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FLUID MECHANICS

UNSTEADY POISEUILLE FLOW OF CARREAU–YASUDA FLUID IN A PIPE. SOLUTION EXISTENCE

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Abstract: The Poiseuille flows of non-Newtonian fluids may have complicated structures depending on the rheological characteristics of the fluid. The Carreau-Yasuda viscosity model is often used to describe the behaviour of many complex fluids. The Poiseuille flow of such fluids has many applications and occurs as a fundamental problem from a practical and theoretical point of view. At high shear rates, the flow may become unstable, then the Poiseuille flow problem may have no solution. In this paper, we prove the condition for the existence of its classical solution.

Key words: Carreau–Yasuda fluid, Poiseuille flow, classical solution, negative power index.

1 Introduction

Although the Newtonian fluid model describes well the behaviour of many fluids of common usages, such as water and air, it is not appropriate for polymer solutions, biofluids, polymer melts, suspensions, etc. The understanding of the dynamics of such fluids is of primary importance for many applications, for example, blood flow in arteries, plastic manufacturing, paint extrusion, nanomaterials depositions, food processing and others. Apart from their experimental study, also theoretical studies are performed to predict their behaviour during the different process cases. These studies rely on adequate rheological models describing the relationship between stress and shear rate.

The generalized Newtonian model can be successfully applied to a major part of the non-Newtonian fluids. Since it is built on a minor modification of the Newtonian

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constitutive equation for viscosity, it occurs useful for solving many practical problems of non-Newtonian flows. In particular, these models are nonlinear, such as the power law, the Carreau model and its generalization - the Carreau-Yasuda model, the Casson model, the Cross model, etc. [1, 2]. Although the power law model is very simple to obtain analytical solutions for fluid flows in different geometrical configurations, it has a limited application for shear-thinning fluids, for example. These fluids are characterized by a decreasing viscosity function of the shear rate with a plateau at a very small shear rate, which is impossible to be described by the power law model. Indeed, the Carreau four-parameter model or its modification - the Carreau-Yasuda model with five parameters is more suitable for further numerical calculations of many fluid flows (analytical solutions are difficult to be found). The Carreau-Yasuda model is given by the apparent viscosity function $\mu_{app}(\dot{\gamma})$:

$$\mu_{app} = \mu_\infty + (\mu_0 - \mu_\infty)[1 + \lambda^\alpha \dot{\gamma}^\alpha]^{(n-1)/\alpha}, \tag{1.1}$$

where $\dot{\gamma}$ is the shear rate, μ_0 is the viscosity at zero shear rate $\dot{\gamma} \to 0$, and μ_∞ is the viscosity at infinite shear rate $\dot{\gamma} \to \infty$, λ is the relaxation time, n is the behavior index and α is a dimensionless parameter ($\alpha = 2$ for the Carreau model). All these parameters are determined after fitting with the results from the rheological experiments.

According to the value of the power index n, the fluids can be grouped as shear thickening at $n > 1$ or shear thinning at $n < 1$. The Newtonian fluid corresponds to $n = 1$. Sometimes at high shear rates, the shear thinning fluids become unstable with a negative slope of stress, which corresponds to $n < 0$ in the Carreau-Yasuda model (1.1). This phenomenon is explained as a shear-banding, for example in [3–5].

The shear thinning fluids have been treated in our previous works as applications for the blood oscillatory flows in arteries, considered as straight channels [6] or pipes [7, 8]. The general unsteady flow of a shear-thinning fluid in a pipe was discussed in [9] at $0 < n < 1$ and in [10] for an arbitrary value of n. In [6, 7, 9] we used the Carreau model, while in [8, 10, 11] – the Carreau-Yasuda model. In these papers, the velocity and its gradient estimates are proved, as well as their differences from the corresponding Newtonian functions. We considered the cases of arbitrary n for unsteady [8, 10] and steady [11] flows and found that the unsteady problem becomes uniformly parabolic, nonuniformly parabolic, degenerate parabolic or backward parabolic depending on the value of n. For the steady case, the problem is elliptic and in [11] it is proved that has a classical solution if a necessary and sufficient condition is fulfilled, which depends on the model parameters.

Thus, the question of a classical solution existence for the general unsteady case (unsteady Poiseuille flow) is still an open problem, and we shall try to resolve it in the present paper. In some sense, this paper is a prolongation of our previous papers [8, 9] and [11]. We shall recall the existence theorem for the steady case from [11] as Theorem 2.1 in Section 2, while in Section 3 we shall prove the existence of a classical solution for the general unsteady case as Theorem 3.1.

2 PRELIMINARIES

The unsteady Poiseuille flow is considered in a pipe with radius R_p in cylindrical coordinates (x, r, φ), where $0 \le r \le R_p$ and $-\infty < x < \infty$. The flow is driven by an arbitrary pressure gradient $Af(t)$ along the pipe axis with a dimensional constant A and dimensionless function $f(t)$ of time t. The flow is supposed unsteady laminar with velocity vector $(v_x(t,r), 0, 0)$, which follows from the continuity and momentum equations. As a result, the axial velocity $v_x(t,r)$ is subjected to the equation:

$$\rho \frac{\partial v_x}{\partial t} = Af(t) + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu_{app}\left(\dot{\gamma}\right) r \frac{\partial v_x}{\partial r}\right], \tag{2.1}$$

where ρ is the density and the apparent viscosity function $\mu_{app}(\dot{\gamma})$ is given by (1.1), with $\dot{\gamma} = \dfrac{\partial v_x}{\partial r}$.

The initial condition as a smooth function of r, and the boundary conditions will be given in their dimensionless form in eq. (2.3).

Using H as a characteristic length ($r = HY$, H can be the pipe radius R_p or some other appropriate length), t_0 as a characteristic time ($t = t_0 T$) and $B = AH^2/\mu_0$ as a characteristic velocity ($v_x = BU$), the dimensionless form of eq. (2.1) becomes

$$P_0 U = 8\beta^2 U_T - \frac{1}{Y}\frac{\partial}{\partial Y}\left\{\left[1 - c + c(1 + Cu^{\alpha} \mid U_Y \mid^{\alpha})^{\frac{n-1}{\alpha}}\right] Y U_Y\right\} = f(T), \tag{2.2}$$
$$\text{in} \quad Q = \{(T, Y); T > 0, Y \in (0, R)\}$$

with boundary and initial conditions,

$$U_Y(T, 0) = U(T, R) = 0 \ \text{ for } \ T \ge 0, \quad U(0, Y) = \Psi(Y) \ \text{ for } \ Y \in [0, R], \tag{2.3}$$

where $R = \dfrac{R_p}{H}$ is the dimensionless pipe radius; $8\beta^2 = ReSt$, where Re is the Reynolds number, $Re = \dfrac{\rho B H}{\mu_0}$ and St is the Strouhal number, $St = \dfrac{H}{B\tau}$; $Cu = \dfrac{\lambda B}{H} \ge 0$ is the Carreau number (Weissenberg number) [2]; $c \in [0, 1)$ is the viscosity ratio: $c = 1 - \dfrac{\mu_\infty}{\mu_0}$; $\alpha > 0$ and $n \in \mathbb{R}$ are empirically determined constants. We suppose that $\Psi(Y) \in C^4([0, R])$ satisfies the compatibility conditions,

$$\Psi'(0) = \Psi(R) = 0 \quad \Psi'(R) = 0, \quad \Psi''(R) - f(0) = 0 \tag{2.4}$$

and

(2.5) $$f(T) \in C^2([0,\infty)), \quad |f(T)| \le f_0, \quad |f'(T)| \le f_1 \quad \text{for } T \ge 0, \\ f(T) \not\equiv 0, \quad f_0, f_1 \quad \text{are positive constants.}$$

For the function

(2.6) $$\begin{aligned}\Phi(\eta) &= (1-n)\,(1+Cu^\alpha\eta^\alpha)^{\frac{n-1-\alpha}{\alpha}} + n\,(1+Cu^\alpha\eta^\alpha)^{\frac{n-1}{\alpha}} \\ &= (1+nCu^\alpha\eta^\alpha)\,(1+Cu^\alpha\eta^\alpha)^{\frac{n-1-\alpha}{\alpha}} \quad \text{for} \quad \eta \ge 0\,,\end{aligned}$$

eq. (2.2) in non-divergent form is

(2.7) $$\begin{aligned}P_0U = 8\beta^2 U_T - [1-c+c\Phi(|\,U_Y\,|)]\,U_{YY} \\ -\frac{1}{Y}\left[1-c+c\,(1+Cu^\alpha\,|\,U_Y\,|^\alpha)^{\frac{n-1}{\alpha}}\right]U_Y = f(T) \quad \text{in} \quad Q\,.\end{aligned}$$

The eq. (2.2) is not uniformly parabolic for $c=1$ and the problem (2.2) – (2.5) has not a global classical solution for $T \in [0,\infty)$ [12]. The type of eq. (2.7) is different for different values of n and Cu, when $\alpha > 0$. More precisely [10]:

(2.8) $$\text{for} \quad n > 1, Cu \ne 0, c \in (0,1)\,,$$

eq. (2.7) is singular, strictly nonuniformly parabolic equation, because

$$0 < 1-c \le 1-c+c\Phi(\eta) \le 1-c+cn\,(1+Cu^\alpha\eta^\alpha)^{\frac{n-1}{\alpha}}\,;$$

(2.9) $$\text{for} \quad n\in[0,1], Cu>0, c\in(0,1) \quad \text{or} \quad n\in\mathbb{R}, Cu=0, c\in(0,1) \\ \text{or} \quad n\in\mathbb{R}, Cu\ge 0, c=0\,,$$

eq. (2.7) is singular, uniformly parabolic equation, because

$$\begin{aligned} &1-c+c\Phi(\eta) \equiv 1 \quad \text{for} \quad Cu=0 \quad \text{or} \quad c=0, \\ \text{while} \quad &0<1-c\le 1-c+c\Phi(\eta) \le 1-c+c\,(1+nCu^\alpha\eta^\alpha) \\ &\times(1+Cu^\alpha\eta^\alpha)^{\frac{n-1-\alpha}{\alpha}} \le 1-c+cn \le 1 \\ \text{for} \quad &n\in[0,1], Cu\ge 0, c\in(0,1) \quad \text{and every} \quad \eta\ge 0;\end{aligned}$$

(2.10) $$\text{for} \quad n<0, Cu\ne 0, c\in(0,1) \quad \text{and}$$

(2.10a) $$\alpha\left(1-\frac{\alpha+1}{n}\right)^{\frac{n-1}{\alpha}} \le \frac{1-c}{c}$$

eq. (2.7) is singular, degenerate parabolic equation, because

$$0 \le 1 - c + \alpha c \left(1 - \frac{\alpha+1}{n}\right)^{\frac{n-1}{\alpha}} \le 1 - c + c\Phi(\eta) \le 1 \quad \text{for} \quad \eta \ge 0;$$

or

$$\alpha \left(1 - \frac{\alpha+1}{n}\right)^{\frac{n-1-\alpha}{\alpha}} > \frac{1-c}{c} \tag{2.10b}$$

eq. (2.7) is singular, degenerate parabolic equation for $\eta \in [0, \eta_1] \cup [\eta_2, \infty)$ and singular, backward parabolic one for $\eta \in (\eta_1, \eta_2)$.

Here $0 < \eta_1 < \eta_2 < \infty$ are the roots of the equation

$$h(\eta) = 1 - c + c\Phi(\eta). \tag{2.11}$$

In order to prove the classical solvability of the problem (2.2)–(2.5), we regularized equation (2.2), and hence equation (2.7), with a small positive parameter $\varepsilon \in [0, \varepsilon_0], \varepsilon_0 \ll R$, i.e.

$$\begin{aligned} (2.12)\quad P_\varepsilon(U^\varepsilon) = 8\beta^2 U^\varepsilon_T - [1 - c + c\Phi\left(\mid U^\varepsilon_Y \mid\right)] U^\varepsilon_{YY} \\ - \frac{1}{Y+\varepsilon}\left[1 - c + c\left(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha\right)^{\frac{n-1}{\alpha}}\right] U^\varepsilon_Y = f(T) \quad \text{in} \quad Q\,, \\ U^\varepsilon_Y(T,0) = U^\varepsilon(T,R) = 0 \ \text{ for } \ T \ge 0\,, \quad U^\varepsilon(0,Y) = \Psi(Y) \ \text{ for } \ Y \in [0,R] \\ \text{and} \quad \varepsilon \in (0, \varepsilon_0], \quad \varepsilon_0 \ll R\,. \end{aligned}$$

Further on, we will prove apriori estimates for $U^\varepsilon_Y(T,Y)$ and its derivatives for every $\varepsilon \in (0, \varepsilon_0]$ at $\varepsilon_0 \ll R$ with constants independent of ε. Thus after the limit $\varepsilon \to 0$, we get the solvability of the problem (2.2)–(2.5). For this purpose we need the following results for the steady Poiseuille flow of Carreau-Yasuda fluid given in [11], i.e.

$$\begin{aligned} (2.13)\qquad LV^\varepsilon = \frac{1}{Y+\varepsilon}\frac{\partial}{\partial Y}\left\{\left[1 - c + c\left(1 + Cu^\alpha \mid V^\varepsilon_Y \mid^\alpha\right)^{\frac{n-1}{\alpha}}\right](Y+\varepsilon)V^\varepsilon_Y\right\} \\ = [1 - c + c\Phi\left(\mid V^\varepsilon_Y \mid\right)] V^\varepsilon_{YY} - \frac{1}{Y+\varepsilon}\left[1 - c + c\left(1 + Cu^\alpha \mid V^\varepsilon_Y \mid^\alpha\right)^{\frac{n-1}{\alpha}}\right] V^\varepsilon_Y = f_0. \end{aligned}$$

Theorem 2.1. *Suppose $\alpha > 0$, $Cu > 0$, $c \in (0,1)$, $n < 0$. The problem (2.13) has a unique classical solution $V^\varepsilon(Y) \in C^2([0,R])$ if:*

$$\left(1 - \frac{\alpha+1}{n}\right)^{\frac{n-1-\alpha}{\alpha}} < \frac{1-c}{\alpha c}; \tag{2.14}$$

or

$$\left(1-\frac{\alpha+1}{n}\right)^{\frac{n-1-\alpha}{\alpha}}=\frac{1-c}{\alpha c}\quad \textit{and}$$

$$\left[1-c+c\left(1-\frac{\alpha+1}{n}\right)^{\frac{n-1}{\alpha}}\right]Cu^{-1}\left(-\frac{\alpha+1}{n}\right)^{\frac{1}{\alpha}}<\frac{f_0(R+\varepsilon)}{2};\tag{2.15}$$

or

$$\left(1-\frac{\alpha+1}{n}\right)^{\frac{n-1-\alpha}{\alpha}}>\frac{1-c}{\alpha c}\quad \textit{and}$$

$$\frac{f_0(R+\varepsilon)}{2}<\left[1-c+c\left(1+Cu^{\alpha}\eta^{\alpha}\right)^{\frac{n-1}{\alpha}}\right]\eta_1,\tag{2.16}$$

where η_1 *is the first positive root of* (2.11)*. Moreover,*

$$V^{\varepsilon}(Y)=-\int_Y^R F^{-1}\left(\frac{f_0(s+\varepsilon_0)}{2}\right)ds\quad \textit{satisfies the estimate}\tag{2.17}$$

$$0\leq V^{\varepsilon}_Y(Y)\leq F^{-1}\left(\frac{f_0(R+\varepsilon_0)}{2}\right)\quad \textit{for}\quad Y\in[0,R],\varepsilon\in(0,\varepsilon_0)\tag{2.18}$$

and $F^{-1}(\zeta)$ *is the inverse function of*

$$F(\zeta)=\left[1-c+c\left(1+Cu^{\alpha}\eta^{\alpha}\right)^{\frac{n-1}{\alpha}}\right]\zeta\quad \textit{for}\quad \zeta\geq 0.\tag{2.19}$$

The proof of Theorem 2.1 is identical with the proof of the Theorems 2 – 4 in [11] and will be omitted here.

3 Apriori Estimates for $U^{\varepsilon}_Y(T,Y)$

In this section we prove apriori estimates for the solution $U^{\varepsilon}_Y(T,Y)$ of (2.12), (2.4) and for their derivatives with constants independent of ε.

Lemma 3.1. *Suppose* $U^{\varepsilon}_Y(T,Y)\in C^2(Q)\cap C^1(\overline{Q})$ *is a solution of* (2.12)*,* (2.4)*,* $\alpha>0$*,* $c\in[0,1)$ *and one of the conditions* (2.8) *or* (2.9) *holds. Then the estimates*

$$|U^{\varepsilon}(T,Y)|\leq K_1\left(R^2-Y^2\right)\leq K_1R^2,\tag{3.1}$$

$$|U^{\varepsilon}_Y(T,R)|\leq 2K_1R\tag{3.2}$$

are satisfied for $T\geq 0$*,* $Y\in[0,R]$*, where*

$$K_1=\max\left\{\frac{|\,\Psi(Y)\,|}{R^2-Y^2},\frac{f_0}{2(1-c)}\right\}.\tag{3.3}$$

Proof. The operator

$$(3.4)\quad PW = 8\beta^2 W_T - [1 - c + c\Phi(| U_Y^\varepsilon |)]\, W_{YY} - \frac{1}{Y+\varepsilon}\left[1 - c + c\,(1 + Cu^\alpha \mid U_Y^\varepsilon \mid^\alpha)^{\frac{n-1}{\alpha}}\right] W_Y$$

is strictly parabolic one under conditions (2.8) or (2.9). For the function $H(T,Y) = K_1(R^2 - Y^2)$ we get from (2.8) or (2.9) and (3.3) the estimate

$$PH = 2K_1\,[1 - c + c\Phi\,(| U_Y^\varepsilon |)] + \frac{2K_1 Y}{Y+\varepsilon}\left[1 - c + c\,(1 + Cu^\alpha \mid U_Y^\varepsilon \mid^\alpha)^{\frac{n-1}{\alpha}}\right] \geq 2K_1(1-c) \geq f_0 \geq f(T) = PU^\varepsilon \quad \text{for} \quad (T,Y) \in Q.$$

Since $H(0,Y) - U^\varepsilon(0,Y) = K_1(R^2 - Y^2) - \Psi(Y) \geq 0$ for $Y \in [0,R]$, $H(T,R) - U^\varepsilon(T,R) = 0$ and $H_Y(T,0) - U_Y^\varepsilon(T,0) = 0$ for $T \geq 0$, from the interior and strong maximum principle (see [13], [14]), it follows that $H(T,Y) \geq U^\varepsilon(T,Y)$ for $T \geq 0$, $Y \in [0,R]$. The estimate from below for $U^\varepsilon(T,Y)$ holds by means of the auxiliary function $-H(T,Y)$ and (3.1) is proved.

The estimate (3.2) follows trivially from (3.1). □

Lemma 3.2. *Suppose $U_Y^\varepsilon(T,Y) \in C^2(\overline{Q})$ is a solution of* (2.12)*,* (2.4)*, $\alpha > 0$, $Cu > 0$, $c \in (0,1)$,*

$$(3.5)\qquad | \Psi(Y) | < -V^0(Y) \quad \textit{for} \quad Y \in [0,R],$$

*where $V^0(Y)$ is defined in **Theorem 2.1** for $\varepsilon = 0$ and one of the conditions* (2.8) *or* (2.9) *holds. Then the estimates*

$$(3.6)\qquad | U^\varepsilon(T,Y) | \leq (R - Y)F^{-1}\Big(\frac{f_0(R+\varepsilon_0)}{2}\Big),$$

$$(3.7)\qquad | U_Y^\varepsilon(T,R) | \leq F^{-1}\Big(\frac{f_0(R+\varepsilon_0)}{2}\Big)$$

are satisfied for $T \geq 0$, $Y \in [0,R]$, $\varepsilon \in [0,\varepsilon_0]$ and the function F is defined in (2.19).

Proof. For the function $V_Y^\varepsilon(T,Y)$ defined in **Theorem 2.1**, simple computations give us

$$(3.8)\quad PV^\varepsilon = -\, f_0 - cV_{YY}^\varepsilon\,[\Phi(| U_Y^\varepsilon |) - \Phi(| V_Y^\varepsilon |)] - \frac{c}{Y+\varepsilon}\Big[(1 + Cu^\alpha \mid U_Y^\varepsilon \mid^\alpha)^{\frac{n-1}{\alpha}} - (1 + Cu^\alpha \mid V_Y^\varepsilon \mid^\alpha)^{\frac{n-1}{\alpha}}\Big]V_Y^\varepsilon = -\, f_0 - cA_1\left[(U_Y^\varepsilon)^2 - (V_Y^\varepsilon)^2\right],$$

where

$$A_1 = \frac{1}{2} C u^\alpha \int_0^1 \left[\theta (U_Y^\varepsilon)^2 + (1-\theta)(V_Y^\varepsilon)^2\right]^{\frac{\alpha-2}{2}} d\theta \Big\{ n(n-1) \left(U_{YY}^\varepsilon + \frac{U_Y^\varepsilon}{Y+\varepsilon} \right)$$
$$\times \int_0^1 \left[\tau \left(1 + C u^\alpha \mid U_Y^\varepsilon \mid^\alpha\right) + (1-\tau)\left(1 + C u^\alpha \mid V_Y^\varepsilon \mid^\alpha\right)\right]^{\frac{n-1-\alpha}{\alpha}} d\tau$$
$$+ V_{YY}^\varepsilon (1-n)(n-1-\alpha) \int_0^1 \left[\tau \left(1 + C u^\alpha \mid U_Y^\varepsilon \mid^\alpha\right)\right.$$
$$\left. + (1-\tau)\left(1 + C u^\alpha \mid V_Y^\varepsilon \mid^\alpha\right)\right]^{\frac{n-1-2\alpha}{\alpha}} d\tau \Big\},$$

The function $Z^\varepsilon(T,Y) = U^\varepsilon(T,Y) + V^\varepsilon(T,Y)$ satisfies the problem

$$\begin{aligned} &PZ^\varepsilon(T,Y) - cA_1 \left(U_Y^\varepsilon - V_Y^\varepsilon\right) Z_Y^\varepsilon = -f_0 + f(T) \le 0 \quad \text{in Q} \\ &Z_Y^\varepsilon(T,0) = Z^\varepsilon(T,R) = 0 \quad \text{for} \quad T \ge 0, \\ &\text{and} \quad Z^\varepsilon(0,Y) = \Psi(Y) + V^\varepsilon(Y) \le 0 \quad \text{for} \quad Y \in [0,R]. \end{aligned} \tag{3.9}$$

From the interior and strong maximum principle it follows that $Z^\varepsilon(T,Y)$ has no positive maximum in $\overline{Q}$, i.e., $Z^\varepsilon(T,Y) \le 0$ and hence $U^\varepsilon(T,Y) \le -V^\varepsilon(T,Y)$ for $T \ge 0, Y \in [0,R]$. In the same way the function $Z_1^\varepsilon(T,Y) = U^\varepsilon(T,Y) - V^\varepsilon(T,Y)$ has no negative minimum in $\overline{Q}$, i.e., $Z_1^\varepsilon(T,Y) \ge 0$ and hence $U^\varepsilon(T,Y) \ge V^\varepsilon(T,Y)$ for $T \ge 0, Y \in [0,R]$. Then $\mid U^\varepsilon(T,Y) \mid \le -V^\varepsilon(T,Y)$.

The estimate (3.6) follows from (2.17) and the monotonicity of $F^{-1}(\zeta)$, while (3.7) is a trivial consequence of (3.6). □

Lemma 3.3. *Suppose $U_Y^\varepsilon(T,Y) \in C^3(Q) \cap C^2(\overline{Q})$ is a solution of (2.12), (2.4), $\alpha > 0$. If:*

(i) $c \in [0,1)$ and (2.8) or (2.9) holds;

or

(ii) $c \in (0,1)$, $Cu > 0$, $n < 0$, (3.5) and one of the conditions (2.14), or (2.15) or (2.16) with

$$\sup_{Y \in [0,R]} \mid \Psi'(Y) \mid < \eta_1 \tag{3.10}$$

is satisfied, then the estimate

$$\mid U_Y^\varepsilon(T,Y) \mid \le K_2 \quad \textit{for} \quad T \ge 0, Y \in [0,R], \varepsilon \in (0,\varepsilon_0) \quad \textit{holds} \tag{3.11}$$

where

$$K_2 = \max\Big\{ 2K_1 R, \sup_{Y \in [0,R]} \mid \Psi'(Y) \mid \Big\} \quad \textit{in case (i) and}$$

$$K_2 = \max\left\{F^{-1}\left(\frac{f_0(R+\varepsilon_0)}{2}\right), \sup_{Y\in[0,R]} \mid \Psi'(Y) \mid\right\} \quad \textit{in case (ii).}$$

Proof. Differentiating (2.12) with respect to Y we get that $Z^\varepsilon(T,Y) = U^\varepsilon_Y(T,Y)$ satisfies the problem

$$\text{(3.12)} \qquad RZ^\varepsilon = 0 \quad \text{in} \quad Q, \quad Z^\varepsilon(0,Y) = \Psi'(Y) \quad \text{for} \quad Y \in [0,R]$$
$$Z^\varepsilon(T,0) = 0 \quad \text{for} \quad T \geq 0,$$

where

$$\text{(3.13)} \qquad RW = 8\beta^2 W_T - [1 - c + c\Phi(|U^\varepsilon_Y|)]\, W_{YY} - A_2 W_Y + \frac{1}{(Y+\varepsilon)^2}\left[1 - c + c\,(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)^{\frac{n-1}{\alpha}}\right] W, \quad \text{in} \quad Q\,,$$

$$\text{(3.14)} \qquad A_2 = (n-1)Cu^\alpha U^\varepsilon_{YY} U^\varepsilon_Y \mid U^\varepsilon_Y \mid^{\alpha-2} \left[(\alpha + 1 + nCu^\alpha \mid U^\varepsilon_Y \mid^\alpha)\, U^\varepsilon_{YY} + \frac{1}{Y+\varepsilon}\,(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)\right](1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)^{\frac{n-1-2\alpha}{\alpha}} \; .$$

From the interior and the strong maximum principle, $Z^\varepsilon(T,Y)$ attains its positive maximum and negative minimum in the part of the parabolic boundary, i.e., on $\Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{(0,Y), Y \in [0,R]\}$ and $\Gamma_2 = \{(T,R), T \geq 0\}$. The estimate (3.11) follows from the choice of K_2 and (3.2) in the case (i) and (3.7) in the case (ii). □

Lemma 3.4. *Suppose* $U^\varepsilon_Y(T,Y) \in C^3(Q) \cap C^2(\overline{Q})$ *is a solution of* (2.12), (2.4). *Under the assumptions of Lemma 3.3 the estimate*

$$\text{(3.15)} \qquad \mid U^\varepsilon_T(T,Y) \mid \leq K_3 \exp(T) \quad \textit{for} \quad T \geq 0, Y \in [0,R] \quad \textit{holds}$$

where

$$\text{(3.16)} \qquad K_3 = \max\left\{\sup_{T\geq 0} \mid f'(T) \mid, \frac{1}{8\beta^2}\left[\sup_{Y\in[0,R]} |[1 - c + c\Phi(\mid \Psi'(Y) \mid^\alpha)]\Psi''(Y)| + \sup_{Y\in[0,R]}\left|\left[1 - c + c(1 + Cu^\alpha \mid \Psi'(Y) \mid^\alpha)^{\frac{n-1}{\alpha}}\right]\frac{\mid \Psi'(Y) \mid}{Y}\right|\right]\right\}.$$

Proof. The function $Z^\varepsilon(T,Y) = U^\varepsilon_T(T,Y)$ is a solution of the problem

$$\text{(3.17)} \qquad R_1 Z^\varepsilon = f'(T) \quad \text{in} \quad Q, \quad Z^\varepsilon_Y(T,0) = 0, \quad Z^\varepsilon(T,R) = 0 \quad \text{for} \quad T \geq 0,$$

$$Z^\varepsilon(0,Y) = \frac{1}{8\beta^2}\Big\{ \big(1 - c + c\Phi(\mid \Psi'(Y) \mid^\alpha\big)\, \Psi''(Y) + \frac{1}{Y+\varepsilon}\Big[1 - c + c\big(1 + Cu^\alpha \mid \Psi'(Y) \mid^\alpha\big)^{\frac{n-1}{\alpha}}\Big]\, \Psi'(Y)\Big\} \text{ for } Y \in [0,R],$$

where

$$R_1 W = 8\beta^2 W_T - [1 - c + c\Phi(|U^\varepsilon_Y|)]\, W_{YY} - A_3 W_Y \tag{3.18}$$

and

$$A_3 = (n-1)Cu^\alpha U^\varepsilon_Y \mid U^\varepsilon_Y \mid^{\alpha-2} \Big[\alpha + 1 + nCu^\alpha \mid U^\varepsilon_Y \mid^\alpha + \frac{1}{Y+\varepsilon}\,(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)\Big]\,(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)^{\frac{n-1-2\alpha}{\alpha}} . \tag{3.19}$$

Hence the function $Z^\varepsilon_1(T,Y) = U^\varepsilon_T(T,Y) - K_3\exp(T)$ satisfies the problem

$$R_1 Z^\varepsilon_1 = f'(T) - 8\beta^2 K_3 \exp(T) \le 0 \quad \text{in} \quad Q\,, \tag{3.20}$$

$$(Z^\varepsilon_1)_Y(T,0) = 0, \quad Z^\varepsilon_1(T,R) = -K_3\exp(T) \le 0 \quad \text{for} \quad T \ge 0, \quad \text{and} \quad Z^\varepsilon_1(0,Y) = Z^\varepsilon(0,Y) - K_3 \le 0 \quad \text{for} \quad Y \in [0,R] \tag{3.21}$$

from the choice of K_3.

From lemma 3.3 the estimate (3.11) holds and the equation (3.18) is parabolic one. The estimate $U^\varepsilon_T(T,Y) \le K_3\exp(T)$ follows from the interior and the strong boundary maximum principle and eq. (3.21). By means of the function $Z^\varepsilon(T,Y) + K_3\exp(T)$, in the same way, the opposite inequality $U^\varepsilon_T(T,Y) \ge -K_3\exp(T)$ holds, which proves (3.15). □

Lemma 3.5. *Under the assumptions of Lemma 3.3 the estimate*

$$\Big[1 - c + c(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)\Big]^{\frac{n-1}{\alpha}} \mid U^\varepsilon_Y(T,Y) \mid \le K_4(Y+\varepsilon) \tag{3.22}$$

holds for $T \ge 0, \quad Y \in [0,R],$ *where*

$$K_4 = 4\beta^2 K_3 \exp(T) + \frac{1}{2} f_0\,. \tag{3.23}$$

Proof. Integrating the regularized equation (2.2), i.e.,

$$P_\varepsilon U^\varepsilon = 8\beta^2 U^\varepsilon_T - \frac{1}{Y+\varepsilon}\frac{\partial}{\partial Y}\Big\{\Big[1 - c + c\,(1 + Cu^\alpha \mid U^\varepsilon_Y \mid^\alpha)^{\frac{n-1}{\alpha}}\Big]\,(Y+\varepsilon)U^\varepsilon_Y\Big\} = f(T) \tag{3.24}$$

from 0 to $Y \in (0, R]$ we get the estimate

$$\left|(Y+\varepsilon)\,[1-c+c(1+Cu^{\alpha} \mid U_Y^{\varepsilon} \mid^{\alpha})]^{\frac{n-1}{\alpha}}\, U_Y^{\varepsilon}(T,Y)\right|$$
$$= \left| \int_0^Y (s+\varepsilon)\left[8\beta^2 U_T^{\varepsilon}(T,s) - f(T)\right] ds \right| \le \frac{1}{2}\left[8\beta^2 K_3 \exp(T) + f_0\right](Y+\varepsilon)^2,$$

which proves (3.22). □

Lemma 3.6. *Under the assumptions of Lemma 3.3 we obtain the estimate*

$$\mid U_{YY}^{\varepsilon}(T,Y) \mid \le K_5 \quad \textit{for} \quad T \ge 0, \quad Y \in [0,R], \quad \textit{where} \tag{3.25}$$

$$K_5 = \left[8\beta^2 K_3 \exp(T) + K_4 + f_0\right][1-c+c\Phi(K_2)]^{-1}. \tag{3.26}$$

Proof. The estimate (3.25) follows from (2.2), (3.11), (3.15), (3.22). □

Lemma 3.7. *Under the assumptions of Lemma 3.3 the estimates*

$$\left| \frac{\partial^{\gamma}}{\partial Y} \frac{\partial^{\mu}}{\partial T} U^{\varepsilon}(T,Y) \right| \le K_6 \exp(T), \tag{3.27}$$

hold for $1 \le \gamma + \mu \le 3$, $T \ge 0$, $Y \in [\delta, R]$, $0 < \delta < R$, where the constant K_6 is independent of ε.

Proof. The proof of (3.27) follows from the Schauder estimates [15], [14] for the equation (2.2) and Lemmas 3.1– 3.6. □

Theorem 3.1. *Suppose $\alpha > 0$. If*

(i) $c \in [0,1)$ and (2.8) or (2.9) hold;

or

(ii) $c \in (0,1)$, $Cu > 0$, $n < 0$, (3.5) and one of the conditions (2.14), or (2.15), or (2.16) together with (3.10) is satisfied.

Then the problem (2.2) - (2.5) has a unique classical solution $U(T,Y) \in C^2(Q_0) \cap C^1(\overline{Q}_0)$, $Q_0 = \left\{(T,Y); T \in (0,T_0), Y \in (0,R)\right\}$ for every $T_0 < \infty$.

Proof. From Lemma 3.3 the equation (2.12) is uniformly parabolic in $\overline{Q}$. By means of the continuity on the parameter $\kappa \in [0,1]$ of the problem

$$S_{\kappa}(U^{\varepsilon}) = \kappa P_{\varepsilon}(U^{\varepsilon}) + (1-\kappa)\left[8\beta^2 U_T^{\varepsilon} - U_{YY}^{\varepsilon}\right] = f(T) \quad \text{in} \quad Q \tag{3.28}$$

with boundary conditions given in (2.12) and Schauder apriori estimates [15], [14], it follows that the problem (2.12) has a unique classical solution $U^{\varepsilon}(T,Y) \in C^3(Q) \cap$

$C^2(\overline{Q})$. Indeed the set $\Omega = \Big\{\kappa \in [0,1]\Big\}$, for which (3.28) has a classical solution is a relatively open set from the theory of the inverse functions and closed set from the apriori estimates in Lemmas 3.1-3.7. Hence $\Omega \equiv [0,1]$ and from the solvability of (3.28) for $\kappa = 0$, it follows its solvability for $\kappa = 1$.

From Lemma 3.7 the sequence in ε of

$$\{U^\varepsilon(T,Y)\}, \quad \{U_Y^\varepsilon(T,Y)\}, \quad \{U_T^\varepsilon(T,Y)\}, \quad \{U_{YY}^\varepsilon(T,Y)\} \quad \text{for} \quad \varepsilon \in (0,\varepsilon_0]$$

is equicontinuous and uniformly bounded for $T \in [0,T_0]$, $Y \in [\delta,R]$, $0<\delta<R$. Moreover, $\{U^\varepsilon(T,Y)\}$, $\{U_Y^\varepsilon(T,Y)\}$ are equicontinuous and uniformly bounded in $\overline{Q_0}$ with constant independent of ε. By means of the Arzela-Ascoli theorem [16] and diagonalization argument there exists a subsequence $\{U^{\varepsilon_i}(T,Y)\}$, $\varepsilon_i \to 0$, which converges to the desired solution of (2.2) - (2.5) for $\varepsilon_i \to 0$. □

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