.
\]

(26)

The form of the continuity equation suggests using the stream function $\psi = \psi(x, y, z)$ and the $x$-component $\omega = \omega(x, y, z)$ of the vorticity $\Omega = \text{rot } V$ which are defined by the relations

\[
\begin{align*}
  v &= \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y}, \quad \omega \equiv \Omega_x = -\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},
\end{align*}
\]

where the stream function and vorticity are scaled by $LW_0$ and $W_0/L$ respectively. The last expressions are defined in any plane $x = x_d$, where $x_d \in (0, 1)$.

Introducing the expressions (27) into the equations (24) and eliminating the pressure, one arrives at the following system:

\[
\begin{align*}
  \Delta_1 \psi &= -\omega, \\
  \frac{\partial \psi}{\partial z} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial z} &= \frac{1}{\delta Re} \left( \frac{\partial^2 \omega}{\partial x^2} + \delta^2 \Delta_1 \omega \right) + \frac{Gr}{\delta Re^2} \frac{\partial \Theta}{\partial y}, \\
  \frac{\partial \psi}{\partial z} \frac{\partial \Theta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Theta}{\partial z} &= \frac{1}{Pr Re} \left( \frac{\partial^2 \Theta}{\partial x^2} + \delta^2 \Delta_1 \Theta + H Q e^{-N x} \right).
\end{align*}
\]

(28)
Correspondingly, the boundary conditions and the flow rate balance equation are re-written in the following form:

\begin{align}
  x = 0 : & \quad \psi = \psi_0 \equiv \text{const}, \quad \Theta = r_T, \\
  x = 1 : & \quad \psi = \psi_d \equiv \text{const}, \quad \Theta = 1, \\
  y = -\frac{1}{2} : & \quad \psi = \psi_1 \equiv \text{const}, \quad \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \Theta}{\partial y} = 0, \\
  y = \frac{1}{2} : & \quad \psi = \psi_2 \equiv \text{const}, \quad \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \Theta}{\partial y} = 0, \\
  z = 0 : & \quad \frac{\partial \psi}{\partial y} = -1, \quad \frac{\partial \psi}{\partial z} = 0, \quad \Theta = 0, \\
  z = \epsilon : & \quad \frac{\partial \psi}{\partial z} = 0, \quad \frac{\partial^2 \psi}{\partial y \partial z} = 0, \quad \frac{\partial \Theta}{\partial z} = 0,
\end{align}

\( (29) \)

\begin{align}
  \int_0^1 \left[ \psi(x, -\frac{1}{2}, z) - \psi(x, \frac{1}{2}, z) \right] dx = 1.
\end{align}

\( (30) \)

From the first condition at \( z = 0 \) one has \( \psi(x, y, 0) = -y + C(x) \) and the flow rate balance is satisfied. This function is equal to \( \psi_1 = \frac{1}{2} + C(0) = \psi_0 \) at the edge \((0, -\frac{1}{2})\) and \( \psi_2 = -\frac{1}{2} + C(1) = \psi_d \) at the edge \((1, \frac{1}{2})\). Because the stream function is determined to an arbitrary constant, we assume that \( \psi_0 = 1 \) and \( \psi_d = 0 \), which means that \( C(0) = C(1) = \frac{1}{2} \). As the thickness \( d \) is sufficiently small compared to the channel length we assume \( C(x) \) to remain constant, namely, \( C(x) = \text{const} = \frac{1}{2} \).

To solve numerically the equations one needs to formulate the boundary conditions for the vorticity on the lateral walls. A certain way to obtain such conditions is to develop the stream function in series near the channel boundaries using the boundary conditions \((29)\) and to apply the Poisson equation for the stream function from \((28)\) on the walls (see, \cite{25}).

Since \( \delta \ll 1 \), the terms \( O(\delta^2) \) in \((28)\) are neglected and the system is reduced to the parabolic-type form

\begin{align}
  \frac{\partial \psi}{\partial z} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial z} = \frac{1}{\delta Re} \frac{\partial^2 \omega}{\partial x^2} + \frac{Gr}{\delta Re^2} \frac{\partial \Theta}{\partial y}, \\
  \frac{\partial \psi}{\partial z} \frac{\partial \Theta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Theta}{\partial z} = \frac{1}{\delta Pr Re} \left( \frac{\partial^2 \Theta}{\partial x^2} + H_Q e^{-N x} \right).
\end{align}

\( (31) \)

The product \( \delta Re \) is of the order of 1. For example, at typical values for water flow glazing chamber and flow regimes, the Reynold number is about 100 and the ratio \( \delta \) is about 0.02.
The last equations could be solved by different numerical methods and we are working on the realization of the finite difference method (FDM). The numerical procedure is under execution.

Finally, it is noteworthy that the flow approaches the profiles of a two-dimensional case around the central plane $y = 0$ when the lateral walls are separated by a great distance $L (\delta \to 0)$.

4 Conclusions

The paper reviews recent studies of the present authors dedicated to natural and mixed convection in a vertical slender channel exposed to solar radiation. The studies are motivated by the development of a perspective facade technology, namely, water flow glazing facade elements. Three physical cases are communicated for laminar convection with a volumetric heat source based on the Navier-Stokes equations and the Beer-Lambert law: two cases of fully developed flows are considered, and a third case of a developing flow is formulated. Three different temperature boundary conditions and two radiation spectra are applied.

Analytical solutions for the velocity and temperature functions are presented for a stationary flow between two vertical infinite transparent plates. The influence of the heat source in the visible and near-infrared spectra is discussed, as well as the influence of the wall temperatures on the flow behavior. Formulas for the existence of reverse flow(s) are derived, together with a critical value for the Reynolds number above which no reverse flows exist. Some important thermal characteristics such as the bulk liquid temperature, the net power outlet, and the Nusselt number are formulated and results for the last one are reported.

The flat channel case is extended to a more complex vertical rectangular chamber under solar radiation. Based on the analytical solutions derived in another paper, some graphical results are shown for different aspect ratios. The influence of the lateral walls on the temperature and velocity fields, as well as on the channel thermal characteristics, is discussed. It is shown that for slender channels with very high aspect ratios, the results approach, therefore confirm, the ones obtained for the two-dimensional flat channel.

Finally, a third problem is formulated for a developing flow in a vertical slender channel in the vorticity-stream function statement. The equations and boundary conditions are presented in dimensionless form and their numerical solution is in progress.

The review confirms the good potential for the aforementioned facade technology and presents a good basis for future research on the mathematical modeling and accurate prediction of the thermal characteristics and flow behavior in the water flow glazing.
ACKNOWLEDGEMENTS

This work is dedicated to the 90th anniversary of Prof. Zapryan Zapryanov for his valuable contribution to the development of theoretical fluid mechanics in Bulgaria. Under his leadership, the Department of Fluid Mechanics at the Scientific Center for Mathematics and Mechanics has achieved great results in fluid mechanics and gas dynamics well-known in the scientific communities from the USA, Russia, France, Germany, Spain, Belgium, etc.

The work is supported by the National Science Fund of Bulgaria under the program “Fundamental research for junior researchers and postdocs - 2022”, grant: KII-06 IIM62/3/15.12.2022.

REFERENCES


[22] M. Rashevski, S. Slavtchev (2023) Natural and mixed convection in a vertical rectangular duct under solar radiation Submitted for publication.


