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# CLOSED FORM SOLUTION OF VERTICAL CONCENTRATION DISTRIBUTION EQUATION: REVISITED WITH HOMOTOPY PERTURBATION METHOD

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ABSTRACT: The present study revisits the solution of Hunt [1] equation to determine the vertical concentration distribution in a turbulent flow carrying sediment using Homotopy Perturbation Method (HPM). It presents a closed (compact) form series solution for sediment concentration and convergence of series solution is also stated briefly. Finally, the solution obtained by HPM is validated by comparing it with the implicit solution and the numerical solution as well as with existing experimental data.

KEY WORDS: Open channel flow, Turbulent flow, Sediment diffusion coefficient, Turbulent diffusion coefficient, Settling velocity, Homotopy Perturbation Method, Convergence

## 1 INTRODUCTION

Sediment transport in fluvial hydraulics is a key element as it is closely linked with many real life problems such as reservoir sedimentation, water resources management, environmental safety etc. Difficulty in understanding the complicated sediment transport mechanism, make this area an important topic of research. Sediment is transported by the flow in one of the two principal modes: as bedload transport and as suspended load transport. Suspended load transport refers to the particles or grains of sediment that move along a river within and is wholly supported by the flow. In most of the rivers, the bulk of transported sediment, often 90% or more, moves as suspended load. One of the critical parts of suspended load study, is to obtain the distribution of suspended sediment concentration as it is directly connected to the sediment discharge rate, bedload layer thickness etc. This study focuses on the vertical distribution of suspended sediment concentration in open channel turbulent flow.

Since long, several models have been developed to predict accurately the vertical distribution of sediment concentration in an open channel flow. Rouse [2] was

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the pathfinder in this field who provided an analytical solution for suspended sediment concentration by balancing the upward flux and downward settling of sediment, the solution being commonly known as Rouse equation. But this model has major drawback as it cannot predict the sediment concentration well throughout the channel height specially near the bed and near the surface and also in case of high concentrated flows. Later, Hunt [1] modified the Rouse [2] model by treating the fluid phase and the solid phase separately which can overcome the drawbacks of Rouse [2] model. For simplicity, Hunt [1] assumed turbulent diffusion coefficient and sediment diffusion coefficient to be equal, which is not physically realistic. Following the work of Hunt [1], numerous investigations have been undertaken to predict the vertical distribution of sediment concentration incorporating different turbulent mechanism. But inclusion of turbulent effects make the model so complicated that researchers had to go for numerical solution only.

A mathematical problem can be solved in two different ways: numerically and analytically. Both the methods have some advantages and some disadvantages over the other and no one can be claimed to be better than the other. The present study adopts an analytical method named Homotopy Perturbation Method (HPM) to solve Hunt [1] equation for vertical concentration distribution in a turbulent flow carrying sediments. This method has several advantages over the other existing analytical methods like Lyapunov's small artificial parameter method [3], Adomial decomposition method [4], perturbation method [5] etc. This method is closed to Homotopy Analysis Method (HAM) which was introduced by Liao in 1992. HAM has been successfully applied to different non-linear problems in science and engineering [6–9] and recently it has been applied by Kumbhakar et al. [10] to solve Hunt [1] equation. In sediment transport field, application of HAM is too limited [10–12] and HPM has not been applied yet. HPM was first proposed by He [13] who developed the method as a combination of homotopy of topology and classical perturbation. He [13] applied HPM to solve a wide variety of equations like Lighhill equation [13], Dulting equation [14], non-linear oscillators Equation [15], Zakharov-Kuznetsov equations [16] and many others. He [17] compared also the two methods Homotopy Analysis Method (HAM) and Homotopy Perturbation Method (HPM) and revealed that the latter is more powerful than the former. The major advantage in applying HPM over HAM is that unlike HAM, HPM does not need to determine the convergence of the solution through convergence control parameter. HPM is basically a new perturbation method, searching an asymptotic solution of a problem with a few terms [14]. The objective of the present study is to explore the use of HPM in sediment transport field by revisiting the solution of Hunt [1] equation through HPM where the Hunt [1] equation is considered in three different forms (1) with linear form of eddy diffusivity and constant settling velocity, (2) with parabolic form of eddy diffusivity and constant settling velocity and (3) with parabolic form of eddy diffusivity and a concentration dependent settling velocity. In all the three cases, the solutions have been presented in closed form. Validation of the solutions have been done with implicit and numerical solution of the problem as well as by comparing with laboratory channel data.

## 2 Mathematical Model

Suspended sediment distribution along a vertical in open channel turbulent flow is displayed in Fig. 1, in which the time averaged suspended sediment concentration decreases monotonically from $C_a$ (reference concentration at reference level $z = a$) to the flow depth $h$ . The generalized governing equation in steady uniform flow for suspended sediment concentration can be written as [1]

$$\varepsilon_s \frac{dC}{dz} + C(\varepsilon_t - \varepsilon_s)\frac{dC}{dz} + C(1 - C)w_0 = 0\,, \tag{1}$$

where $C$ is the dimensionless suspended sediment concentration, $z$ is the vertical distance from bed, $\varepsilon_s$ is the sediment diffusion coefficient, $\varepsilon_t$ is the turbulent diffusion

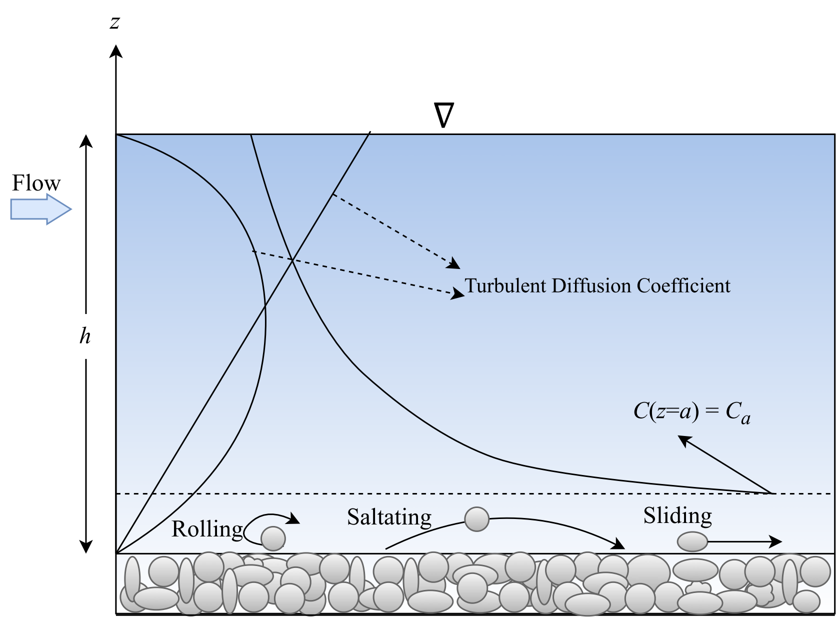



Fig. 1: Schematic diagram of sediment transport process along a vertical in open channel turbulent flow.

coefficient and $w_0$ denotes the settling velocity of sediment particles in clear fluid. The ratio of sediment diffusion coefficient $(\varepsilon_s)$ to turbulent diffusion coefficient $(\varepsilon_t)$ is known as inverse Schimdt number and denoted by the parameter $\beta$. This parameter $\beta$ is assumed as 1 by Hunt [1] to solve Eq. (1) analytically, although it is experimentally revealed that $\beta \approx 1$ only for finer sediment particles, and $\beta < 1$ for coarse sediment particle [18, 19]. Also, Graf and Cellino [20] experimentally determined $\beta < 1$ without bed form and $\beta > 1$ with bed form over an erodible sediment bed. Accordingly, the assumption $\beta = 1$ may not be physically reasonable.

It is clear from the Eq. (1) that to determine the vertical concentration distribution $C(z)$ , one should know the expression for $\varepsilon_s(z)$ and $\varepsilon_t(z)$. Three types of profiles (constant, linear and parabolic-type [20, 21]) for turbulent diffusion coefficient have been found in literature. The present study uses the linear and parabolic-type profiles which are given as follows :

(2)
$$\varepsilon_t(z) = \begin{cases} \kappa u_* z \\ \kappa u_* z\left(1 - \dfrac{z}{h}\right) \end{cases},$$

where $\kappa$ is the von-Karman constant and $u_*$ denotes the shear velocity. We can rearrange Eq. (1) in non-dimensional form as:

(3)
$$H(\hat{z})(1 + \delta C)\frac{dC}{d\hat{z}} + \zeta C(1 - C) = 0$$

with the boundary condition

$$C(\hat{z} = \hat{a}) = C_{\hat{a}} ,$$

where

$$H(\hat{z}) = \begin{cases} \hat{z} \\ \hat{z}(1 - \hat{z}) \end{cases}, \quad \hat{z} = \frac{z}{h}, \quad \delta = \frac{(1 - \beta)}{\beta}, \quad \zeta = \frac{w_0}{\beta \kappa u_*}$$

is the Rouse number and $C_{\hat{a}}$ is the reference concentration at reference level $\hat{a} = a/h$.

Implicit solution of the above equation for linear profile of $\varepsilon_t(z)$ is given as

(4)
$$\left(\frac{C}{C_{\hat{a}}}\right)\left(\frac{1 - C_{\hat{a}}}{1 - C}\right)^{1+\delta} = \left(\frac{\hat{a}}{\hat{z}}\right)^{\zeta}$$

and implicit solution of the above equation for parabolic-type profile of $\varepsilon_t(z)$ is given as

(5)
$$\left(\frac{C}{C_{\hat{a}}}\right)\left(\frac{1 - C_{\hat{a}}}{1 - C}\right)^{1+\delta} = \left(\frac{1 - \hat{z}}{\hat{z}}\,\frac{\hat{a}}{1 - \hat{a}}\right)^{\zeta}.$$

A similar implicit solution can be found in Kumbhakar et al. [10] with an arbitrary turbulent diffusion coefficient that was kept under integral sign. In this work, we aim to find an explicit solution of the non-linear differential Eq. (1) for linear and parabolic-type profiles of $\varepsilon_t(z)$ with a constant $w_0$ by HPM.

## 3 Homotopy Perturbation Method

To explain the basic concept of Homotopy Perturbation Method as proposed by He [13], we consider the following non-linear system of differential equations:

$$\mathscr{A}(U) - f(\mathbf{r}) = 0, r \in \Omega \tag{6}$$

with boundary condition

$$\mathscr{B}(U, \frac{\partial U}{\partial n}) = 0, r \in \Gamma\,, \tag{7}$$

where $A$ is a differential operator, $f(\mathbf{r})$ is a known analytic function, $\Gamma$ is the boundary of the domain $\Omega$ and $\mathscr{B}$ is a boundary operator. The differential operator $\mathscr{A}$ can generally be divided into two parts $\mathscr{L}$ and $\mathscr{N}$, where $\mathscr{L}$ is linear and $\mathscr{N}$ is a non-linear operator. Therefore Eq. (6) can be rewritten as follows:

$$\mathscr{L}(U) + \mathscr{N}(U) - f(\mathbf{r}) = 0\,. \tag{8}$$

We can construct a homotopy $V(\mathbf{r}, q) : \Omega \times [0,1] \longrightarrow \mathbb{R}$, which satisfies

$$\mathscr{H}(V, q) = (1-q)\,[\mathscr{L}(V) - \mathscr{L}(u_0)] + q\,[\mathscr{A}(V) - f(\mathbf{r})] = 0\,, \tag{9}$$

or

$$\mathscr{H}(V, q) = \mathscr{L}(V) - \mathscr{L}(u_0) + q\,[\mathscr{L}(u_0) + \mathscr{N}(V) - f(\mathbf{r})] = 0\,, \tag{10}$$

where $q \in [0,1]$ is an embedding parameter and $u_0$ is an initial approximation for the solution of Eq. (6). It is better to choose an initial approximation that satisfies the initial data. Eq. (10) is called homotopy equation. Clearly we have,

$$\mathscr{H}(V, 0) = \mathscr{L}(V) - \mathscr{L}(u_0) = 0\,, \tag{11}$$

$$\mathscr{H}(V, 1) = \mathscr{A}(V) - f(\mathbf{r}) = 0\,. \tag{12}$$

The changing process of $q$ from zero to unity is just that of $V(r, q)$ from $u_0$ to $U(r)$ which is called deformation. If the embedding parameter $q$ ($q \in [0,1]$) is considered as a 'small parameter', then one can apply the classic perturbation method to get the solution to Eqs. (9) and (10) as a power series in $q$:

$$V = v_0 + v_1 q^1 + v_2 q^2 + \cdots = \sum_{i=0}^{\infty} v_i q^i \,. \tag{13}$$

Considering $q = 1$, the approximate solution of Eq. (6) can be obtained as follows:

$$U = \lim_{q \to 1} V = v_0 + v_1 + v_2 + \cdots \tag{14}$$

In many of the problems, the series Eq. (14) is convergent and leads to the exact solution of Eq. (6). For obtaining approximate solutions, one can take the closed form or truncate the series.

## 4 Solution Obtained by HPM

In this section, we obtain the closed form series solution of the non-linear differential Eq. (3) for linear and parabolic-type profile of turbulent diffusion coefficient. Equation (3) can be written as follows:

$$H(\hat{z})(1+\delta C)\frac{dC}{d\hat{z}} + \zeta C(1-C) = 0\,, \quad C(\hat{z} = \hat{a}) = C_{\hat{a}}\,. \tag{15}$$

There are several choices for linear and non-linear operators. Here we select the simplest one as $\mathscr{L}(C) = H(\hat{z})\frac{dC}{d\hat{z}}$ and $\mathscr{N} = H(\hat{z})\delta C\frac{dC}{d\hat{z}} + \zeta C(1-C)$. We can construct the following homotopy $V(\hat{z}, q) : \mathbf{D} \times [0,1] \to \mathbb{R}$ which satisfies

$$(1-q)\Big(H(\hat{z})\frac{\partial V}{\partial \hat{z}} - H(\hat{z})\frac{\partial u_0}{\partial \hat{z}}\Big) + q\Big(H(\hat{z})(1+\delta V)\frac{\partial V}{\partial \hat{z}} + \zeta V(1-V)\Big) = 0, \quad q \in [0,1],\ \hat{z} \in [\hat{a}, 1] \tag{16}$$

with initial approximation $u_0(\hat{z}) = C_{\hat{a}}$.

Suppose the solution of Eq. (16) has the form

$$V = v_0 + q v_1 + q^2 v_2 + \cdots \tag{17}$$

Substituting Eq. (17) into Eq. (16), and equating the terms with the identical powers of $q$,

$$q^0 : \frac{dv_0}{dz} = 0,\ v_0(\hat{a}) = C_{\hat{a}}. \tag{18}$$

$$q^1 : H(\hat{z})\frac{dv_1}{d\hat{z}} + \delta H(\hat{z}) v_0 \frac{dv_0}{d\hat{z}} + \zeta v_0 - \zeta v_0^2 = 0,\ v_1(\hat{a}) = 0. \tag{19}$$

(20) $$q^2 : H(\hat{z})\frac{dv_2}{d\hat{z}} + \delta H(\hat{z})\Big(v_0\frac{dv_1}{d\hat{z}} + v_1\frac{dv_0}{d\hat{z}}\Big) + \zeta v_1 - \zeta(2v_0v_1) = 0,\ v_2(\hat{a}) = 0.$$

$$\vdots$$

(21) $$q^j : H(\hat{z})\frac{dv_j}{d\hat{z}} + \mathscr{R}_{j-1} = 0,\ v_j(\hat{a}) = 0\,,$$

where

(22) $$\mathscr{R}_{j-1} = \delta H(\hat{z})\sum_{m=0}^{j-1} v_m\frac{dv_{j-1-m}}{d\hat{z}} + \zeta\Big(v_{j-1} - \sum_{m=0}^{j-1} v_m v_{j-m-1}\Big).$$

So we derive the following recurrence relation:

(23) $$v_j = \int_{\hat{a}}^{\hat{z}} \Big( -\delta\sum_{m=0}^{j-1} v_m\frac{dv_{j-1-m}}{d\hat{z}} - \zeta\frac{v_{j-1}}{H(\hat{z})} + \zeta\sum_{m=0}^{j-1}\frac{v_m v_{j-m-1}}{H(\hat{z})}\Big) d\hat{z}\,.$$

After solving Eq. (23), we get the following:

$$v_1 = \begin{cases} C_{\hat{a}}\zeta(C_{\hat{a}}-1)\log\dfrac{\hat{z}}{\hat{a}} & \text{for } H(\hat{z}) = \hat{z} \\ C_{\hat{a}}\zeta(C_{\hat{a}}-1)\log\Big(\dfrac{\hat{z}}{1-\hat{z}}\,\dfrac{1-\hat{a}}{\hat{a}}\Big) & \text{for } H(\hat{z}) = \hat{z}(1-\hat{z}) \end{cases}$$

$$v_2 = \begin{cases} C_{\hat{a}}^2\Big(\dfrac{2}{2!}\zeta^2(C_{\hat{a}}-1)\Big(\log\dfrac{\hat{z}}{\hat{a}}\Big)^2 - \dfrac{1}{1!}\delta\zeta(C_{\hat{a}}-1)\log\dfrac{\hat{z}}{\hat{a}}\Big) \\ \quad + C_{\hat{a}}\Big(\dfrac{-1}{2!}\zeta^2(C_{\hat{a}}-1)\Big[\log\dfrac{\hat{z}}{\hat{a}}\Big]^2\Big) & \text{for } H(\hat{z}) = \hat{z} \\ \\ C_{\hat{a}}^2\Big\{\dfrac{2}{2!}\zeta^2(C_{\hat{a}}-1)\Big[\log\Big(\dfrac{\hat{z}}{1-\hat{z}}\,\dfrac{1-\hat{a}}{\hat{a}}\Big)\Big]^2 \\ \quad -\dfrac{1}{1!}\delta\zeta(C_{\hat{a}}-1)\log\Big(\dfrac{\hat{z}}{1-\hat{z}}\,\dfrac{1-\hat{a}}{\hat{a}}\Big)\Big\} \\ \quad + C_{\hat{a}}\Big(\dfrac{-1}{2!}\zeta^2(C_{\hat{a}}-1)\Big[\log\Big(\dfrac{\hat{z}}{1-\hat{z}}\,\dfrac{1-\hat{a}}{\hat{a}}\Big)\Big]^2\Big) & \text{for } H(\hat{z}) = \hat{z}(1-\hat{z}) \end{cases}$$

$$v_3 = \begin{cases} C_{\hat a}^3\Big\{\frac{6}{3!}\zeta^3(C_{\hat a}-1)\Big(\log\frac{\hat z}{\hat a}\Big)^3 - \frac{5}{2!}\delta\zeta^2(C_{\hat a}-1)\Big(\log\frac{\hat z}{\hat a}\Big)^2 \\ \quad +\frac{1}{1!}\delta^2\zeta(C_{\hat a}-1)\log\frac{\hat z}{\hat a}\Big\} \\ \quad +C_{\hat a}^2\Big(\frac{-6}{3!}\zeta^3(C_{\hat a}-1)\Big(\log\frac{\hat z}{\hat a}\Big)^3 - \frac{-3}{2!}\delta\zeta^2(C_{\hat a}-1)\Big(\log\frac{\hat z}{\hat a}\Big)^2\Big) \\ \quad +C_{\hat a}\frac{1}{3!}\delta^3\zeta(C_{\hat a}-1)\Big(\log\frac{\hat z}{\hat a}\Big)^3 & \text{for } H(\hat z)=\hat z, \\ C_{\hat a}^3\Big\{\frac{6}{3!}\zeta^3(C_{\hat a}-1)\Big[\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^3 \\ \quad -\frac{5}{2!}\delta\zeta^2(C_{\hat a}-1)\Big[\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^2 \\ \quad +\frac{1}{1!}\delta^2\zeta(C_{\hat a}-1)\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big\} \\ \quad +C_{\hat a}^2\Big\{\frac{-6}{3!}\zeta^3(C_{\hat a}-1)\Big[\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^3 \\ \quad -\frac{-3}{2!}\delta\zeta^2(C_{\hat a}-1)\Big[\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^2\Big\} \\ \quad +C_{\hat a}\frac{1}{3!}\delta^3\zeta(C_{\hat a}-1)\Big[\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^3 & \text{for } H(\hat z)=\hat z(1-\hat z) \end{cases}$$

$$\vdots$$

(24)

$$v_m = \begin{cases} (C_{\hat a}-1)\sum_{i=1}^{m} C_{\hat a}^i \sum_{j=m-i+1}^{m} (-1)^{i+j}\delta^{m-j}\Big(\zeta\log\frac{\hat z}{\hat a}\Big)^j \frac{a_{ij}^m}{j!} & \text{for } H(\hat z)=\hat z \\ (C_{\hat a}-1)\sum_{i=1}^{m} C_{\hat a}^i \sum_{j=m-i+1}^{m} (-1)^{i+j}\delta^{m-j} \\ \qquad \times\Big[\zeta\log\Big(\frac{\hat z}{1-\hat z}\,\frac{1-\hat a}{\hat a}\Big)\Big]^j \frac{a_{ij}^m}{j!} & \text{for } H(\hat z)=\hat z(1-\hat z) \end{cases}$$

where

(25) $$a_{ij}^m = S(m,i)T(i,i+j-m)\,,$$

$S(n,m)$ is the Stirling numbers of the second kind which is defined as the number of ways of partitioning a set of $n$ elements into $m$ non-empty sets (i.e. $m$ set blocks). The Stirling numbers of the second kind are differently denoted by different authors such as by $\mathcal{S}_n^{(m)}$ [22], $S(n,m)$ [23] and Knuth's notation $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ [24]. The Stirling numbers of the second kind can be computed from the sum [25]

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)i \binom{k}{i} (k-i)^n \tag{26}$$

with $\binom{n}{k}$ a binomial coefficient. It satisfies the recurrence relation [26]

$$S(n+1,k) = S(n,k-1) + kS(n,k)\,. \tag{27}$$

For a fixed integer $k$, the Stirling numbers of the second kind have exponential generating function given by

$$\sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} = \frac{(e^x-1)^k}{k!} \tag{28}$$

and $T(n+1,k+1) = e_k(2,3,\ldots,n+1)$, $n \geq 0$, $k = 0,\ldots,n$, with the elementary symmetric function

$$e_k(x_1,x_2,x_3,\ldots,x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}\,. \tag{29}$$

It satisfies the recurrence relation

$$T(n,k) = T(n-1,k) + nT(n-1,k-1)\,. \tag{30}$$

In particular,

$$T(n,n) = n!\,, \tag{31}$$
$$T(n,1) = 1 \tag{32}$$

Finally, the closed form series solution of Eq. (15) can be obtained as

$$C(\hat{z}) = \sum_{m=0}^{\infty} v_m \tag{33}$$

and $n$-th order HPM based approximation of $C(\hat{z})$ can be obtained as

$$C(\hat{z}) = \sum_{m=0}^{n} v_m\,. \tag{34}$$

## 5 Result and Discussion

### 5.1 Selected expressions for the parameters

It is to be noted that the derived closed form series solution of concentration can be plotted only when the proportionality parameter $\beta$ and settling velocity $w_0$ are known. Here the settling velocity of particles is calculated from the widely used expression given by Cheng [27] which is as follows:

$$w_0 = \frac{\nu_f}{D_p}\left(\sqrt{25 + 1.2D_*^2} - 5\right)^{1.5}, \tag{35}$$

where $D_*\left( = \left(\frac{\triangle\, g}{\nu_f^2}\right)^{1/3} D_p\right)$ is the dimensionless particle diameter, $g$ is the gravitational acceleration and $\triangle = s - 1$, $s$ being the specific gravity of sediment particle. In the study of vertical concentration distribution, $\beta$ plays an important role. Many researchers [20, 21, 28, 29] reported that $\beta$ is a function of normalized settling velocity $\frac{w_0}{u_*}$. Pal and Ghoshal [30] established two relations of $\beta$ for dilute and non-dilute flow. Their study claims that $\beta$ does not depend only on normalized settling velocity but also on reference level and reference concentration. They gave the following expressions as:

$$\beta = \begin{cases} 0.033\left(\frac{w_0}{u_*}\right)^{0.931} C_{\hat{a}}^{-0.118}\hat{a}^{-1.196} & \text{for dilute flow} \\ 2.2040\left(\frac{w_0}{u_*}\right)^{0.667} C_{\hat{a}}^{0.017}\hat{a}^{0.178} & \text{for non-dilute flow} \end{cases}. \tag{36}$$

Reference level $\hat{a}$ was reported by Mazumder and Ghoshal [31, 32] as the lowest height given in the observed data set and the concentration there at as $C_{\hat{a}}$ which is followed in the present study.

### 5.2 Experimental data considered

In this section, we discuss about the experiment data used in the model. To check the applicability of the model, a revelent set of experimental data was chosen from Coleman [33] and Einstein and Chien [34]. A summary of selected data sets is given below.

Coleman [33] took a re-circulatory flume of 15 m long and 356 wide to investigate the effect of suspended sediment on the profile of fluid velocity and suspended load by using sediment concentration profile. Total 40 experimental runs were considered. From these data sets, three experiment runs are used in this study and flow parameters are summarized in Table 1.

Table 1: Summary of the selected experimental data from Coleman [33]

| Run | $\beta$ | $\hat{a}$ | $h$ (cm) | $C_a$ (%) | $w_0$ (cm/s) | $u_*$ (cm/s) | $D$ (cm) | $\hat{w}_0$ | $Re$ | $\eta$ | $D_*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 0.71 | 0.035 | 17.2 | 0.17 | 0.66 | 4.1 | 0.0105 | 0.16 | 0.70 | 4.45 | 2.66 |
| 4 | 0.67 | 0.035 | 17.1 | 0.28 | 0.66 | 4.1 | 0.0105 | 0.16 | 0.70 | 4.45 | 2.66 |
| 11 | 0.50 | 0.036 | 16.9 | 1.20 | 0.61 | 4.1 | 0.0100 | 0.15 | 0.61 | 4.47 | 2.53 |

Similarly, experimental data of Einstein and Chien [34] was selected for verification. Einstein and Chien [34] measured concentration values very near to the channel bed and reported 16 different measurement runs, named Run S1 to S16. Their experiments were carried out in a recirculating flume of 12.2 m length and 30.7 cm width for three different particle sizes, namely, 0.274 mm, 940 mm, and 1.3 mm. Summary of the flow parameters for the selected runs are given in Table 2.

Table 2: Summary of the selected experimental data from Einstein and Chien [34]

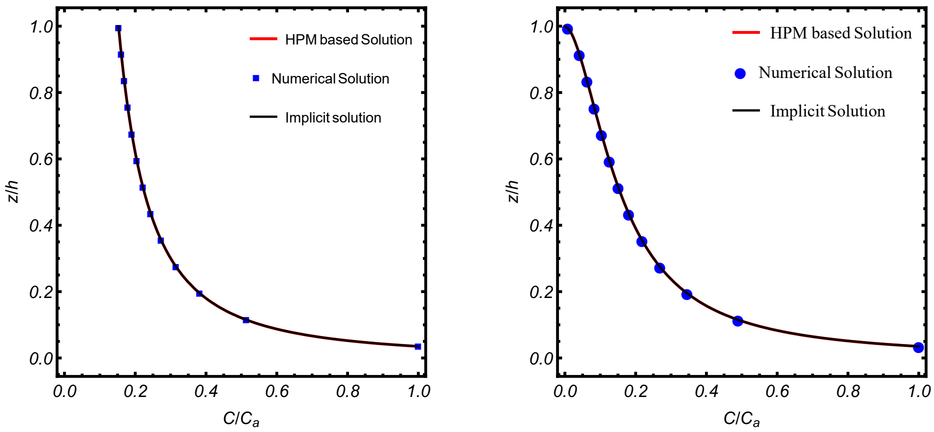
| Run | $\beta$ | $\hat{a}$ | $h$ (cm) | $C_a$ (%) | $w_0$ (cm/s) | $u_*$ (cm/s) | $D$ (cm) | $\hat{w}_0$ | $Re$ | $\eta$ | $D_*$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| S1 | 1.30 | 0.04 | 13.8 | 2.189 | 13.51 | 11.47 | 0.130 | 1.18 | 175.67 | 2.62 | 32.89 |
| S6 | 1.09 | 0.04 | 14.3 | 1.057 | 10.62 | 11.82 | 0.090 | 0.90 | 99.85 | 2.78 | 23.78 |
| S12 | 0.51 | 0.03 | 13.2 | 7.721 | 2.96 | 10.09 | 0.027 | 0.29 | 7.90 | 3.57 | 6.83 |

### 5.3 Verification of the HPM-based solution

HPM based closed form series solution is compared with numerical and implicit solution in this section. To do so, Mathematica software is used on a PC having

Table 3: Comparison of the numerical solution of Eq. (15) ($H(\hat{z}) = \hat{z}$) with the different order of approximations of HPM based solution for Run-3 of Coleman [33] experimental data.

| $z/h$ | Numerical solution | $10^{th}$ | $9^{th}$ | $8^{th}$ | $7^{th}$ | $6^{th}$ | $5^{th}$ | $4^{th}$ | $3^{rd}$ | $2^{nd}$ | $1^{st}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.03500 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.11500 | 0.51374 | 0.51374 | 0.51374 | 0.51374 | 0.51374 | 0.51375 | 0.51362 | 0.51475 | 0.50645 | 0.55585 | 0.33391 |
| 0.19500 | 0.38216 | 0.38216 | 0.38216 | 0.38216 | 0.38214 | 0.38231 | 0.38116 | 0.38811 | 0.35237 | 0.50064 | 0.03822 |
| 0.27500 | 0.31519 | 0.31519 | 0.31519 | 0.31521 | 0.31511 | 0.31569 | 0.31234 | 0.32947 | 0.25558 | 0.51161 | -0.15427 |
| 0.35500 | 0.27316 | 0.27316 | 0.27316 | 0.27320 | 0.27297 | 0.27425 | 0.26758 | 0.29813 | 0.18049 | 0.54369 | -0.29725 |
| 0.43500 | 0.24375 | 0.24375 | 0.24374 | 0.24382 | 0.24339 | 0.24565 | 0.23470 | 0.28105 | 0.11659 | 0.58381 | -0.41104 |
| 0.51500 | 0.22174 | 0.22175 | 0.22172 | 0.22186 | 0.22116 | 0.22466 | 0.20860 | 0.27255 | 0.05958 | 0.62697 | -0.50557 |
| 0.59500 | 0.20450 | 0.20451 | 0.20446 | 0.20467 | 0.20363 | 0.20864 | 0.18676 | 0.26967 | 0.00731 | 0.67096 | -0.58642 |
| 0.67500 | 0.19054 | 0.19055 | 0.19048 | 0.19078 | 0.18934 | 0.19607 | 0.16776 | 0.27069 | -0.04144 | 0.71473 | -0.65706 |
| 0.75500 | 0.17894 | 0.17896 | 0.17887 | 0.17927 | 0.17736 | 0.18603 | 0.15075 | 0.27455 | -0.08743 | 0.75777 | -0.71977 |
| 0.83500 | 0.16912 | 0.16914 | 0.16902 | 0.16955 | 0.16710 | 0.17791 | 0.13520 | 0.28053 | -0.13116 | 0.79983 | -0.77617 |
| 0.91500 | 0.16066 | 0.16069 | 0.16054 | 0.16120 | 0.15817 | 0.17128 | 0.12075 | 0.28812 | -0.17300 | 0.84078 | -0.82740 |




(a) $H(\hat{z}) = \hat{z}$ (b) $H(\hat{z}) = \hat{z}(1-\hat{z})$

Fig. 2: Validation of the HPM-based solution of Eq. (15) with numerical and implicit solution for Run-3 of Coleman [33] experimental data.

configurations as Intel (R) Core (TM) i5-6500 CPU 3.20 GHz 64-bit with 4.00 GB of RAM. Comparison of explicit series solution with numerical and implicit solution is displayed in Fig. 2 for constant and parabolic eddy diffusivity. It can be seen from the figure that the closed form series solution obtained from HPM is closed enough to numerical and implicit solution. To check the accuracy of HPM based solution, two tables are also provided that report numerical solution and HPM based series solution for ($H(\hat{z}) = \hat{z}(1-\hat{z})$). Table 3 shows the comparison of 1$^{\text{st}}$ to 10$^{\text{th}}$ order of approximations of HPM based series solution for concentration profile with

Table 4: Comparison of the numerical solution of Eq. (15) ($H(\hat{z}) = \hat{z}(1-\hat{z})$) with the different order of approximations of HPM based solution for Run-3 of Coleman [33] experimental data.

| $z/h$ | Numerical solution | $20^{th}$ | $19^{th}$ | $18^{th}$ | $17^{th}$ | $16^{th}$ | $15^{th}$ | $14^{th}$ | $13^{th}$ | $12^{nd}$ | $11^{th}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.03500 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.11500 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 | 0.48942 |
| 0.19500 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 | 0.34524 |
| 0.27500 | 0.26851 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 | 0.26852 |
| 0.35500 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 | 0.21794 |
| 0.43500 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18056 | 0.18055 |
| 0.51500 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15078 | 0.15077 |
| 0.59500 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12568 | 0.12569 | 0.12566 |
| 0.67500 | 0.10351 | 0.10351 | 0.10351 | 0.10351 | 0.10351 | 0.10351 | 0.10351 | 0.10351 | 0.10350 | 0.10352 | 0.10346 |
| 0.75500 | 0.08296 | 0.08296 | 0.08296 | 0.08296 | 0.08296 | 0.08296 | 0.08296 | 0.08297 | 0.08295 | 0.08300 | 0.08286 |
| 0.83500 | 0.06282 | 0.06282 | 0.06282 | 0.06282 | 0.06281 | 0.06282 | 0.06281 | 0.06283 | 0.06279 | 0.06287 | 0.06266 |
| 0.91500 | 0.04114 | 0.04114 | 0.04114 | 0.04114 | 0.04114 | 0.04114 | 0.04113 | 0.04112 | 0.04124 | 0.04083 | 0.04148 |

numerical solution for linear profile of turbulent diffusivity and Table 4 shows the same for $11^{th}$ to $20^{th}$ order of approximations for parabolic type turbulent diffusivity.

### 5.4 Convergence of HPM-based solution

The convergence of the method is discussed in Ayati and Biazar [35] and the proof is given in the appendix for convenience of the readers. To check the convergence of the obtained series, Ayati and Biazar [35] mentioned a theorem which is given as follows:

**Theorem 1.** *Let $B$ be a Banach space. $\sum_{i=1}^{\infty} v_i$ obtained by Eqs. (58), converges to $s \in B$, if $\exists\, 0 \leq \lambda < 1$, s.t $(\forall n \in \mathbb{N} \Rightarrow ||v_{n+1}|| \leq \lambda ||v_n||)$*

For the first case (Linear profile of turbulent diffusion coefficient), we take function Banach space $\mathcal{C}(\hat{a}, \hat{z}_m)$ of all continuous functions that are defined in a closed interval $[\hat{a}, \hat{z}_m]$ ($\hat{z}_m$ is the maximum height available in a particular experimental data set) and the norm $||f||$ defined on $\mathcal{C}(\hat{a}, \hat{z}_m)$ is the maximum absolute value of $f$ for $\hat{a} \leq \hat{z} \leq \hat{z}_m$, i.e.

$$||f|| = \max_{\hat{a} \leq \hat{z} \leq \hat{z}_m} \mid f(\hat{z}) \mid$$

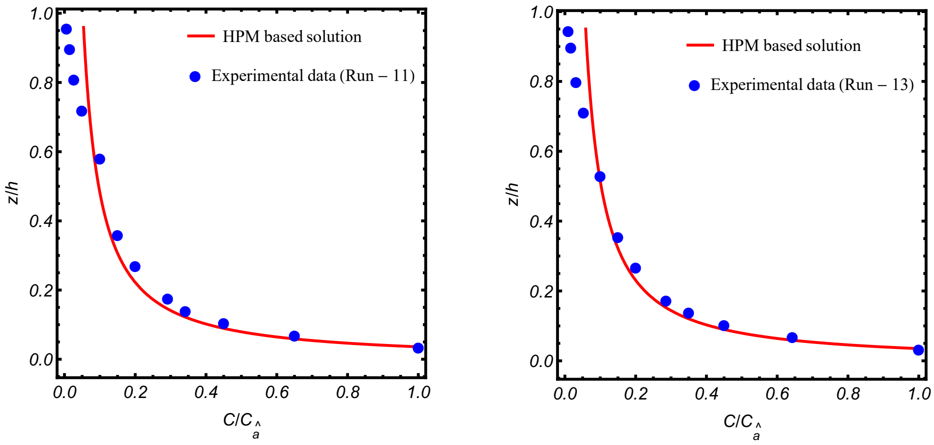
So in this case, using Eqs. (24) (for $H(\hat{z}) = \hat{z}$), we have

$$\begin{aligned}
||v_1|| &= 0.00313426\\
||v_2|| &= 0.00288665\\
||v_3|| &= 0.00176986\\
||v_4|| &= 0.000812154\\
||v_5|| &= 0.000297367\\
||v_6|| &= 0.0000905455\\
&\vdots
\end{aligned}$$

For all $n \in \mathbb{N}, \exists\, \lambda = 0.920999$ s.t $||v_{n+1}|| \leq 0.920999\, ||v_n||$. Hence our series $\sum_{i=1}^{\infty} v_i$ converges to $s \in \mathcal{C}(\hat{a}, \hat{z}_m)$.

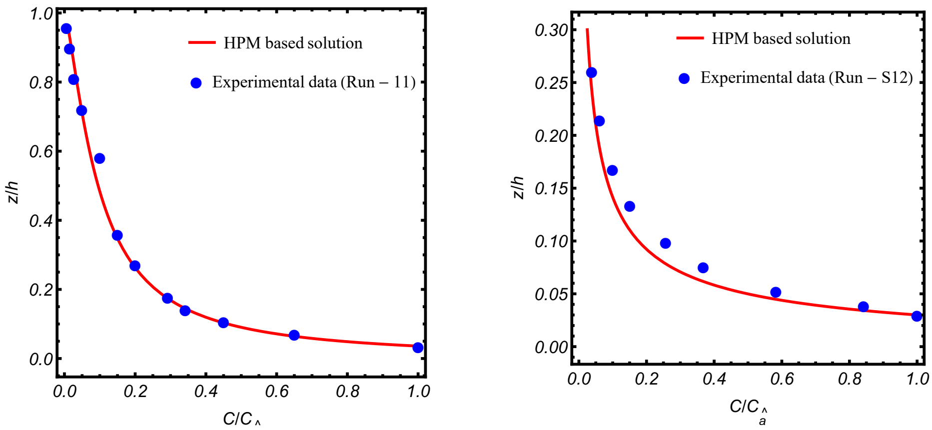
### 5.5 Comparison between solution of Eq. (15) with experimental data

To compare the HPM based solution of Eq. (15) with the experimental data, three data sets from Coleman [33] and one from Einstein and Chien [34] are chosen and the comparison is presented in Figs. 3 and 4. The continuous line stands for HPM based solution and the dots represent the experimental data. Fig. 3 shows that the model does not agree well with experimental data near the water surface; this may be due




(a) $H(\hat{z}) = \hat{z}$ (b) $H(\hat{z}) = \hat{z}$

Fig. 3: Comparison between computed particle concentration profile (Eq. (33)) and observed data of Coleman [33] (a) Run-11 and (b) Run-13.




(a) $H(\hat{z}) = \hat{z}(1 - \hat{z})$ (b) $H(\hat{z}) = \hat{z}(1 - \hat{z})$

Fig. 4: Comparison between computed particle concentration profile (Eq. (33)) and observed data of Coleman [33] (a) Run-11 and Einstein and Chein [34] (b) Run-S12.

to the linear profile chosen for turbulent diffusivity and constant settling velocity. On the other hand, Fig. 4 shows that the model agrees well for Run-11 of Coleman [33] but not for Run-S12 of Einstein and Chien [34] in the main flow region. This may be due the reason that Run-S12 of Einstein and Chien [34] is of high concentration and the constant settling velocity in the model is not capable to match this type of data. So further modification is required regrading settling velocity.

## 6 Further Modification to the Model and Solution

In this section, we modify our model by incorporating hindered settling effect and parabolic-type profile of turbulent diffusion coefficient. The modification has already been done by Kumbhakar et al. [10] and the modified equation has been solved by HAM; but they [10] did not provide the closed form of HAM based series solution and convergence of series. As mentioned previously, turbulent diffusion coefficient plays a significant role in modelling the concentration and have been found to have three types of profiles: constant, linear and parabolic-type. Researchers [20,21] found that the parabolic profile estimates experimental data better than the others. Thus, parabolic-type profile of turbulent diffusion coefficient is chosen here which is given as (also written as a part of Eq. (2))

$$\varepsilon_t(z) = \kappa u_* z\left(1 - \frac{z}{h}\right). \tag{37}$$

An accurate determination of the settling velocity of sediment particles is fundamental to the modelling of sediment suspension. In turbulent flow with suspended particles, a large number of experimental observation revealed that the settling velocity is less as compared to the settling velocity in clear water flow which is the hindered settling effect. Richardson and Zaki [36] proposed the expression for settling velocity which is given as

$$w_s = w_0(1 - C)^{\eta}, \tag{38}$$

where $w_0$ is the settling velocity of particle in clear water and $\eta$ is the exponent of reduction of settling velocity that depends on the particle Reynolds number $Re$ as follows:

$$\eta = \begin{cases} 4.65 & \text{when } Re < 0.2 \\ 4.4Re^{-0.03} & \text{when } 0.2 < Re < 1 \\ 4.4Re^{-0.1} & \text{when } 1 < Re < 500 \\ 2.4 & \text{when } Re > 500 \end{cases}, \tag{39}$$

where $Re = \dfrac{w_0 D_p}{\nu_f}$.

Now the expression of turbulent diffusion coefficient is taken from Eq. (37) and in Eq. (1) settling velocity $w_0$ of sediment particles in the clear water is replaced with the expression of settling velocity $w_s$ of sediment particles in sediment-laden flow [21]. Then using Eq. (38), the non-dimensional form of Eq. (1) becomes

$$\hat{z}(1-\hat{z})(1+\delta C)\frac{dC}{d\hat{z}}+\zeta C(1-C)^{\eta+1}=0\,. \tag{40}$$

As the maximum value of $C$ can be 0.6 [37] and the reference concentration $C_{\hat{a}}$ is much smaller than 0.6, we can write Eq. (40) as

$$\hat{z}(1-\hat{z})(1+\delta C)\frac{dC}{d\hat{z}}+\zeta(C-(\eta+1)C^2)=0\,. \tag{41}$$

Implicit solution of above equation is

$$\Big(\frac{C}{C_{\hat{a}}}\Big)\Big(\frac{1-(\eta+1)C_{\hat{a}}}{1-(\eta+1)C}\Big)^{(\eta+1)+\delta}=\Big(\frac{1-\hat{z}}{\hat{z}}\,\frac{\hat{a}}{1-\hat{a}}\Big)^{\zeta}. \tag{42}$$

A similar solution can be found in Kumbhakar et al. [10] who provided the solution with two more terms in the binomial expansion of Eq. (40) with an arbitrary turbulent diffusion coefficient that was kept under integral sign. Now, we discuss about the HPM based series solution of Eq. (41). There are lots of choices for linear and non-linear operators to solve the non-linear Eq. (41). For this equation, we select the same single term linear operator which is

$$\mathscr{L}(C)=\hat{z}(1-\hat{z})\frac{dC}{d\hat{z}} \tag{43}$$

and non-linear operator as

$$\mathscr{N}=\hat{z}(1-\hat{z})\delta C\frac{dC}{d\hat{z}}+\zeta(C-(\eta+1)C^2). \tag{44}$$

We can construct the following homotopy $W(\hat{z},q):[\hat{a},1]\times[0,1]\longrightarrow\mathbb{R}$ which satisfies

$$(1-q)\Big(\hat{z}(1-\hat{z})\frac{\partial W}{\partial\hat{z}}-\hat{z}(1-\hat{z})\frac{du_0}{d\hat{z}}\Big)+q\Big[\hat{z}(1-\hat{z})(1+\delta W)\frac{\partial W}{\partial\hat{z}}+\zeta\bigg(W-(\eta+1)W^2\bigg)\Big]=0 \tag{45}$$

with the initial approximation $u_0=C_{\hat{a}}$. Suppose the solution of Eq. (45) has the form

$$W=W_0+qW_1+q^2W_2+\cdots \tag{46}$$

Now we follow the same steps which are mentioned in Section 4 and finally get the recurrence relation as

$$
\begin{aligned}
(47)\quad W_j =& \int_{\hat{a}}^{\hat{z}} \Big[ -\delta \sum_{m=0}^{j-1} W_m \frac{dW_{j-1-m}}{d\hat{z}} - \frac{\zeta}{\hat{z}(1-\hat{z})} \Big( W_{j-1} - (\eta+1) \sum_{m=0}^{j-1} W_m W_{j-m-1} \Big) \Big] d\hat{z} \\
=& \left(C_{\hat{a}}(\eta+1) - 1\right) \sum_{i=1}^{m} C_{\hat{a}}^{i} \sum_{j=m-i+1}^{m} (-1)^{i+j} \delta^{m-j} (\eta+1)^{j-(m-i+1)} \\
& \times \left[ \zeta \log \left( \frac{\hat{z}}{1-\hat{z}} \frac{1-\hat{a}}{\hat{a}} \right) \right]^{j} \frac{a_{ij}^{m}}{j!}.
\end{aligned}
$$

Comparison of HPM based series solution of Eq. (41) with numerical and implicit solution is displayed in Fig. 5. It can be seen that the closed form series solution obtained from HPM is closed enough to numerical solution as well as with implicit solution. To check the accuracy of HPM based solution like the previous cases, here also a table is provided that reports numerical solution and HPM based series solution

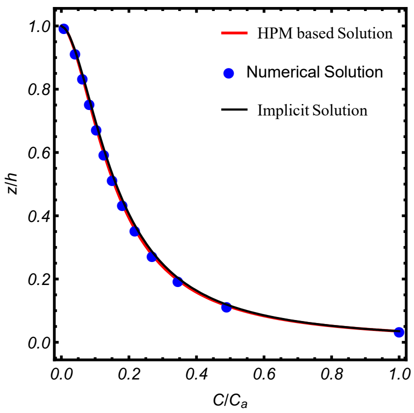



Fig. 5: Validation of HPM-based solution of Eq. (41) with numerical solution and implicit solution for Run-3 of Coleman [33] experimental data.

Table 5: Comparison of the numerical solution of Eq. (41) with different orders of approximation of HPM based solution for Run-3 of Coleman [33] experimental data.

| $z/h$ | Numerical solution | $23^{rd}$ | $22^{nd}$ | $21^{st}$ | $20^{th}$ | $19^{th}$ | $18^{th}$ | $17^{th}$ | $16^{th}$ | $15^{th}$ | $14^{th}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.03500 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.11500 | 0.49132 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 | 0.49133 |
| 0.19500 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 | 0.34697 |
| 0.27500 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 | 0.27001 |
| 0.35500 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 | 0.21924 |
| 0.43500 | 0.18168 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 | 0.18169 |
| 0.51500 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15175 | 0.15176 |
| 0.59500 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12652 | 0.12653 |
| 0.67500 | 0.10421 | 0.10421 | 0.10421 | 0.10421 | 0.10421 | 0.10421 | 0.10421 | 0.10421 | 0.10422 | 0.10421 | 0.10424 |
| 0.75500 | 0.08354 | 0.08354 | 0.08354 | 0.08354 | 0.08354 | 0.08354 | 0.08354 | 0.08354 | 0.08355 | 0.08350 | 0.08363 |
| 0.83500 | 0.06327 | 0.06327 | 0.06326 | 0.06327 | 0.06326 | 0.06327 | 0.06326 | 0.06326 | 0.06334 | 0.06306 | 0.06369 |
| 0.91500 | 0.04144 | 0.04144 | 0.04143 | 0.04148 | 0.04134 | 0.04158 | 0.04132 | 0.04132 | 0.04223 | 0.03940 | 0.04506 |

for different order of approximations. Comparison of $14^{\text{th}}$ to $23^{\text{rd}}$ order of approximations of HPM based series for concentration profile $C(\hat{z})$ with numerical solution is shown in Table. 5.

### 6.1 Comparison between solution of Eq. (41) with experimental data

In this section, we compare the proposed model with experimental data. For that purpose, experimental data of Coleman [33] and Einstein and Chien [34] are used. Comparison of present model with selected runs from the Coleman [33] and Einstein and Chien [34] is shown in Fig. 6. From the figure, it can be noticed that the modified model agrees well with experimental data through out the water depth and the agreement is better than the previous two cases.

## 7 Conclusions

The present study derives the closed form series solution of Hunt [1] equation for suspended sediment using an analytical method called Homotopy Perturbation Method (HPM). The Hunt [1] equation is taken in three different forms. In the first form, the eddy diffusivity is linear and settling velocity is constant; in the second form, the eddy diffusivity is parabolic and settling velocity is constant and in the third form, the eddy diffusivity is parabolic and settling velocity is concentration dependent to deal with high concentration flow. For all the three cases, closed form series solutions are obtained in terms of Striling number and elementary symmetric function. The HPM based series solutions are compared with the implicit solutions as well as with the numerical solutions for all the cases. Convergence of the Homotopy Perturbation Method is discussed and the obtained series solution has been shown to be convergent for the first case. Finally, the solution of modified governing equation with concentration dependent settling velocity and parabolic eddy diffusivity, has been compared with different sets of experimental data and comparison shows good agreement.

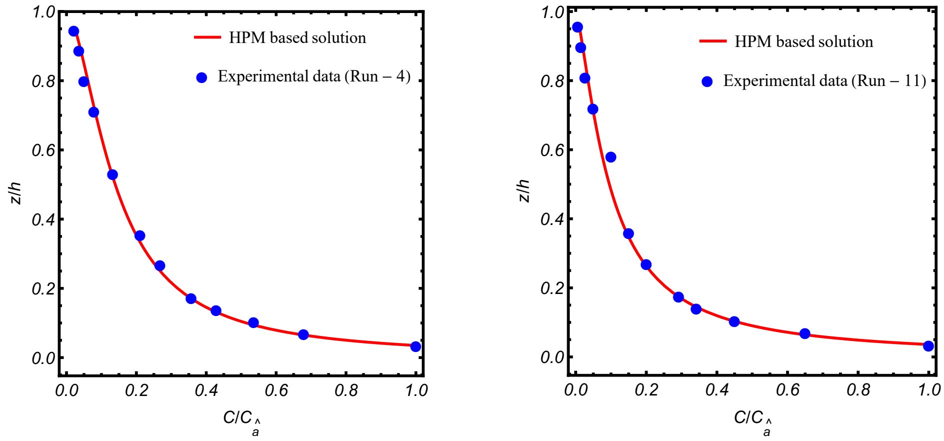
## Acknowledgements

The first author is thankful to Mr. Nitin Kumar, Senior Research Scholar, Dept. of Mathematics, IIT Kharagpur for discussion regarding computation.

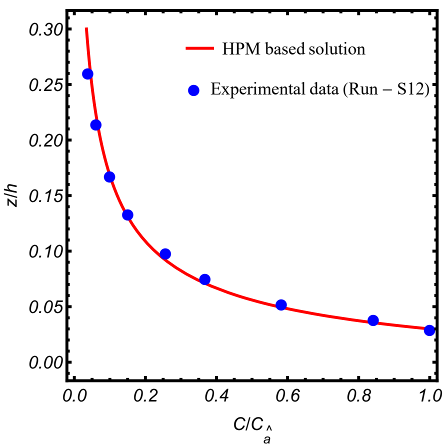
## Appendix: Convergence of the method

Convergence of HPM as discussed in Ayati and Biazar [35] is given as follows: Let us rewrite Eq. (10) as follows:

$$\mathscr{L}(V) - \mathscr{L}(u_0) = q[f(r) - \mathscr{L}(u_0) - \mathscr{N}(V)]. \tag{48}$$




(a) (b)




(c)

Fig. 6: Comparison between computed particle concentration profile (Eq. (46)) and observed data of Coleman [33]: (a) Run-4; (b) Run-11 and Einstein and Chien [34]; (c) Run-S12.

Substituting Eq. (13) into Eq. (48) leads to

$$\mathscr{L}\left(\sum_{i=0}^{\infty} v_i q^i\right) - \mathscr{L}(u_0) = q\left[f(r) - \mathscr{L}(u_0) - \mathscr{N}\left(\sum_{i=0}^{\infty} v_i q^i\right)\right]. \tag{49}$$

So

$$\sum_{i=0}^{\infty} \mathscr{L}(v_i) q^i - \mathscr{L}(u_0) = q\left[f(r) - \mathscr{L}(u_0) - \mathscr{N}\left(\sum_{i=0}^{\infty} v_i q^i\right)\right]. \tag{50}$$

According to Maclaurin's expansion of $\mathscr{N}\left(\sum_{i=0}^{\infty} v_i q^i\right)$ with respect to $p$, we have

$$\mathscr{N}\left(\sum_{i=0}^{\infty} v_i q^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial q^n}\sum_{i=0}^{\infty} v_i q^i\right)_{q=0} q^i\,. \tag{51}$$

From Ghorbani [38], we get

$$\left(\frac{\partial^n}{\partial q^n}\mathscr{N}\Big(\sum_{i=0}^{\infty} v_i q^i\Big)\right)_{q=0} = \left(\frac{\partial^n}{\partial q^n}\mathscr{N}\Big(\sum_{i=0}^{n} v_i q^i\Big)\right)_{q=0} \tag{52}$$

Then

$$\mathscr{N}\left(\sum_{i=0}^{\infty} v_i q^i\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial q^n}\mathscr{N}\Big(\sum_{i=0}^{n} v_i q^i\Big)\right)_{q=0} q^i\,. \tag{53}$$

We set

$$\mathscr{R}_n(v_0, v_1, \ldots, v_n) = \left(\frac{1}{n!}\frac{\partial^n}{\partial q^n}\mathscr{N}\Big(\sum_{i=0}^{n} v_i q^i\Big)\right)_{q=0}, \quad n = 0, 1, 2, \ldots, \tag{54}$$

where $\mathscr{R}_n$'s are the so-called He's polynomials [38]. Then

$$\mathscr{N}\Big(\sum_{i=0}^{\infty} v_i q^i\Big) = \sum_{n=0}^{\infty}\mathscr{R}_n q^n\,. \tag{55}$$

Substituting Eq. (55) into Eq. (50), we derive

$$\sum_{i=0}^{\infty}\mathscr{L}(v_i)q^i - \mathscr{L}(u_0) = p\Big[f(r) - \mathscr{L}(u_0) - \sum_{n=0}^{\infty}\mathscr{R}_n q^n\Big]. \tag{56}$$

By equating the terms with identical powers in $p$

$$\begin{cases} q^0 & : \mathscr{L}(v_0) - \mathscr{L}(u_0) = 0, \\ q^1 & : \mathscr{L}(v_1) = f(r) - \mathscr{L}(u_0) - \mathscr{R}_0, \\ q^2 & : \mathscr{L}(v_2) = -\mathscr{R}_1, \\ \quad\vdots & \\ q^{n+1} & : \mathscr{L}(v_{n+1}) = -\mathscr{R}_n, \\ \quad\vdots & \end{cases} \tag{57}$$

So we derive

$$\begin{cases} v_0 = u_0, \\ v_1 = \mathscr{L}^{-1}[f(r)] - u_0 - \mathscr{L}^{-1}(\mathscr{R}_0), \\ v_2 = -\mathscr{L}^{-1}(\mathscr{R}_1), \\ \vdots \\ v_{n+1} = -\mathscr{L}^{-1}(\mathscr{R}_n), \\ \vdots \end{cases} \tag{58}$$

**Theorem 2.** *Let $B$ be a Banach space. $\sum_{i=1}^{\infty} v_i$ obtained by Eq. (58), converges to $s \in B$, if $\exists\ 0 \leq \lambda < 1$, s.t* $(\forall n \in \mathbb{N} \Rightarrow ||v_{n+1}|| \leq \lambda ||v_n||)$

*Proof.* We have

$$||s_{n+1} - s_n|| = ||v_{n+1}|| \leq \lambda ||v_n|| \leq \lambda^2 ||v_{n-1}|| \leq \cdots \leq \lambda^n ||v_1|| \,. \tag{59}$$

For any $n, m \in \mathbb{N}$, $n \geq m$, we derive

$$\begin{aligned} ||s_n - s_m|| &= ||(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)|| \\ &\leq ||(s_n - s_{n-1})|| + ||(s_{n-1} - s_{n-2})|| + \cdots + ||(s_{m+1} - s_m)|| \\ &\leq \lambda^{n-1} ||v_1|| + \lambda^{n-2} ||v_1|| + \cdots \lambda^m ||v_1|| \\ &\leq (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1} + \cdots) ||v_1|| \\ &\leq \lambda^m (1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + \cdots) ||v_1|| \\ &= \frac{\lambda^m}{1 - \lambda} ||v_1|| \end{aligned} \tag{60}$$

So

$$\lim_{n,m \to \infty} ||s_n - s_m|| = 0 \,. \tag{61}$$

Then $\{s_n\}$ is Cauchy sequence in Banach space and it is convergent, i.e., $\exists\ s \in \mathbf{B}$, s.t

$$\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} v_n = s \,. \tag{62}$$

□

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