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FLUID MECHANICS

NOTES ON THE RESEARCH AT THE DEPARTMENT OF FLUID MECHANICS OF THE INSTITUTE OF MECHANICS OF THE BULGARIAN ACADEMY OF SCIENCES

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ABSTRACT: In this article, we present several remarks on selected topics of research in fluid mechanics at the Institute of Mechanics of the Bulgarian Academy of Sciences. A large part of the discussed results are from the years when Professor Stefan Radev was the Chair of the Department of Fluid Mechanics at the Institute of Mechanics. The topics include the optimum theory of turbulence, methodology for solving differential equations related to the motion of waves in fluids, and the application of time series analysis to problems in environmental fluid mechanics. Special attention is given to Professor Radev's participation in this research.

KEY WORDS: Optimum theory of turbulence, Simple Equations Method, Water waves, Blood flow, Extreme water levels, Fluid flow in window systems.

1 INTRODUCTION

This short review is dedicated to the 80th anniversary of Professor Stefan Radev, a corresponding member of the Bulgarian Academy of Sciences. It presents, very briefly, a part of the research conducted at the Department of Fluid Mechanics of the Institute of Mechanics of the Bulgarian Academy of Sciences, namely the research devoted to the optimum theory of turbulence, research on the nonlinear waves in fluids, as well as a small part of the research on the time series analysis which is connected to fluid systems. We begin the review with the results from the research on the optimum theory of turbulence. This variational theory is based on solutions of the Euler-Lagrange equations obtained from variational problems based on the Navier-Stokes equations, Boussinesq equations, etc. The description of this research is in Section 2 of the article. In Section 3, we describe briefly a method for obtaining exact solutions of nonlinear differential equations: Simple Equations Method (SEsM).

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The method is developed at the Department of Fluid Mechanics of the Institute of Mechanics. We mention several applications of SEsM for obtaining exact solutions of nonlinear differential equations, some of which are connected to nonlinear water waves. In Section 4, we discuss part of the research on time series from natural and technological systems connected to fluid mechanics. In Section 5, we mention the joint book of Professor Radev and Professor Vitanov. Several concluding remarks are summarized in Section 6.

2 Research on the Optimum theory of Turbulence

Variational methods are popular in the fluid mechanics of turbulent flows, where the extremum of some quantity can give information about the observed flow. In the optimum theory of turbulence, instead of dealing directly with the Navier-Stokes equations, one constructs a variational problem on the basis of several integral relations derived from the equations of motion. Thus, all velocity fields that satisfy the equations of fluid motion are included in the class of fields among which we seek those leading to extremal values of some quantities of interest, such as the convective heat transport. Some "unphysical" fields can also satisfy these integral relations (plus possibly extra constraints). Because of this, the obtained extremal values of the flow quantities are considered as bounds on these turbulent flow quantities. Additional integral relations can be added in order to tighten the bounds by excluding some undesired fields [1].

There are two methods in the optimum theory of turbulence. The classical method is the Howard-Busse method [2, 3]. In 1963, Howard obtained the first upper bounds on the transport of heat by convection through a variational problem based on two integral relations (power integrals) obtained from the Boussinesq equations. Howard showed that the addition of more constraints to the variational problem can lower the upper bound (for an example, see [4]). The main statement of the optimum theory of turbulence is that the value of the corresponding integral property of the turbulent flow cannot be higher than the value obtained by applying the methods of the optimum theory of turbulence.

The solution of Howard was called the $1-\alpha$-solution of the Euler-Lagrange equations of the variational problem. In 1969, Busse constructed the multi$-\alpha$ solutions of the variational problem of Howard. Each of these solutions gives upper bounds on the convective heat transport in some interval of Rayleigh numbers. For very large values of the Rayleigh number R, the bound on the Nusselt number is given by the $N-\alpha$-solution of the variational problem with $N \to \infty$. This bound is $Nu \propto R^{1/2}$.

The second method of the optimum theory of turbulence was proposed by Doering and Constantin in 1992 [5]. The idea here is to decompose the velocity field into a steady background field carrying the inhomogeneous boundary conditions, and a field of homogeneous fluctuations. The background field satisfies certain spectral constraints, and an appropriate choice of this field leads quickly to estimates of bounds on the turbulent quantity under study.

The Howard-Busse method was further refined by Chan [6], and an energy balance parameter modification of the Doering-Constantin method was proposed by Grossmann, Nicodemus, and Holthaus [7]. The optimum theory of turbulence has applications, such as transport in fluid layers [2, 3, 8], particularly [24], heat transport in rotating and porous layers [25, 28], energy dissipation in shear flows [29, 34], upper bound for plasma turbulence, and other applications of the optimum theory of turbulence [35–37].

Several useful reviews on the optimum theory of turbulence are [38–40]. Below, we mention some results obtained in the years when Professor Stefan Radev was the Head of the Department of Fluid Mechanics at the Institute of Mechanics of the Bulgarian Academy of Sciences. The results were mainly obtained by Professor Vitanov. He started research on the optimum theory of turbulence in Germany at the research group of Professor F. H. Busse. The first problem to be solved there was to obtain numerical upper bounds on turbulent thermal convection in a layer of fluid, heated from below, and in the presence of stress-free boundary conditions. The results were published in [8]. In the corresponding figures, one can observe the thinning of the thermal and viscous boundary layers connected to the optimal fields for velocity and temperature in the fluid layer. On the basis of the numerical simulations which have been carried out up to Rayleigh numbers of 2.5×10^7, one can conclude that the asymptotic power law for the dependence of the Nusselt number Nu on the Rayleigh number is about $Nu \propto R^{1/2}$. Another interesting conclusion was that the existing theory for the case of rigid boundaries would not be applicable to the case of stress-free boundaries.

The next problem to be solved was the upper bound on the convective heat transport in a horizontal layer of fluid in the presence of stress-free boundary conditions and very large values of the Prandtl number (cases where the Prandtl number can be treated as infinite). The problem was attacked numerically, and then on the basis of the obtained numerical results a theory was built for the case of the solution of the variational problem with a single wave number ($1-\alpha$-solution of the variational problem within the scope of the Howard-Busse methodology) [9]. The results from the numerical study showed that the upper bound on the Nusselt number given by the $1-\alpha$-solution of the variational problem was $Nu \approx 0.3211R^{1/3}$. The analytical theory led to the result $Nu = 0.3254R^{1/3}$, which was in very good agreement with the numerical result. The wave number connected to the optimum fields followed the asymptotic law $\alpha = 0.2011R^{1/6}$, which differed from the case of the rigid boundaries where the corresponding power law was $\alpha = 0.5266R^{1/4}$.

The first use of the multi-α solutions of a variational problem in the optimum theory of turbulence was for the problem of obtaining bounds on convection in a fluid-saturated porous medium. The results were published in [4]. The main results are as follows: for the parameter α of the $1-\alpha$ solution of the variational problem, one obtains the relation $\alpha = \left(\frac{1}{5}\right)^{1/2} R^{1/2}$. For the upper bound on the convective heat transport through the porous layer, one obtains

$$F = \left(\frac{2}{\sigma+\tau}\right)^{4/3} 3^{-1/3} \left(\frac{1}{5}\right)^{5/3} R^{2/3} (\ln R)^{1/3} \left(\frac{1}{1+r}\right)^{1/3}, \tag{1}$$

where

$$r = \frac{3\left\{\ln\left[\frac{A^{2/3}}{(\ln R)^{2/3}}\right] + \ln\ln\left[\frac{R^{1/3}(\ln R)^{2/3}}{A^{2/3}}\right]\right\}}{\ln(R) - 3\left\{\ln\left[\frac{A^{2/3}}{(\ln R)^{2/3}}\right] + \ln\ln\left[\frac{R^{1/3}(\ln R)^{2/3}}{A^{2/3}}\right]\right\}}. \tag{2}$$

$\sigma + \tau \approx 1.0618$ which arises from calculation of two integrals [4]. In addition $A = 3 \cdot 5^{1/2}(\sigma+\tau)/2$.

For the case of the multi$-\alpha$ solutions of the variational problem ($N \geq 2$ below), one obtains for the convective heat transport

$$F_N = \frac{\left[4 \cdot 3^{\frac{(N-3/2)3^{N-1}+1/2}{2\cdot 3^{N-2}}}\right]^{4/3}}{[2(\sigma+\tau)]^{4/3}} \left(\frac{2}{3\beta}\right)^{\frac{2(3^{N-1}-1)}{3^N}} [2\cdot 3^N - 1]^{-\frac{2\cdot 3^N - 1}{3^N}}$$
$$\times R^{\frac{2\cdot(3^N-1)}{2\cdot 3^N}} \left[\ln\left(\frac{1}{g_N^2}\right) - \ln\ln\left(\frac{1}{g_N^2}\right)\right]^{1/3} \prod_{j=1}^{N-1} \left[\ln\left(\frac{1}{g_{N-j}^2}\right) - \ln\ln\left(\frac{1}{g_{N-j}^2}\right)\right]^{3^{-j-1}} \tag{3}$$

in the cases $N = 2, 3, \dots$

The wave numbers of the $N-\alpha-$solution are $\alpha = \left(\frac{1}{5}\right)^{1/2} R^{1/2}$ for $i = 1$ and

$$\alpha_i = 3^{\frac{(i-2)3^{i-2} - (1/2)(3^{i-2}-1)}{3^{i-2}}} \left(\frac{2}{3\beta}\right)^{\frac{3^{i-1}-1}{3^{i-1}}} \left[2\cdot 3^N - 1\right]^{-\frac{2\cdot 3^{i-1}-1}{2\cdot 3^{i-1}}}$$
$$\times R^{\frac{2\cdot 3^{i-1}-1}{2\cdot 3^{i-1}}} \prod_{j=1}^{i-1} \left[\ln\left(\frac{1}{g_{i-j}^2}\right) - \ln\ln\left(\frac{1}{g_{i-j}^2}\right)\right]^{\frac{3^{i-j-1}}{2\cdot 3^{i-1}}} \tag{4}$$

for $i = 2, 3, \dots$

The functions g_i are

$$g_1 = \frac{3^{1/3}5^{1/6}}{2^{1/3}}(\sigma+\tau)^{1/3}R^{-1/6}(\ln R)^{-1/3}(1+r)^{1/3}, \tag{5}$$

for $i = 1$, and for $i = 2, 3, \ldots, N$

$$g_i = 3^{1/3}\chi_i^{-1/6}(R)[\ln \chi_i(R)]^{-1/3}(1+r_i)^{1/3}, \tag{6}$$

where

$$\chi_1(R) = R\,,$$
$$\chi_i^{-1}(R) = \left\{ \frac{R^{-1}}{\prod\limits_{j=1}^{i-1}\left[\ln\left(\frac{1}{g_{i-j}^2}\right) - \ln\ln\left(\frac{1}{g_{i-j}^2}\right)\right]^{-3^{i-j-1}}} \right\}^{\frac{1}{3^{i-1}}}, \tag{7}$$
$$i = 2, 3, \ldots, N,$$

$$r_i = \frac{3\Big\{\ln\Big[\Big(\frac{3}{\ln\chi_i(R)}\Big)^{2/3}\Big] + \ln\ln\Big[\frac{\chi_i(R)^{1/3}(\ln\chi_i(R))^{2/3}}{3^{2/3}}\Big]\Big\}}{\ln\chi_i(R) - 3\Big\{\ln\Big[\Big(\frac{3}{\ln\chi_i(R)}\Big)^{2/3}\Big] + \ln\ln\Big[\frac{\chi_i(R)^{1/3}(\ln\chi_i(R))^{2/3}}{3^{2/3}}\Big]\Big\}} \tag{8}$$

and

$$\beta = \frac{1}{3}\left[2^{-3/4}\int_0^\infty d\gamma\left(\frac{de}{d\gamma}\right)^2 + 2^{1/4}\int_0^\infty d\gamma(1-\gamma e)\right] \approx 0.441\,. \tag{9}$$

The next problem studied was the upper bound on the convective heat transport in a heated from below horizontal fluid layer of infinite Prandtl number with a rigid lower boundary and a stress-free upper boundary. The boundary conditions here are asymmetric, and thus the corresponding solutions of the Euler-Lagrange equations of the variational problem are also asymmetric. The bound on the convective heat transport and the corresponding wave number are between the values for a fluid layer with two rigid boundaries and a fluid layer with two stress-free boundaries. The theory is presented in [11], and the results are as follows. For the $-\alpha$-solution of the variational problem, one obtains

$$F = \left(\frac{24}{D_l}\right)^{6/5}(A^*(R))^{26}13^{-13/10}20^{-1/5}, R^{3/10}(\ln R)^{1/5} \tag{10}$$

and the corresponding wave number is

$$\alpha = (A^*(R))^5 \, (R/13)^{1/4} \ , \tag{11}$$

where

$$A_1 = 24^4 c^{1/2}/(12 D_u D_l^3)\,;$$
$$A_2 = (13 D_l^4/24^4)^{-39/40}(1/20)^{3/20} R^{-1/40}(\ln R)^{3/20}$$

and

$$A^*(R) \approx [A_1/(A_1 + A_2)]^{1/20}\,. \tag{12}$$

R is the Rayleigh number and D_u and D_l are two integrals which approximate values are $D_u = 2.11975$ and $D_l = 2.1222$ [11].

The upper bounds and the wave numbers connected to the $N - \alpha$ solutions of the variational problem are [12]

$$\begin{aligned} F_N = {} & [1/(2(\sigma+\tau)]^{6/5}\{10^{(1/3)[N-1-(10/9)(1-1/10^{N-1}]}(2/\beta^*)^{(1/3)(1-1/10^{N-1})} \\ & \times b_1^{(1/3)(4-1/10^{N-1})^{2/5}}(120 b_1^4 10^{N-2})^{6/5}\}(1/2)^{(2/9)(1-10^{-N})} \\ & \times 10^{-(1/405)[10(9N-1)]10^{1-N}} R^{(1/3)(1-10^{-N})}(\ln R)^{(2/9)(1-10^{-N})}\,. \end{aligned} \tag{13}$$

The bound for $N \to \infty$ is

$$\begin{aligned} F \propto {} & [1/(2(\sigma+\tau))]^{6/5}(3/(2\beta^*))^{2/15}(9/10)^{6/5}10^{-38/135}(1/3)^{2/9}R^{1/3} \\ & \approx (1/6)R^{1/3}\,. \end{aligned} \tag{14}$$

For the wave numbers α_n ($n = 1, \ldots, N$) connected to the last solution, the result is

$$\alpha_n = b_n \prod_{k=1}^{n-1} (\ln(1/\delta_k^l))^{10^k/2\cdot 10^n} R^{(1/6)(2-5/10^n)}\,, \tag{15}$$

where

$$\begin{aligned} \delta_n^l = {} & \{R^{-1} \prod_{k=1}^{n-1} [\ln(1/\delta_k^l)]^{6\cdot 10^{k-1}}\}^{1/(2\cdot 10^n)} \\ & \times \{[2\cdot 10^n/\ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^l)]^{-6\cdot 10^{k-1}}\}][1/(1-\Omega_n)]\}^{1/5}, \end{aligned} \tag{16}$$

and

$$\Omega_n = \{2 \cdot 10^n / \{5 \ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^l)]^{-6 \cdot 10^{k-1}}\}\}\} \times \ln\{2 \cdot 10^n / \{\ln\{R \prod_{k=1}^{n-1} [\ln(1/\delta_k^l)]^{-6 \cdot 10^{k-1}}\}\}\}. \tag{17}$$

In addition $b_1 = [(3/4)/(10^N - 1/4)]^{1/4}$, and

$$b_{n+1} = 10^{(1/3)[n-(10/9)(1-10^{-n})]} (2/\beta^*)^{(1/3)(1-10^{-n})} b_1^{(1/3)(4-10^{-n})}. \tag{18}$$

β^* and $\sigma + \tau$ are integrals which approximate values are $\beta^* = 0.21835$ and $\sigma + \tau = 1.110646$.

Another studied case was the case of convection in a horizontal layer of fluid which was under the action of rotation. The corresponding theory was developed in [15]. The obtained results were for the case of fluid of infinite Prandtl number and for various cases of boundary conditions and for bounds corresponding to the $1-\alpha$ solution of the corresponding variational problem. The rotation was associated with a dimensionless number called the Taylor number $Ta = (2/E)^2$, where E was the Ekman number $E = \nu/(\Omega d^2)$. d was the thickness of the fluid layer and ν was the kinematic viscosity. Ω was the angular velocity of the rotation of the fluid layer. There were two cases with respect to rotation when one could obtain analytical results. The first case was the case of intermediate rotation $\alpha_1^4 \ll Ta \ll \alpha_1^6$. The second case was the case of strong rotation where $Ta \gg \alpha_1^6$.

For the case of intermediate rotation, the obtained analytical results for the upper bound on the heat transport were as follows. For the case of a rigid lower boundary and a stress-free upper boundary, the bound on the heat transport was

$$F_1 = 2^{-7/15} (D_u + D_l)^{-6/5} Ta^{1/10} R^{1/5} \left[\ln\left(2Ta^{1/2}\right) - \ln\ln\left(2Ta^{1/2}\right)\right]^{1/5} \tag{19}$$

and the corresponding wave number was

$$\alpha_1 = 2^{-1/5} (D_u + D_l)^{1/5} Ta^{2/5} R^{-1/5} \times \left[\ln\left(2Ta^{1/2}\right) - \ln\ln\left(2Ta^{1/2}\right)\right]^{3/10} [\ln(Ta) - \ln\ln(Ta)]^{-1/2}. \tag{20}$$

Above $D_l = I_l + 2J_l$ and $D_u = I_u + 2J_u$. The values of the integrals I and J are : $I_l \approx 0.9255$; $I_u \approx 0.79635$; $J_l \approx 0.1851$; $J_u \approx 0.2635$.

For the case of fluid layer with two stress-free boundaries and for the case of intermediate rotation, the upper bound is

$$F_1 = 2 \cdot 3^{-1/3} \cdot 5^{-5/3} D^{-4/3} R^{2/3} Ta^{-1/3} \times \left[\ln\left(\frac{5Ta^2}{R}\right)\right]^{1/3} \left(1 + \frac{\ln\{\ln[(5Ta^2/R)^{1/6}]\}}{3\ln[(5Ta^2/R)^{1/6}]}\right)^{-1/3}, \tag{21}$$

and the corresponding wave number is $\alpha_1 = (R/5)^{1/4}$. In (21) $D = I + J$, $I = 0.52705$, $J = 1.59270$.

For the case of fluid layer with two rigid boundaries and for the case of intermediate rotation, the upper bound is

$$F_1 \propto Ta^{1/10} R^{1/5} [\ln(Ta) - \ln\ln(Ta)]^{1/5}, \tag{22}$$

and the corresponding waven umber $\alpha \propto Ta^{1/6}$.

The case of large Taylor numbers was considered in [41]. Here, we have again three cases with respect to the boundary conditions. For any case of boundary conditions, we have two subcases. In Subcase 1, the Ekman layer of the optimum field is thicker than the boundary layer. In Subcase 2, the boundary layer is thicker than the Ekman layer.

For the case of a fluid layer with a rigid lower boundary and a stress-free upper boundary, and if the Ekman layer is thicker than the boundary layer with values of the wave number $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$, the upper bound on the heat transport is given by

$$F_1 \propto 2^{-3/5} (D_u + D_l)^{-6/5} R^{1/5} Ta^{1/10} [\ln(2Ta^{1/6}) - \ln\ln(2Ta^{1/6})]^{1/5} \tag{23}$$

for $\alpha_1 \propto O(Ta^{1/6})$. Here and below $D_l = 1.1106$ and $D_u = D = 1.05985$.

For the case when the Ekman layer is thicker than he boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$ the upper bound on the upper bound convective heat transport is

$$F_1 \propto 2^{-7/10} (D_u + D_l)^{-6/5} Ta^{1/10} R^{1/5} \tag{24}$$

for $\alpha_1 \to O(Ta^{1/8})$.

For the case when the boundary layer is thicker than the Ekman layer and $O((R \ln R)^{4/3}) \ll O(Ta) \ll O(R^{3/2})$, the upper bound on the convective transport is

$$F = \frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta} \Big[\ln\Big(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\Big) - \ln\ln\Big(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\Big)\Big] \tag{25}$$

for $\alpha_1 = (R/5)^{1/4}$.

Next, we discuss the case of a fluid layer with rigid lower boundary and stress-free upper boundary and when the boundary layer is thicket than the Ekman layer and for $O(R^{1/4}) \gg O(Ta^{1/6}) \gg \alpha_1 \gg O((Ta/R)^{1/2}) \gg O(R^{1/8})$ the upper bound of the convective heat transport is

$$F = \frac{4R\alpha_1^2}{\pi^2 Ta}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right]. \tag{26}$$

Next, we discuss the case of fluid layer with two stress-free boundaries. For the case when the Ekman layer is thicker than the boundary layer and $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$ the upper bound of the convective heat transport is

$$F_1 \propto (2D)^{-4/3} R^{1/3} [\ln(Ta^{1/6}/2) - \ln\ln(Ta^{1/6}/2)]^{1/3}, \tag{27}$$

for $\alpha_1 \propto O(Ta^{1/6})$.

For the case when the Ekman layer is thicker than the boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$ the upper bound on the convective heat transport is

$$F \propto 2^{-11/6} D^{-4/3} R^{1/3} Ta^{-1/6}, \tag{28}$$

for $\alpha_1 \to O(Ta^{1/8})$. For the case when the boundary layer is thicker than the Ekman layer and $O((R\ln R)^{4/3}) \ll Ta \ll O(R^{3/2})$ the upper bound on the convective heat transport is

$$F = \frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta}\left[\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right) - \ln\ln\left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta}\right)\right], \tag{29}$$

for $R = (R/5)^{1/4}$.

For the case when the boundary layer is thicker than the Ekman layer and $O(R^{1/4}) \gg O(Ta^{1/6}) \gg \alpha_1 \gg O((Ta/R)^{1/2}) \gg O(R^{1/8})$ the upper bound on the convective heat transport is

$$F = \frac{4R\alpha_1^2}{\pi^2 Ta}\left[\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right) - \ln\ln\left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2}\right)\right]. \tag{30}$$

Finally we consider the case when the fluid layer has two rigid boundaries. The first subcase is when Ekman layer is thicker than the boundary layer and $O(Ta^{1/8}) \ll \alpha_1 \ll O(R^{1/4})$. The upper bound on the convective heat transport

$$F_1 = 2^{-9/5} D^{-6/5} R^{1/5} (1 - \alpha_1^4/R)^{6/5} Ta^{1/10} \times \left[\ln \left(\frac{2\alpha_1^4}{Ta^{1/2}} \right) - \ln \ln \left(\frac{2\alpha_1^4}{Ta^{1/2}} \right) \right]^{1/5} \tag{31}$$

for $\alpha_1 \propto Ta^{1/6}$.

For the subcase when the Ekman layer is thicker than the boundary layer and $O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/8})$ the upper bound on the convective heat transport is

$$F_1 \propto 2^{-19/10} D^{-6/5} R^{1/5} Ta^{1/10} \tag{32}$$

for $\alpha_1 = (R/5)^{1/4}$.

For the subcase when the boundary layer is thicker than the Ekman layer and $O((R \ln R)^{4/3}) \ll Ta \ll O(R^{3/2})$ the upper bound of the convective heat transport is

$$F = \frac{64R^{3/2}}{25\sqrt{5}\pi^2 Ta} \left[\ln \left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta} \right) - \ln \ln \left(\frac{64R^{3/2}}{5\sqrt{5}\pi^2 Ta} \right) \right] \tag{33}$$

for $\alpha_1 = (R/5)^{1/4}$.

For the subcase when the boundary layer is thicker than the Ekman layer and $O(R^{1/8}) \ll O((Ta/R)^{1/2}) \ll \alpha_1 \ll O(Ta^{1/6}) \ll O(R^{1/4})$ the upper bound on the convective heat transport is

$$F = \frac{4R\alpha_1^2}{\pi^2 Ta} \left[\ln \left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2} \right) - \ln \ln \left(\frac{4R^2\alpha_1^4}{\pi^2 Ta^2} \right) \right]. \tag{34}$$

Next, we mention the extensive numerical simulations on the problems of optimum theory of turbulence from 2001 onwards. Busse had an idea of imposing an additional constraint on the class of optimum fields for the case of a thermal convection of a fluid layer under the action of rotation. The assumption was that this additional constraint would lead to lower upper bounds on the heat transport. This was indeed the case in the study in [1] where stress-free boundary conditions are discussed. However, the corresponding Euler-Lagrange equations were quite complicated and an analytical asymptotic solution was no longer possible. Because of this, the bounds had to be studied numerically.

The article [1] initiated an extensive numerical research on the upper bounds of convective heat transport. It was realized that no numerical simulations had been performed for the problem of upper bounds of turbulent convection in a fluid layer under the action of rotation. A large numerical study on this problem for the case of a fluid layer of infinite Prandtl number and stress-free boundary conditions is [16]. An advantage of the numerical simulation is that one can obtain upper bounds on the Nusselt number for large areas of Rayleigh and Taylor numbers where no analytical asymptotic upper bounds are available. If the computers are powerful enough, then one can reach the areas of Rayleigh and Taylor numbers where the numerical upper bounds can be compared to the analytical asymptotic upper bounds. Such a comparison is made in [16], and the results show that the assumptions of the analytical asymptotic theory are correct. The convection in a fluid layer under the action of rotation is connected to a rich dynamics of numerous boundary layers which occur and vanish in this situation. The dynamics of the boundary layers was discussed in [17]. More numerical results on this problem can be seen in [18], and results for convection in rapidly rotating fluid layers are available in [19–21].

The classical case of the optimum theory of turbulence: the bound on the heat transport in a layer of fluid with rigid boundaries and absence of rotation was discussed in [14] by means of a numerical simulation. In this article, we obtained bounds by means of the solutions of the Euler-Lagrange equations of a variational problem possessing up to three wave numbers. We show that for low and intermediate Rayleigh numbers, the numerical bounds are positioned below the analytical asymptotic bounds obtained by Howard and Busse. For the case of large Rayleigh numbers, the numerical bounds approach the analytical asymptotic bounds. Thus, we confirmed numerically the bound obtained by Howard for the case of one-wave-number solution of the Euler-Lagrange equations. In the area of the numerical study of the optimum fields, we note especially the joint article with Prof. S. Radev on the spectral method for study of the optimum fields [22]. Another study is connected to the variational problem for the upper bounds on the solute transport by double-diffusive convection in a layer of fluid. In this study, an amended version of the variational problem is proposed with respect to the previous version of the problem considered by Strauss [23].

Numerical studies of thermal convection were not confined to the methodology of the optimal theory of turbulence. Chaotic states connected to the Bnard-Marangoni convection have been studied in [13].

The next analytical results are for the upper bounds on the energy dissipation in Couette-Ekman flow [32]. This flow is a flow in an infinite layer of a homogeneous incompressible fluid between two parallel walls. The walls move in the x-direction with velocities $\pm U_0/2$. The distance between the walls is h and the coordinate system rotates with a constant angular velocity Ω about the z-axis. In addition, ν is the kinematic viscosity. The boundary conditions at $z = \pm h/2$ are $\mathbf{U}(x, y, \mp h/2) = \pm(U_0/2)\mathbf{i}$, where $\mathbf{i}$ is the unit vector in the x-direction. The boundary conditions on $\mathbf{u}$ and p are periodic in $x-$ and $y-$ direction. The obtained upper bounds are for the time averaged energy dissipation rate

$$S = \nu \langle | \nabla \mathbf{U} |^2 \rangle, \tag{35}$$

which is make dimensionless as $s = \frac{S}{U_0^3 h^{-1}}$. The results are obtained by background-fluctuation decomposition of the velocity field and are for two cases: (i) when the background velocity fields is a gradient; and (ii) when the background velocity field is not a gradient. For the case when the background velocity field is a gradient, the upper bound on the energy dissipation is

$$\nu \langle | \nabla \mathbf{U} |^2 \rangle \leq \frac{\nu U_0^2}{h^2} C\,, \tag{36}$$

where

$$C = \frac{1}{4} \frac{\sqrt{\Omega^+}\{\sin(\sqrt{\Omega^+}) + \sinh(\sqrt{\Omega^+})\}}{\sin^2(\sqrt{\Omega^+}/2) + \sinh^2(\sqrt{\Omega^+}/2)}$$

is a coefficient which accounts for the influence of the rotation. In addition $\Omega^+ = \frac{\Omega h^2}{\nu^*}$ is the inverse Ekman number and $\nu^* = \nu \frac{a-2}{a}$ is the viscosity rescalled by means of an appropriate balance parameter a. Thus, the upper bound depends on a and optimization with respect to this parameter must be made when the values of the other parameters are fixed.

The validity of the obtained bound is given by a special relation called spectral constraint. The relation is for $Re(\Omega^+)$, where $Re = \frac{U_0 h}{\nu^*}$ is the Reynolds number [32].

For the case when the background velocity field is not a gradient there are two possibilities: (i) the background velocity field is without developed boundary layer; and (ii) the background velocity field is with developed boundary layer. For the case of background velocity field without developed boundary layer, the upper bound is

$$s \leq \frac{1}{Re} \frac{{\Omega^+}^4 + 7824{\Omega^+}^2 + 967680}{144(17{\Omega^+}^2 + 6720)} \tag{37}$$

and it has a limit $1/Re$ for the case without rotation. The critical Ω^+ for the profile is $\Omega^+ < 8\sqrt{(105)}/5$.

For the case of background velocity field with developed boundary layer, the upper bound on the energy dissipation is

$$s \leq \frac{1}{Re} + \frac{a^2}{4(a-1)Re}[\frac{Re}{18\sqrt{2}} \frac{a}{a-1} - 1] - \frac{a^2 {\Omega^+}^2}{Re(a-1)}[7 - 8\frac{(a-1^2)}{a^2 Re^2} + \frac{4992\sqrt{2}}{35} \frac{(a-1)^3}{a^3 Re^3} - \frac{7488}{5} \frac{(a-1)^4}{a^4 Re^4}]\,. \tag{38}$$

3 SEsM and Water Waves

Water waves are an research area in which we invest much effort. These waves are usually modelled by nonlinear differential equations and have been our research area for many years. We have had a successful collaboration with experimentalists in the area of water waves [42], analysing various aspects of rogue waves in the North Sea. This sparked our interest in the exact solutions of the nonlinear Schrödinger equation and led to a study devoted to the use of the nonlinear Schrödinger equation for modelling deep water waves [43, 44].

The exact analytical relationships for water waves are connected to the problem of obtaining exact analytical solutions of nonlinear differential equations. Here, we make several remarks on one such methodology developed at the Department of Fluid Mechanics of the Institute of Mechanics. This methodology is called SEsM (Simple EquationS Method). There are various versions of the methodology; the general case is denoted as SEsM(n,m). In this case, we have to solve n nonlinear differential equations and use the solutions of m more simple equations in order to obtain the desired solutions. SEsM(1,m) means that we solve one differential equation and the most used version is SEsM(1,1), where we solve a single complicated equation by using the solution of one more simple differential equation.

The essence of SEsM(n,m) is as follows. We have to solve a system of n nonlinear differential equations:

$$\mathcal{E}_i[u_{i1}(x, \dots, t), \dots, u_{in}(x, \dots, t)] = 0, \quad i = 1, 2, \dots, n. \tag{39}$$

$\mathcal{E}_i[u_{i1}(x, \dots, t), \dots, u_{in}(x, \dots, t), \dots]$ depends on the functions $u_{i1}(x, \dots, t), \dots, u_{in}(x, \dots, t)$ and some of their derivatives. The functions u_{ik} can depend on several spatial coordinates. We have to transform (39) to

$$\sum_{i=1}^{n} a_{ij}(\dots)F_{ij} = 0, \quad j = 1, 2, \dots, p_i. \tag{40}$$

This happens by representing u_{ik} as composite functions of known analytical solutions to the m simpler equations. The functions F_{ij} are functions of the independent spatial variables and of time, and the quantities a_{ij} are relationships among the parameters of the equation (39), parameters of the solutions, and parameters of the solutions to the simpler equations. p_i is a parameter that is characteristic of the i-th equation from (39). It is important that the relationships a_{ij} contain only parameters, whereas the spatial coordinates and the time are concentrated in the functions F_{ij}. If we manage to reduce the equation (39) to the form (40), then we can set

$$a_{ij}(\dots) = 0. \tag{41}$$

Thus, we obtain a system of nonlinear algebraic equations. This system contains relationships among the parameters of (39), the parameters of the simpler equations used, and the parameters of the solution constructed by the simpler equations. Each nontrivial solution to (41) leads to a solution to the system (39).

Research on the above methodology started more than 30 years ago. Below, we list several of the achievements from the last 15 years. These results were obtained in the Department of Fluid Mechanics of the Institute of Mechanics. The first results were obtained in the area of population waves [45, 46]. Then, we used the ordinary differential equation of Bernoulli as a simple equation [47, 48]. Here, we used the concept of the balance equation and the version of the methodology was SEsM(1,1), called also the Modified Method of Simplest Equation (MMSE) [49, 50]. Up to 2018, our contributions to the methodology and its application were connected to SEsM(1,1) [51–58]. The role of the simplest equation in the methodology of the SEsM(1,1) is discussed in [52], with examples of application of the methodology to the nonlinear partial differential equations of Newell-Whitehead and FitzHugh-Nagumo. The case of using differential equations for the Jacobi elliptic functions as simple equations can be seen in [53]. Special attention to the exact traveling wave solutions of the nonlinear equations that are models for nonlinear water waves is given in [54], where exact traveling wave solutions are obtained for the extended Korteweg-de Vries equation

$$2\frac{\partial \eta}{\partial \tau} + 3\eta\frac{\partial \eta}{\partial \xi} + \frac{1}{3}\delta^2\frac{\partial^3 \eta}{\partial \xi^3} - \frac{3}{4}\epsilon\eta^2\frac{\partial \eta}{\partial \xi} = -\frac{1}{12}\epsilon\delta^2\left(23\frac{\partial \eta}{\partial \xi} + 10\eta\frac{\partial^3 \eta}{\partial \xi^3}\right) \tag{42}$$

(where ϵ and δ are small parameters called amplitude parameter and shallowness parameter) and for the generalized Camassa-Holm equation

$$\frac{\partial u}{\partial t} + p_1\frac{\partial u}{\partial x} + \frac{p_4}{2}\frac{\partial}{\partial x}g(u) - p_2\frac{\partial^3 u}{\partial^2 x\partial t} - 2p_3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - p_3 u\frac{\partial^3 u}{\partial x^3} = 0 \tag{43}$$

for the case when the equation of Bernoulli and Riccati are used for simple equations and for several forms of the polynomial $g(u)$. Above p_1-p_4 are parameters.

Further application of the SEsM(1,1) methodology was presented in [56], where the simple equation $f_\xi^2 = 4(f^2 - f^3)$ was used. This equation has the solution $f(\xi) = \cosh^{-2}(\xi)$ where $\xi = \alpha x + \beta t$. An extension of the methodology with respect to more general simplest made in [57]. There was proposed the use of a more general simple equation with polynomial which defines nonlinearity, a certain special function called the V-function. The developed methodology was applied to the generalized Kortweg-de Vries equation and to second order the Korteweg-de Vries equation (also known as the Olver equation). The research presented in [56] was extended in [58] for the case of simplest equation $f_\xi^2 = n^2(f^2 - f^{(2n+2)/n})$, which has a solution $f(\xi) = \cosh^{-n}(\xi)$, where $\xi = \alpha x + \beta t$.

The methodology SEsM(1,1) was applied in the last years for studying propagation of waves in artery with aneurism [59–62]. Since 2018, the SEsM methodology has been developed in direction of SEsM(m,n). The SEsM based on two simple equations was used in [63]. Further discussions on SEsM can be seen at [64–74]. Applications of specific cases of SEsM can be seen in [75–82].

4 Fluid Mechanics, Time Series Analysis and Renewable Energy

Many engineering applications of fluid mechanics are connected to heat transfer. It is connected to the generation, use, conversion, and exchange of heat and it is facilitated by thermal: (i) conduction; (ii) radiation; (iii) convection, etc. [83, 84]. Below we mention two applications time series analysis to problems of fluid mechanics: analysis of extreme water levels and an application connected to the heat transfer in buildings [85] with special attention on the use of windows system to collect the energy of the Sun [86].

There are several publications on the extreme water levels of various rivers. This research started with a study of the probability of extreme waves in the Nord sea [42]. This was motivated by the evaluation of the risks for the German oil platforms in the North sea. The research continued with a study of probability of extreme water levels in the river Elba in Germany [87] in connection with the possibility of flooding of some parts of the city of Hamburg, Germany. We note the study of extreme water levels of a large river in China – Huang He river [88], and for several large rivers in the Indian subcontinent: Indus, Ganges and Brahmaputra rivers [89]

Next we discuss an environmental application: fluid mechanics for windows heat transfer system. Several processes are of importance for such a system. They are: (i) the convective heat transfer connected to the fluid between the windows in this system; and (ii) heat transfer connected with solar energy. The systems have many sensors, which implies much use of the methodology of the time series analysis in order to analyze the obtained data. Nonlinear differential equations model the convective heat transfer [90]. With increasing Rayleigh number, various instabilities arise and a turbulent motion occurs at relatively small values of the characteristic number. Below we describe briefly the experimental facility and discuss some of the time series for quantities of interest for fluid mechanics aspects of the windows system. The experimental facility was built within the scope of the InDeWaG project [91]. It contains a pavilion with special windows, filled with liquid which is warmed by solar radiation. The liquid circulates and thus the solar energy can be collected and used. The windows system consists of Fluid Flow Glazing (FFG) elements which are

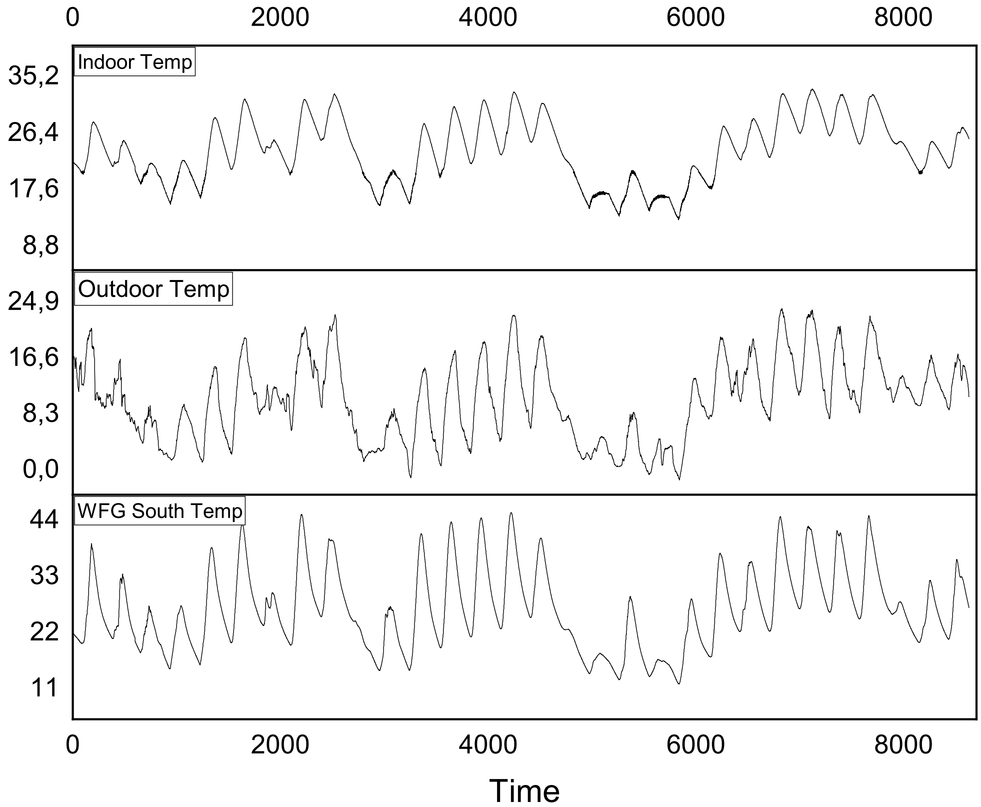

Fig. 1: Indoor, outdoor and south water flow glazing (WFG) temperature distribution for April,2022

collectors absorbing heat from the outside and from the inside of the pavilion. The facility has many sensors and sophisticated system for controlling of the flow of fluid and the flow of heat.

Figure 1 shows the flow glazing temperature. According to the Bulgarian regulations for the optimal office conditions, the temperature is between 21 and 23°C. In order to control the fluid temperature in FFG modules, they are connected to a heat pump through plate heat exchangers mounted on each module. During the day when the temperature of the fluid is higher than the set temperature of the heat pump, the heating of the fluid by the heat pump stops. In this way, we save energy using the energy absorbed by the FFG modules.

Figure 2 presents the irradiance data for the whole month of April, 2022. As it can be seen, most of the month the incoming radiation was comparatively high and stable. There are only 4 days where the weather was unstable and the radiation varied

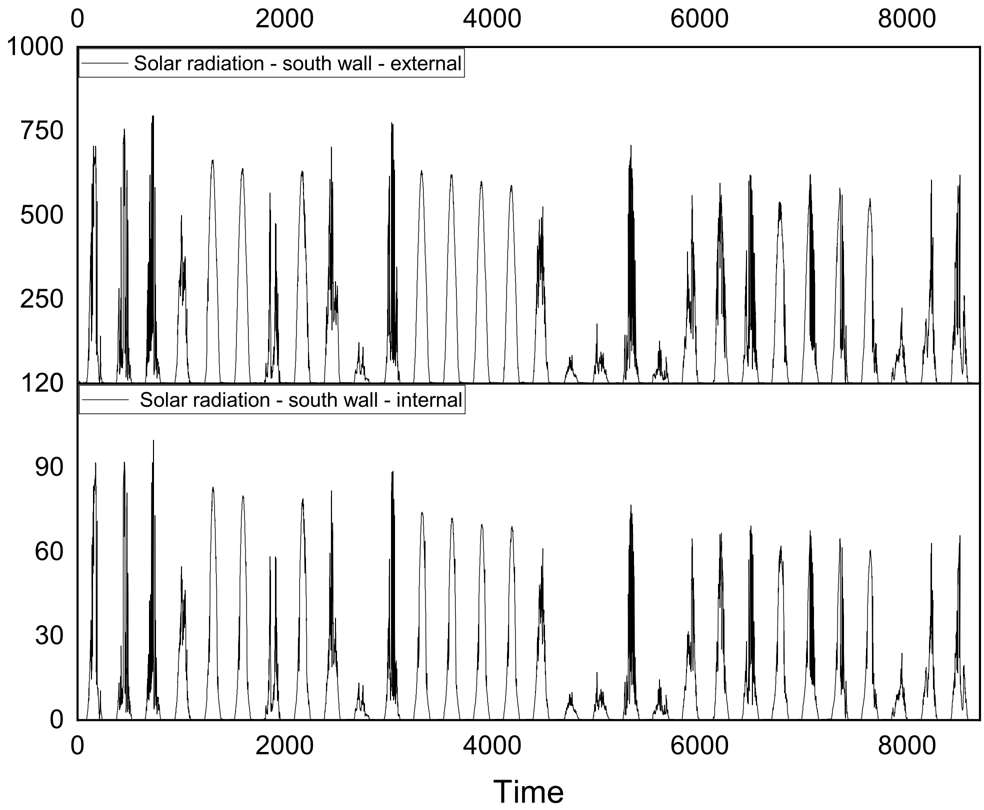

Fig. 2: Indoor and outdoor solar radiation for April,2022.

from 50 to 120 W/m^2. In Fig. 2 the upper part presents the incoming radiation. The bottom part is the radiation transmitted through the FFG module. It can be observed that about 60% of the incoming radiation determines the energy performance of the water-flow glazing and most of the solar radiation is captured by the WFG. The data is collected from two pyrometers: one mounted outside the plain of FFG modules; and the other one inside, measured every 5 minutes. The difference between the two measured irradiances is around 500 W/m^2.

Many other sensors are available in the experimental pavilion. On the basis of obtained data one can calculate various quantities which can be used to monitor and control the conditions for work of the windows glazing system [92]. In addition, one can do further studies on the time series recorded by the facility. Simulations can be done about the functioning of the window system and especially simulations on the processes connected to fluid flow in the water layer between the window plates of the studied window system.

5 The Book

We mention especially the book *Instability, Chaos and Turbulence* by S. Panchev, S. Radev and N. K. Vitanov, published by the Marin Drinov Academic Publishing House of the Bulgarian Academy of Sciences in 2012 [93]. The book has 4 chapters. Chapter 1 is written by Professor Stefan Radev. This chapter is devoted to the theory of hydrodynamic stability with an application to flows which are connected to surfaces between different phases. The second chapter of the book is written by the academician Stoicho Panchev. This chapter discusses chaos and turbulence in fluid mechanical systems. The last two chapters are written by Professor Nikolay K. Vitanov. Chapter 3 is devoted to the optimum theory of turbulence. Chapter 4 discusses the stochastic modeling of turbulence.

6 Concluding Remarks

In this article, we present a brief overview of a part of the research devoted to the problems of fluid mechanics and carried out at the Department of Fluid Mechanics of the Institute of Mechanics of the Bulgarian Academy of Sciences. Prof. Radev was the first head of this department since its creation in 2010 after the reorganization of the structure of the Institute. Prof. Radev spent a lot of time and efforts in order to make the organization of the research and administrative functioning of the department. In the following years the department started to grow and an experimental laboratory – the Open Laboratory for Experimental Mechanics (OLEM) became part of the department. The theoretical part of the department also has a significant growth and today the department is a high level research unit of the Institute of Mechanics.

We wish good health, long live and many successes in the coming years to our founding head of the Department of Fluid Mechanics prof. Stefan Radev, Corresponding member of the Bulgarian Academy of Sciences.

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