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In memory of JORDAN G. BRANKOV

ON DYNAMICS OF AN OPEN BOSON SYSTEM

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ABSTRACT: The core property of generator and the trace-preserving property of a minimal dynamical semigroup constructed by regularisation à la Kato are scrutinised for a simple boson model in the framework of the Gorini–Kossakowski–Lindblad–Davies approach to the open systems.

KEY WORDS: Dynamical semigroup, perturbation, Kato regularisation, positivity preserving.

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1 INTRODUCTION

Construction of stochastic semigroup for a time homogeneous Markov chain with an infinite matrix of transition probabilities goes back to T. Kato [1]. To this aim he proposed a convergent *one-parameter* regularisation family of approximants, which later was extended by E.B. Davies [2] for construction of quantum *dynamical semigroups*.

In the recent paper [4] a new step to generalisation of the one-parameter Kato–Davies regularisation was suggested by introducing a *functional* version of regularisation family for *bounded* approximants. Given that, the boson multi-mode particle-number cut-off regularisation family is a *net*. Using the functional regularisation it was proved in [4] that (similar to [1] and [2]) the semigroup constructed within this general setting has the property to be *minimal*. This property is related to the *core condition* for the generator of the minimal semigroup, which implies that it is Markovian. Note that applying Kato's one-parametric regularisation, this result was established in [2] for a special set of quantum dynamical semigroups with generators that have a *canonical* Gorini–Kossakowski–Lindblad–Davies (GKLD) form.

We recall that for the minimal quantum GKLD semigroups the conditions formulated in [2] are *sufficient* and *necessary* to promise the Markov property of this semigroup. For review and discussion see [5].

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The aim of the present paper is an inspection of a dynamical semigroup Markov property within the GKLD *ansatz*. We explore it, including a *critical* regime for parameters of the semigroup generator. To this end we consider in Section 2 a generator of dynamical semigroup for the open quantum model of bosons, which is motivated by [6]. For this system the functional regularisation reduces to the one-mode particle-number cut-off in the boson Fock space. In the framework of the GKLD *ansatz* we consider the core and the Markov properties of the GKLD semigroup in Sections 2 and 3. Taking into account that the GKLD evolution of our system is *quasi-free* and *Gaussian*, we scrutinise dynamics of this system for different sets of initial conditions and regimes in Section 4.

2 Open Boson Model

2.1 The GKLD approach to open systems

We consider a simple example of a general *functional regularisation* à la Kato, proposed in [4], which in our case is a particle-number cut-off in a boson Fock space $\mathscr{F}$. In this way one can construct on the space of the trace-class operators (including quantum statistical *density matrices*) an evolution owing to *dynamical semigroup*, which is *minimal*, *positivity-preserving* and *Markovian* (trace-preserving). As a basic model we scrutinise the one-mode quantum resonator (cf. boson models studied in [6] and [7]) within the Gorini–Kossakowski–Lindblad–Davies (GKLD) *approach* to open systems by means of *dissipative* extension of autonomous dynamics. For an exhaustive review see, e.g., [8].

Let b and b^* be boson annihilation and creation operators defined in the Fock space $\mathscr{F}$ generated by a cyclic vector Ω. That is, $\mathscr{F}$ is a Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$ with $e_0 = \Omega$. The Bose operators b, b^* are defined by

$$b\, e_n = \sqrt{n}\, e_{n-1} \quad \text{and} \quad b^*\, e_n = \sqrt{n+1}\, e_{n+1}\,,$$

for all $n \in \mathbb{N}_0$, with domain $\mathrm{dom}(b) = \mathrm{dom}(b^*) = \{\psi \in \mathscr{F} : \sum_{n=0}^{\infty} n\, |(\psi, e_n)_{\mathscr{F}}|^2 < \infty\}$, here we set $e_{-1} = 0$. The Bose operators satisfy the Canonical Commutation Relation (CCR): $(b\, b^* - b^*\, b)\psi = \psi$ and $(b\, b - b\, b)\psi = (b^*\, b^* - b^*\, b^*)\psi = 0$ for all $\psi \in \mathrm{dom}(b^*\, b)$.

As an *isolated* system we consider the *one-mode* quantum *resonator* with equidistant discrete spectrum with spacing $E > 0$ defined by self-adjoint Hamiltonian

$$h := E\, b^*\, b\,, \tag{2.1}$$

on domain $\mathrm{dom}(h) = \{\psi \in \mathscr{F} : \sum_{n=0}^{\infty} n^2\, |(\psi, e_n)_{\mathscr{F}}|^2 < \infty\}$. The *boson-number* operator $\hat{n} := b^*\, b$ on $\mathrm{dom}(\hat{n}) = \mathrm{dom}(h) \subset \mathscr{F}$, counts the number of bosons

$(\hat{n}\,\psi, \psi)_{\mathscr{F}}$ in a quantum *vector-state* of resonator $\psi \in \mathscr{F}$, for normalised vector $\|\psi\|_{\mathscr{F}} = 1$.

Let $\mathfrak{C}_1 := \mathfrak{C}_1(\mathscr{F})$ be a complex Banach space of *trace-class* operators on $\mathscr{F}$ with *trace-norm* $\|\phi\|_1 = \mathrm{Tr}\,(\sqrt{\phi^*\phi})$, $\phi \in \mathfrak{C}_1(\mathscr{F})$. Recall that its *dual* space $\mathfrak{C}_1^*(\mathscr{F})$ is *isometrically isomorphic* to the Banach space of bounded operators $\mathcal{L}(\mathscr{F})$. The corresponding dual pair is determined by the bilinear *trace functional*:

$$\langle \phi \mid A \rangle_{\mathfrak{C}_1(\mathscr{F})\times\mathcal{L}(\mathscr{F})} := \mathrm{Tr}\,(\phi\, A)\,, \quad \phi \in \mathfrak{C}_1(\mathscr{F})\,,\;\; A \in \mathcal{L}(\mathscr{F})\,. \tag{2.2}$$

The quantum-mechanical Hamiltonian evolution of *isolated* system (2.1):

$$\partial_t \rho \,=\, -i\,[h, \rho]\,, \tag{2.3}$$

for *density matrix*: $\rho \in \mathfrak{C}_1^+ := \{\rho \in \mathfrak{C}_1(\mathscr{F}) : \rho \geq 0\}$ and $\|\rho\|_{\mathfrak{C}_1} = 1$, is determined by the unitary group $\{U_{ih}(t)\}_{t\in\mathbb{R}}$, where $U_{ih}(t) = e^{-ith} \in \mathcal{L}(\mathscr{F})$ and $t \in \mathbb{R}$. For all $t \in [0, \infty)$ we define the mapping $W_t \colon \mathfrak{C}_1^{\mathrm{sa}} \to \mathfrak{C}_1^{\mathrm{sa}}$, by

$$W_t\rho = U_{ih}(t)\,\rho\,U_{ih}(t)^*\,, \quad \rho \in \mathfrak{C}_1^+\,. \tag{2.4}$$

Then $\{W_t\}_{t\geq 0}$ (2.4) is a *strongly* continuous (C_0-) on the Banach space $\mathfrak{C}_1(\mathscr{F})$ *contraction* semigroup, which is *positivity* preserving and *trace* preserving, that is:

$$W_t : \mathfrak{C}_1^+ \to \mathfrak{C}_1^+ \;\;\text{and}\;\; \mathrm{Tr}\,(W_t\rho) = \mathrm{Tr}\,(\rho)\,.$$

Definition 2.1. A positivity-preserving contraction C_0-semigroup on the Banach space $\mathfrak{C}_1(\mathscr{F})$ is called a *dynamical semigroup*, cf. [2], §7. If this semigroup is also trace-preserving, then it is called a *Markov* dynamical semigroup (see Ch. 2.4 in [3]).

As a consequence, the positivity-preserving contraction *strongly* continuous on $\mathfrak{C}_1(\mathscr{F})$ and trace-preserving semigroup $\{W_t\}_{t\geq 0}$ (2.4) is the Markov dynamical semigroup for evolution of the isolated system (2.1). Let L be the generator of semigroup $\{W_t = e^{-t\,L}\}_{t\geq 0}$ and the mapping $\Psi\colon \mathfrak{C}_1^{\mathrm{sa}} \to \mathfrak{C}_1^{\mathrm{sa}}$, be defined by $\Psi(\rho) = (\mathbb{1} + \hat{n})^{-1}\,\rho\,(\mathbb{1} + \hat{n})^{-1}$. Then $\Psi(\mathfrak{C}_1^{\mathrm{sa}}) \subset \mathrm{dom}(L)$ and

$$L\Psi(\rho_0) = i\,h\,(\mathbb{1} + \hat{n})^{-1}\,\rho_0\,(\mathbb{1} + \hat{n})^{-1} - i\,(\mathbb{1} + \hat{n})^{-1}\,\rho_0\,(\mathbb{1} + \hat{n})^{-1}\,h\,, \tag{2.5}$$

for all $\rho_0 \in \mathfrak{C}_1^{\mathrm{sa}}$. We recall that if $\Gamma(L)$ and $\Gamma(i\,[h, \cdot])$ are the *graphs* of operators L and $i\,[h, \cdot]$, then (2.5) implies $\Gamma(L) \supset \Gamma(i\,[h, \cdot])$. Following [9], Ch.VIII.1, we denote this inclusion as

$$L\rho \supset i\,[h, \rho]\,, \quad \rho \in \Psi(\mathfrak{C}_1^{\mathrm{sa}})\,.$$

To introduce the *open* system corresponding to resonator (2.1), we consider a model when this system is in contact with an *external* reservoir (environment) of bosons b, b^* via *pumping* and *leaking*. Then to describe evolution of this open system we follow the GKLD *approach* to a *dissipative* extension of the *Hamiltonian* positivity-preserving evolution (2.4) up to the *non-Hamiltonian* positivity-preserving evolution.

Fix the pumping *in* and leaking *out* parameters: $\sigma_+, \sigma_- \in [0, \infty)$. Define the operator $\widetilde{\mathcal{Q}}_\sigma \colon \operatorname{dom}(\widetilde{\mathcal{Q}}_\sigma) \to \mathfrak{C}_1^{\rm sa}$ with domain $\operatorname{dom}(\widetilde{\mathcal{Q}}_\sigma) = \Psi(\mathfrak{C}_1^{\rm sa})$ by

$$\begin{aligned}\widetilde{\mathcal{Q}}_\sigma \rho =& \sigma_- \left(b(\mathbb{1}+\hat{n})^{-1}\right) \rho_0 \left(b(\mathbb{1}+\hat{n})^{-1}\right)^* \\ &+ \sigma_+ \left(b^* (\mathbb{1}+\hat{n})^{-1}\right) \rho_0 \left(b^*(\mathbb{1}+\hat{n})^{-1}\right)^* ,\end{aligned} \tag{2.6}$$

where $\rho_0 \in \mathfrak{C}_1^{\rm sa}$ is such that $\rho = \Psi(\rho_0)$. Note that according the graph inclusion one gets

$$\widetilde{\mathcal{Q}}_\sigma \rho \supset \sigma_- \, b \, \rho \, b^* + \sigma_+ \, b^* \, \rho \, b \,, \tag{2.7}$$

and that the mapping: $\rho_0 \mapsto \widetilde{\mathcal{Q}}_\sigma \, \Psi(\rho_0)$, is trace-norm continuous from $\mathfrak{C}_1^{\rm sa}$ into $\mathfrak{C}_1^{\rm sa}$. Since operator $\hat{n} \geq 0$ is densely defined, we obtain that $\Psi(\mathfrak{C}_1^{\rm sa}) \cap \mathfrak{C}_1^+ = \Psi(\mathfrak{C}_1^+)$. Hence $\widetilde{\mathcal{Q}}_\sigma$ is a positivity-preserving operator, that is, $\widetilde{\mathcal{Q}}_\sigma : \Psi(\mathfrak{C}_1^+) \to \mathfrak{C}_1^+$.

Using the bilinear trace functional (2.2), one gets that *dual* operator $\widetilde{\mathcal{Q}}_\sigma^*$ acts in $\mathcal{L}(\mathscr{F})$. It is defined by relation: $\langle \widetilde{\mathcal{Q}}_\sigma \, \rho \mid A \rangle_{\mathfrak{C}_1(\mathscr{F}) \times \mathcal{L}(\mathscr{F})} = \langle \rho \mid \widetilde{\mathcal{Q}}_\sigma^*(A) \rangle_{\mathfrak{C}_1(\mathscr{F}) \times \mathcal{L}(\mathscr{F})}$. As a consequence, if $A_0 \in \mathcal{L}(\mathscr{F})$ and $A = (\mathbb{1}+\hat{n})^{-1} A_0 (\mathbb{1}+\hat{n})^{-1}$, then $A \in \operatorname{dom}(\widetilde{\mathcal{Q}}_\sigma^*)$ and

$$\widetilde{\mathcal{Q}}_\sigma^*(A) \supset \sigma_- \, b^* \, A \, b + \sigma_+ \, b \, A \, b^* .$$

Note that if for pumping *in* and leaking *out* parameters one has $\sigma_+ + \sigma_- > 0$, then clearly $\mathbb{1} \notin \operatorname{dom}(\widetilde{\mathcal{Q}}_\sigma^*)$.

Formally, the non-Hamiltonian evolution equation for the *open* resonator is defined in the framework of the GKLD *approach* as

$$\partial_t \rho \;=\; -i \, [h, \rho] - \frac{1}{2} (\widetilde{\mathcal{Q}}_\sigma^*(\mathbb{1}) \rho + \rho \, \widetilde{\mathcal{Q}}_\sigma^*(\mathbb{1})) + \; \widetilde{\mathcal{Q}}_\sigma \rho \,. \tag{2.8}$$

Here (again formally) we define operator $\widetilde{\mathcal{Q}}_\sigma^*(\mathbb{1}) = \sigma_- \, b^* \, b + \sigma_+ \, b \, b^*$. Therefore, a *formal* GKLD generator $\widetilde{L}_\sigma$ of the evolution semigroup for the open system (2.8) has the form

$$\widetilde{L}_\sigma \rho := i \, [h, \rho] + \frac{1}{2} \Big((\sigma_- \, b^* \, b + \sigma_+ \, b \, b^*) \rho + \rho \, (\sigma_- \, b^* \, b + \sigma_+ \, b \, b^*) \Big) - \; \widetilde{\mathcal{Q}}_\sigma \rho \,. \tag{2.9}$$

A mathematical sense of the operator (2.9) and definition of the corresponding semigroup are elucidated in the next subsection.

2.2 The GKLD generator and cut-off regularisation

To proceed we consider the operator $h_\sigma :\ \mathrm{dom}(\hat{n}) \to \mathscr{F}$, defined by

$$h_\sigma = i\,h + \tfrac{1}{2}\,(\sigma_-\,b^*\,b + \sigma_+\,b\,b^*)\,. \tag{2.10}$$

Then h_σ is an m-accretive operator. Define $U_{h_\sigma}(t) = e^{-t\,h_\sigma} \in \mathcal{L}(\mathscr{F})$ for all $t \in [0,\infty)$. Then similarly to (2.4) the contraction C_0-semigroup $\{U_{h_\sigma}(t)\}_{t\geq 0}$ induces on the Banach space $\mathfrak{C}_1^{\rm sa}$ a positivity-preserving contraction C_0-semigroup $\{S_t^\sigma\}_{t\geq 0}$ given by

$$S_t^\sigma\,\rho = U_{h_\sigma}(t)\,\rho\,U_{h_\sigma}(t)^*. \tag{2.11}$$

Let H_σ be the generator of the semigroup $\{S_t^\sigma = e^{-\,t\,H_\sigma}\}_{t\geq 0}$. Then $\mathrm{dom}(H_\sigma) \supset \Psi(\mathfrak{C}_1^{\rm sa})$. If $\rho \in \Psi(\mathfrak{C}_1^{\rm sa})$, then (2.10) and (2.11) yield

$$H_\sigma \rho \supset i\,[h,\rho] + \frac{1}{2}\Big((\sigma_-\,b^*\,b + \sigma_+\,b\,b^*)\rho + \rho\,(\sigma_-\,b^*\,b + \sigma_+\,b\,b^*)\Big)\,, \tag{2.12}$$

where H_σ is the *unperturbed* part of the formal GKLD generator $\widetilde{L}_\sigma = H_\sigma - Q_\sigma$ (2.9). Moreover, the map $\rho \mapsto H_\sigma \Psi(\rho)$ is trace-norm continuous from $\mathfrak{C}_1^{\rm sa}$ into $\mathfrak{C}_1^{\rm sa}$. Also, if $\rho \in \Psi(\mathfrak{C}_1^{+})$, then $\mathrm{Tr}\,H_\sigma\rho \geq 0$. Since S_t^σ commutes with the operator Ψ, one deduces that

$$S_t^\sigma\,\Psi(\mathfrak{C}_1^{\rm sa}) \subset \Psi(\mathfrak{C}_1^{\rm sa})\,.$$

On that account, by the *Nelson* theorem (see, e.g., [10], Theorem 6.1.18) $\Psi(\mathfrak{C}_1^{\rm sa})$ is a *core* for operator H_σ, that is, the closure of restriction: $\overline{H_\sigma \restriction \Psi(\mathfrak{C}_1^{\rm sa})} = H_\sigma$. We denote this as $\Psi(\mathfrak{C}_1^{\rm sa}) = \mathrm{core}(H_\sigma)$.

Note that whenever $\sigma_- + \sigma_+ > 0$, the semigroup $\{S_t^\sigma\}_{t\geq 0}$ is *not* trace-preserving. Indeed, if $\rho \in \mathfrak{C}_1^{+}$ is given by $\rho(\varphi) = (\varphi, e_1)_{\mathscr{F}}\,e_1$, then $H_\sigma\rho = (\sigma_- + 2\sigma_+)\rho$. Hence $S_t^\sigma\,\rho = e^{-(\sigma_- + 2\sigma_+)t}\rho$ and $\mathrm{Tr}\,(S_t^\sigma\,\rho) = e^{-(\sigma_- + 2\sigma_+)t}$ for all $t > 0$.

Remark 2.2. We also note that operator H_σ is *not* positivity-preserving, even although the semigroup $\{S_t^\sigma\}_{t\geq 0}$ is positivity-preserving. For a proof let for simplicity assume that $E = 1$. Then using the commutation relation $(b\,b^* - b^*\,b)\psi = \psi$ for all $\psi \in \mathrm{dom}(b^*\,b)$, one deduces that

$$H_\sigma\rho \supset i(\hat{n}\rho - \rho\hat{n}) + \frac{1}{2}(\sigma_- + \sigma_+)(\hat{n}\rho + \rho\hat{n}) + \sigma_+\rho$$

for all $\rho \in \Psi(\mathfrak{C}_1^{\rm sa})$. Let $k \in \mathbb{N}$ and $\lambda > 0$. Choose $\psi := e_1 + i\,\lambda\,e_k$ and define $\rho \in \Psi(\mathfrak{C}_1^{\rm sa})^+$ by $\rho\,\varphi = (\varphi,\psi)_{\mathscr{F}}\,\psi$. Then

$$(H_\sigma\rho)\varphi = i\,((\varphi,\psi)_{\mathscr{F}}\hat{n}\psi - (\hat{n}\varphi,\psi)_{\mathscr{F}}\psi) \\ + (\sigma_- + \sigma_+)\,((\varphi,\psi)_{\mathscr{F}}\hat{n}\psi + (\hat{n}\varphi,\psi)_{\mathscr{F}}\psi) + \sigma_+(\varphi,\psi)_{\mathscr{F}}\psi$$

for all $\varphi \in \mathrm{dom}(\hat{n})$. So

$$((H_\sigma\rho)\varphi,\varphi)_{\mathscr{F}} = -2\,\mathrm{Im}\,((\varphi,\psi)_{\mathscr{F}}(\hat{n}\psi,\varphi)_{\mathscr{F}}) \\ + (\sigma_- + \sigma_+)\,\mathrm{Re}\,((\varphi,\psi)_{\mathscr{F}}(\hat{n}\psi,\varphi)_{\mathscr{F}}) + \sigma_+|(\varphi,\psi)_{\mathscr{F}}|^2.$$

Now choose $\varphi = e_1 + e_k/\sqrt{k}$. Then

$$(\varphi,\psi)_{\mathscr{F}}\,(\hat{n}\,\psi,\varphi)_{\mathscr{F}} = (1 - i\,\lambda/\sqrt{k})(1 + i\,\sqrt{k}\,\lambda) = 1 + \lambda^2 + i\,(k-1)\,\lambda/\sqrt{k}\,.$$

As a consequence,

$$((H_\sigma\rho)\varphi,\varphi)_{\mathscr{F}} = -2(k-1)\,\lambda/\sqrt{k} + (\sigma_- + \sigma_+)(1+\lambda^2) + \sigma_+\,(1 + \lambda^2/k)\,.$$

Then $((H_\sigma\rho)\varphi,\varphi)_{\mathscr{F}} < 0$, for any fixed $\lambda > 0$ and $\sigma_\pm \in \mathbb{R}$, if $k \in \mathbb{N}$ is sufficiently large. Therefore, the operator $H_\sigma\rho$ is not positive and thus the operator H_σ is not positivity-preserving.

It is for that reason, the perturbation $(-\widetilde{\mathcal{Q}}_\sigma)$ in representation $\widetilde{L}_\sigma = H_\sigma - \widetilde{\mathcal{Q}}_\sigma$ (2.9) serves to restore the *trace-preserving* property of the evolution semigroup with a well-defined generator L_σ, keeping at the same time for this semigroup the *positivity-preserving* property. The next assertion is the first key step towards the proof of these properties.

Proposition 2.3. *The operator* $\widetilde{\mathcal{Q}}_\sigma$ (2.6) *has a unique extension in* $\mathrm{dom}(H_\sigma)$ *to continuous operator* $\mathcal{Q}_\sigma\colon\ \mathrm{dom}(H_\sigma) \to \mathfrak{C}_1^{\mathrm{sa}}$*, where* $\mathrm{dom}(H_\sigma)$ *is provided with the graph norm. Moreover, operator* $\mathcal{Q}_\sigma$ *is positivity-preserving and such that*

$$\mathrm{Tr}\,(H_\sigma\rho - \mathcal{Q}_\sigma\rho) = 0\,, \tag{2.13}$$

and $\|\mathcal{Q}_\sigma\rho\|_{\mathfrak{C}_1} \le \|H_\sigma\rho\|_{\mathfrak{C}_1}$ *for all* $\rho \in \mathrm{dom}(H_\sigma)$.

For the proof we refer to [4], Lemma 3.2.

For regularisation we use a general approach developed in [4], Section 2. To this aim we consider a regularisation generated by the family of *projections* $(P_N)_{N\in\mathbb{N}_0}$, where for all $N \in \mathbb{N}_0$ the projection $P_N\colon \mathscr{F} \to \mathscr{F}$ is given by

$$P_N\psi := \sum_{n=0}^{N}(\psi,e_n)_{\mathscr{F}}\,e_n.$$

Note that the number of bosons in the subspace $P_N\mathscr{F}$ is bounded because the boson number operator satisfies $\hat{n}(P_N\psi) \leq N\,\|\psi\|^2_{\mathscr{F}}$ for all $\psi \in \mathscr{F}$.

Obviously $\lim_{N\to\infty} P_N\psi = \psi$ for all $\psi \in \mathscr{F}$. For all $N \in \mathbb{N}_0$ define the *particle number cut-off* regularisation $\mathcal{Q}_{\sigma,N} \in \mathcal{L}(\mathfrak{C}_1^{\mathrm{sa}})$ of the operator $\mathcal{Q}_\sigma$ by

$$\mathcal{Q}_{\sigma,N}\,\rho = \sigma_-\,(b^*\,P_N)^*\,\rho\,(b^*\,P_N) + \sigma_+\,(b\,P_N)^*\,\rho\,(b\,P_N). \tag{2.14}$$

Note that $\mathcal{Q}_{\sigma,N}\,\rho = P_N\,(\mathcal{Q}_\sigma\rho)\,P_N$ for all $\rho \in \Psi(\mathfrak{C}_1^{\mathrm{sa}})$ by (2.6). Therefore $\|\mathcal{Q}_{\sigma,N}\,\rho\|_{\mathfrak{C}_1} \leq \|\mathcal{Q}_\sigma\rho\|_{\mathfrak{C}_1}$ for all $\rho \in \Psi(\mathfrak{C}_1^{\mathrm{sa}})$ and then by density $\|\mathcal{Q}_{\sigma,N}\,\rho\|_{\mathfrak{C}_1} \leq \|\mathcal{Q}_\sigma\rho\|_{\mathfrak{C}_1}$ for all $\rho \in \mathrm{dom}(H_\sigma)$.

We next verify that $(\mathcal{Q}_{\sigma,N})_{N\in\mathbb{N}_0}$ is a functional regularisation of $\mathcal{Q}_\sigma$. Clearly $\mathcal{Q}_{\sigma,N}$ is positivity-preserving for all $N \in \mathbb{N}_0$. The definition of $\mathcal{Q}_{\sigma,N}$ implies the estimate

$$\|\mathcal{Q}_{\sigma,N}\,\rho\|_{\mathfrak{C}_1} \leq (\sigma_-(N+1) + \sigma_+ N)\,\|\rho\|_{\mathfrak{C}_1},$$

for all $\rho \in \mathfrak{C}_1^{\mathrm{sa}}$. Since $\sigma_\pm \geq 0$, the regularisation (2.14) is monotone increasing as a sequence of positivity-preserving maps in $\mathfrak{C}_1^{\mathrm{sa}}$, and bounded by $\mathcal{Q}_\sigma$. Finally we show, [4], that $\lim_{N\to\infty}((\mathcal{Q}_{\sigma,N}\,\rho)\psi,\psi)_{\mathscr{F}} = ((\mathcal{Q}_\sigma\rho)\psi,\psi)_{\mathscr{F}}$ for all $\rho \in \mathrm{dom}(H_\sigma)$ and $\psi \in \mathscr{F}$. Let $\psi \in \mathscr{F}$. Let $\rho \in \Psi(\mathfrak{C}_1^{\mathrm{sa}})$. Then

$$((\mathcal{Q}_\sigma\rho)\psi,\psi)_{\mathscr{F}} := \lim_{N\to\infty}((\mathcal{Q}_{\sigma,N}\,\rho)\psi,\psi)_{\mathscr{F}} = \lim_{N\to\infty}((\mathcal{Q}_\sigma\rho)\,P_N\psi,P_N\psi)_{\mathscr{F}} \tag{2.15}$$

for all $\rho \in \Psi(\mathfrak{C}_1^{\mathrm{sa}})$. Since $\Psi(\mathfrak{C}_1^{\mathrm{sa}})$ is dense in $\mathrm{dom}(H_\sigma)$ and $\|\mathcal{Q}_{\sigma,N}\,\rho\|_{\mathfrak{C}_1} \leq \|\mathcal{Q}_\sigma\rho\|_{\mathfrak{C}_1}$ for all $\rho \in \mathrm{dom}(H_\sigma)$ and $N \in \mathbb{N}_0$, one deduces that $\lim_{N\to\infty}((\mathcal{Q}_{\sigma,N}\,\rho)\psi,\psi)_{\mathscr{F}} = ((\mathcal{Q}_\sigma\rho)\psi,\psi)_{\mathscr{F}}$ for all $\rho \in \mathrm{dom}(H_\sigma)$ and $\psi \in \mathscr{F}$.

Consequently, the family $\{\mathcal{Q}_{\sigma,N}\}_{N\in\mathbb{N}_0}$ is a functional regularisation of the operator $\mathcal{Q}_\sigma$. For all $N \in \mathbb{N}$ define the operator $L_{\sigma,N}$ by

$$L_{\sigma,N} = H_\sigma - \mathcal{Q}_{\sigma,N}$$

with domain $\mathrm{dom}(L_{\sigma,N}) = \mathrm{dom}(H_\sigma)$. Let $\{T^\sigma_{t,N}\}_{t\geq 0}$ be the semigroup generated by $-L_{\sigma,N}$. Then it follows from Theorem 2.1, [4], that $\{T^\sigma_{t,N}\}_{t\geq 0}$, each $N \geq 1$, is a positivity-preserving contraction C_0-semigroup on $\mathfrak{C}_1$, so it is a dynamical semigroup. Moreover, for all $t > 0$ and $\rho \in \mathfrak{C}_1^{\mathrm{sa}}$ the limit

$$T^\sigma_t\rho = \lim_{N\to\infty} T^\sigma_{t,N}\,\rho\,, \tag{2.16}$$

exists on $\mathfrak{C}_1$ and $\{T^\sigma_t\}_{t\geq 0}$ is a positivity-preserving contraction C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$.

Let L_σ be the generator of semigroup $\{T^\sigma_t\}_{t\geq 0}$ (2.16). Then by Theorem 2.1(7), [4], L_σ is a closed extension of the operator $H_\sigma - \mathcal{Q}_\sigma$, that is, $L_\sigma := (H_\sigma - \mathcal{Q}_\sigma)_{ext}$.

Remark 2.4. By Theorem 2.2, [4], the semigroup $\{T_t^\sigma\}_{t\geq 0}$ constructed by the cut-off regularisation (2.16) is *minimal* in the following sense:
If $(\widehat{T}_t^\sigma)_{t\geq 0}$ is a positivity-preserving C_0-semigroup with generator $\widehat{L}_\sigma$, which is an *extension* of operator $(H_\sigma - \mathcal{Q}_\sigma)$, that is, $\widehat{L}_\sigma := (H_\sigma - \mathcal{Q}_\sigma)\widehat{\ }$, then $\widehat{T}_t^\sigma \geq T_t^\sigma$ for all $t > 0$.

2.3 Core property and trace-preserving

A priori, it is unclear whether the constructed by means of the cut-off regularisation (*minimal*) dynamical semigroup $\{T_t^\sigma\}_{t\geq 0}$ (2.16) is *trace-preserving* and hence is the Markov dynamical semigroup. On the other hand, due to [2], Theorem 3.2, and [4], Theorems 1.1 and 2.3, one has the following general result.

Proposition 2.5. *Let H be the generator of a positivity-preserving contraction C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$. Let $K\colon \operatorname{dom}(H) \to \mathfrak{C}_1^{\mathrm{sa}}$ be a positivity-preserving operator such that for all $u \in \operatorname{dom}(H)^+$ one has $\operatorname{Tr}(Ku) \leq \operatorname{Tr}(Hu)$. Here $\operatorname{dom}(H)^+ = \operatorname{dom}(H) \cap \mathfrak{C}_1^+$.*

Let $(K_\alpha)_{\alpha\in J}$ be a functional regularisation of K. Set $L_\alpha = H - K_\alpha$ for all $\alpha \in J$. Then the following holds:
(a) *For all $\alpha \in J$ the operator L_α is the generator of a positivity-preserving contraction C_0-semigroup $\{T_t^\alpha\}_{t\geq 0}$ on $\mathfrak{C}_1^{\mathrm{sa}}$.*
(b) *If $t > 0$, then $\lim_\alpha T_t^\alpha u$ exists in $\mathfrak{C}_1^{\mathrm{sa}}$ for all $u \in \mathfrak{C}_1^{\mathrm{sa}}$.*
For all $t > 0$ define $T_t\colon \mathfrak{C}_1^{\mathrm{sa}} \to \mathfrak{C}_1^{\mathrm{sa}}$ by $T_t u = \lim_\alpha T_t^\alpha u$.
(c) *The family $\{T_t\}_{t\geq 0}$ is a positivity-preserving minimal contraction C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$ for which the generator L is an extension of the operator $(H - K)$, that is $L = (H - K)_{ext}$.*

If in addition we suppose that for all $u \in \operatorname{dom}(H)$ holds

$$\operatorname{Tr}(Hu - Ku) = 0\,, \tag{2.17}$$

and that $\operatorname{dom}(H)$ is a core for generator L, then:
(d) *The constructed in* (a)-(c) *minimal dynamical semigroup $\{T_t\}_{t\geq 0}$ is trace-preserving, that is, Markovian.*

Proof. For the proof of assertions (a)-(c) we refer to [4], Theorem 2.1 and Theorem 2.2.
Proof of (d). Thanks to definition of operators K and L condition (2.17) provides that $\operatorname{Tr} Lu = 0$ for all $u \in \operatorname{dom}(H)$. Because $\operatorname{dom}(H) = \operatorname{core}(L)$, one deduces that $\operatorname{Tr} Lu = 0$ for all $u \in \operatorname{dom}(L)$. Let $u \in \operatorname{dom}(L)$. Seeing that the semigroup $\{T_t\}_{t\geq 0}$ maps $\operatorname{dom}(L)$ into $\operatorname{dom}(L)$, one also gets $\operatorname{Tr} L\, T_t u = 0$ for all $t > 0$. Then differentiability of the function $t \mapsto T_t u$ from $(0, \infty)$ into $\mathfrak{C}_1$ yields $\partial_t \operatorname{Tr} T_t u =$

$-\mathrm{Tr}\, L\, T_t u = 0$ for all $t > 0$. Hence $\mathrm{Tr}\, T_t u = \mathrm{Tr}\, u$ for all $t > 0$. Since $\mathrm{dom}(L)$ is dense in $\mathfrak{C}_1^{\mathrm{sa}}$, the latter also holds for all $u \in \mathfrak{C}_1^{\mathrm{sa}}$. □

Owing to definition of operators H_σ (2.12) and $\mathcal{Q}_\sigma$ (2.13) we obtain by Proposition 2.3 that $\mathrm{Tr}\,(H_\sigma \rho - \mathcal{Q}_\sigma \rho) = 0$ for all $\rho \in \mathrm{dom}(H_\sigma)$. As a consequence, if $\mathrm{dom}(H_\sigma) = \mathrm{core}(L_\sigma)$, then we can use Proposition 2.5(d), to conclude that dynamical semigroup $\{T_t^\sigma\}_{t\geq 0}$ (2.16) is trace-preserving. One can show that this is the case if $\sigma_+ < \sigma_-$.

Proposition 2.6. ([4], Theorem 3.1.)
If $\sigma_+ < \sigma_-$, then the domain $\mathrm{dom}(H_\sigma)$ is a core for L_σ.

Corollary 2.7. *If $\sigma_+ < \sigma_-$, then dynamical semigroup $\{T_t^\sigma\}_{t\geq 0}$ (2.16) is trace-preserving.*

Proof. This follows from Proposition 2.3, Proposition 2.5 and Proposition 2.6. □

Corollary 2.8. *If $\sigma_+ < \sigma_-$, then the set $\Psi(\mathfrak{C}_1^{\mathrm{sa}})$ is a core for the operator L_σ.*

Proof. The set $\Psi(\mathfrak{C}_1^{\mathrm{sa}})$ is dense in $\mathrm{dom}(H_\sigma)$. Moreover, $\mathrm{dom}(H_\sigma)$ is dense in $\mathrm{dom}(L_\sigma)$ by Proposition 2.6. For that reason $\Psi(\mathfrak{C}_1^{\mathrm{sa}})$ is dense in $\mathrm{dom}(L_\sigma)$, and as a consequence: $\Psi(\mathfrak{C}_1^{\mathrm{sa}}) = \mathrm{core}(L_\sigma)$. □

Remark 2.9. The proof of Proposition 2.6 is considerably based on the strict inequality $\sigma_+ < \sigma_-$ for non-negative contact parameters. In the next Section 3 we study the *critical regime*: $\sigma_- = \sigma_+ > 0$, as well as: $\sigma_+ > \sigma_-$. It turns out that to answer to this particular questions one needs a more general setting that allows also to elucidate the statement, which in the framework of the GKLD *approach* is *converse* to Proposition 2.5(d). This more general GKLD setting is based on the concept of the GKLD *ansatz* for evolution of open systems which is formulated in subsection 3.3.

3 Duality, GKLD Ansatz and Markov Property

3.1 Dual semigroup

We recall that by duality relation (2.2) one can define for *strongly continuous* (on Banach space $\mathfrak{C}_1(\mathscr{F})$) semigroup $\{T_t^\sigma\}_{t\geq 0}$ (2.16) a family of operators $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on the dual space $\mathfrak{C}_1^*(\mathscr{F})$ such that:

$$\langle T_t^\sigma u \,|\, A\rangle = \langle u \,|\, T_t^{\sigma\,*}(A)\rangle \,. \tag{3.1}$$

Here $u \in \mathfrak{C}_1(\mathscr{F})$ and $A \in \mathfrak{C}_1^*(\mathscr{F}) \simeq \mathcal{L}(\mathscr{F})$ (that is, $\mathfrak{C}_1^*(\mathscr{F})$ is isometrically isomorphic to the space of bounded operators $\mathcal{L}(\mathscr{F})$), and we use for short $\langle \cdot \mid \cdot \rangle :=$

$\langle \cdot \mid \cdot \rangle_{\mathfrak{C}_1(\mathscr{F})\times\mathfrak{C}_1^*(\mathscr{F})}$. Owing to (3.1), the family $\{T_t^{\sigma\,*}\}_{t\geq 0}$ is a semigroup of operators on the Banach space $\mathcal{L}(\mathscr{F})$.

In general the dual semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ is *not* strongly continuous on $\mathcal{L}(\mathscr{F})$, but it is always weak*-continuous (that is, continuous in the Banach topology $\sigma(\mathfrak{C}_1^*(\mathscr{F}), \mathfrak{C}_1(\mathscr{F}))$ on $\mathcal{L}(\mathscr{F})$) since

$$w^*-\lim_{t\downarrow 0}(T_t^{\sigma\,*}(A) - A) := \lim_{t\downarrow 0}\langle u \,|\, T_t^{\sigma\,*}(A) - A\rangle = \lim_{t\downarrow 0}\langle T_t^{\sigma} u - u \,|\, A\rangle = 0\,, \tag{3.2}$$

for any $A \in \mathcal{L}(\mathscr{F})$ and all $u \in \mathfrak{C}_1(\mathscr{F})$. Here $\lim_{t\downarrow 0}\|T_t^{\sigma}u - u\|_1 = 0$ by C_0-continuity of semigroup $\{T_t^{\sigma}\}_{t\geq 0}$. Then following the standard scheme one defines the weak*-generator L_σ° of $\{T_t^{\sigma\,*}\}_{t\geq 0}$ by

$$\operatorname{dom}(L_\sigma^{\circ}) := \{A \in \mathcal{L}(\mathscr{F}) : \exists\, w^*-\lim_{t\downarrow 0}\frac{1}{t}(A - T_t^{\sigma\,*}(A))\},$$
$$L_\sigma^{\circ}(A) := w^*-\lim_{t\downarrow 0}\frac{1}{t}(A - T_t^{\sigma\,*}(A)).$$

By duality relations (2.2), (3.1) one gets that the mapping: $A \geq 0 \mapsto T_t^{\sigma\,*}(A) \geq 0$, on the Banach space $\mathcal{L}(\mathscr{F})$ of bounded operators, is positivity-preserving. We note that a positivity-preserving weak*-continuous on $\mathcal{L}(\mathscr{F})$) dual semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ is called a (*quantum*) dynamical semigroup, see, e.g., review articles in [8]. Moreover, for a trace-preserving dynamical semigroup $\{T_t^{\sigma}\}_{t\geq 0}$ relations (2.2), (3.1) yield identity

$$\operatorname{Tr}(u) = \operatorname{Tr}(T_t^{\sigma} u\, \mathbb{1}) = \operatorname{Tr}(u\, T_t^{\sigma\,*}(\mathbb{1})) \quad \forall\, u \in \mathfrak{C}_1(\mathscr{F})\,.$$

That is, $T_t^{\sigma\,*}(\mathbb{1}) = \mathbb{1}$ and therefore semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ is *unity-preserving*. Then it is called a *quantum Markov* semigroup, see [5,8].

Recall that if L_σ is a densely defined generator of dynamical semigroup $\{T_t^{\sigma}\}_{t\geq 0}$ on $\mathfrak{C}_1(\mathscr{F})$, then the adjoint operator $-L_\sigma^*$ in the Banach space $\mathfrak{C}_1^*(\mathscr{F}) \simeq \mathcal{L}(\mathscr{F})$ is defined as follows. First let the set

$$\operatorname{dom}(L_\sigma^*) := \{A \in \mathcal{L}(\mathscr{F}) : u \mapsto \langle L_\sigma u \mid A\rangle \text{ is continuous for } u \in \operatorname{dom}(L_\sigma)\}\,.$$

Since for every $A \in \operatorname{dom}(L_\sigma^*)$, the mapping $F_A : u \mapsto \langle L_\sigma u \mid A\rangle$ is continuous on the dense domain $\operatorname{dom}(L_\sigma)$, one can extend F_A to a *unique* continuous linear functional $\widetilde{F}_A$ on $\mathfrak{C}_1(\mathscr{F})$. Hence, $\widetilde{F}_A \in \mathfrak{C}_1^*(\mathscr{F})$ and one denotes it by $L_\sigma^*(A)$. This construction decodes the relation

$$\langle L_\sigma u \mid A\rangle = \langle u \mid L_\sigma^*(A)\rangle\,, \tag{3.3}$$

for all $u \in \operatorname{dom}(L_\sigma)$ and $A \in \operatorname{dom}(L_\sigma^*)$. There is a relation between generator L_σ° and operator, L_σ^* [11], Ch.II, Sec.2.5, which is expressed by the following statement.

Proposition 3.1. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $\mathfrak{B}$ with generator L. Then the weak*-generator L° with $\mathrm{dom}(L^\circ)$ of the dual semigroup $\{T_t^*\}_{t\geq 0}$ on the dual space $\mathfrak{B}^*$ coincides with operator L^* with domain $\mathrm{dom}(L^*) \subset \mathfrak{B}^*$.*

Corollary 3.2. *For dual semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ the generator L_σ° coincides with operator L_σ^*. From definition of $\widetilde{L}_\sigma$ (2.9) and extension of this operator to L_σ (Section 2.2) one gets by duality relation (3.3) that for any $A \in D(L_\sigma^*)$*

$$
\begin{aligned}
(3.4)\quad L_\sigma^*(A) \supset Ah_\sigma + h_\sigma^* A - (F_1^* A F_1 + F_2^* A F_2) = i[A,h] \\
+ \frac{1}{2}[A(\sigma_- b^* b + \sigma_+ bb^*) + (\sigma_- b^* b + \sigma_+ bb^*)A] - (\sigma_- b^* A b + \sigma_+ b A b^*),
\end{aligned}
$$

where $h_\sigma := i\,h + \frac{1}{2}\,(\sigma_-\, b^*\, b + \sigma_+\, b\, b^*)$ and $F_1 := \sigma_-^{1/2}\; b$, $F_2 := \sigma_+^{1/2}\; b^*$, i.e., $\{T_t^{\sigma\,*} = e^{-t\,L_\sigma^*}\}_{t\geq 0}$.

On account of (2.8), (3.1) and by virtue of (3.4) the evolution on the dual space $\mathcal{L}(\mathscr{F})$ is determined for generator L_σ^* by equation

$$
(3.5)\qquad \partial_t T_t^{\sigma\,*}(A) = -\,L_\sigma^*\,(T_t^{\sigma\,*}(A)) = -\,T_t^{\sigma\,*}\,(L_\sigma^*(A))\,, \quad A \in D(L_\sigma^*)\,.
$$

To elucidate (3.5) it is sufficient to consider semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on some appropriate subset $\mathfrak{A} \subset \mathcal{L}(\mathscr{F})$, see comments below and in subsection 3.2.

Note that by duality relation (2.2) the Banach space of bounded operators $\mathcal{L}(\mathscr{F})$ admits the *predial* Banach space $\mathcal{L}_*(\mathscr{F}) \simeq \mathfrak{C}_1(\mathscr{F})$. Let the Banach space of bounded operators $\mathcal{L}(\mathscr{F})$ be endowed with the weak*-topology $\sigma(\mathcal{L}(\mathscr{F}), \mathcal{L}_*(\mathscr{F}))$. Let $(\mathcal{G}_t\}_{t\geq 0}$ be a weak*-continuous semigroup on $\mathcal{L}(\mathscr{F})$. Then by duality relation each of the maps S_t admit a predual map $S_{t\,*}$ on $\mathcal{L}_*(\mathscr{F})$. The maps $(\mathcal{G}_{t\,*}\}_{t\geq 0}$ inherit the property to be a semigroup. Moreover, the relation

$$
(3.6)\qquad \langle u\,|\,\mathcal{G}_t(A)\rangle = \langle \mathcal{G}_{t\,*} u\,|\,A\rangle\;,
$$

for any $A \in \mathcal{L}(\mathscr{F})$ and all $u \in \mathcal{L}_*(\mathscr{F})$, implies that semigroup $(\mathcal{G}_{t\,*}\}_{t\geq 0}$ is continuous in the *weak* Banach space topology topology $\sigma(\mathcal{L}_*(\mathscr{F}), \mathcal{L}(\mathscr{F}))$ on $\mathcal{L}_*(\mathscr{F})$. Since weakly continuous in a Banach space semigroups are also strongly continuous in this space, the semigroup $(\mathcal{G}_{t\,*}\}_{t\geq 0}$ is strongly continuous on the Banach space $\mathcal{L}_*(\mathscr{F}) \simeq \mathfrak{C}_1(\mathscr{F})$.

Remark 3.3. The weak*-continuity of semigroup $(\mathcal{G}_t\}_{t\geq 0}$ on $\mathcal{L}(\mathscr{F})$ implies the strong continuity of predual semigroup $(\mathcal{G}_{t\,*}\}_{t\geq 0}$ on $\mathfrak{C}_1(\mathscr{F})$. Identifying the semigroup $(\mathcal{G}_t\}_{t\geq 0}$ with $\{T_t^{\sigma\,*}\}_{t\geq 0}$ we conclude that predual semigroup $(\{T_t^{\sigma\,*})_*\}_{t\geq 0} = \{T_t^\sigma\}_{t\geq 0}$ on $\mathfrak{C}_1(\mathscr{F})$ as it is in (3.1).

Our aim is to study the action of the weak*-continuous (quantum) dynamical semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on the space of bounded operators $\mathcal{L}(\mathscr{F})$ endowed with the weak*-topology $\sigma(\mathcal{L}(\mathscr{F}), \mathcal{L}_*(\mathscr{F}))$ and to profit the fact that $\mathcal{L}(\mathscr{F})$ is the *von Neumann* algebra. We note that in fact it is sufficient to consider $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on an *appropriate* subset $\mathfrak{A} \subset \mathcal{L}(\mathscr{F})$. We choose this subset in such a way that:

Note (a). The subset $\mathfrak{A}$ is a *unital* $*$-subalgebra of $\mathcal{L}(\mathscr{F})$, that is, $\mathbb{1} \in \mathfrak{A}$.

Note (b). The action of the semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on $\mathfrak{A}$ is (in a certain sense) *easy* to analyse. The sense of the *easy* depends on the choice of $\mathfrak{A}$ and on details of dynamics generated by (3.4). We shall discuss them below.

Remark 3.4. What concerning the Note (a), we recall that if $\mathfrak{A}$ is a unital $*$-subalgebra of $\mathcal{L}(\mathscr{F})$, then by the von Neumann *density theorem* it is weakly (strongly) dense in its *double commutant* $\mathfrak{A}''$. In particular, it is also true in the weak*-topology on $\mathcal{L}(\mathscr{F})$, which is stronger than the weak operator topology. Therefore, $\mathfrak{A}''$ coincides with the closure of $\mathfrak{A}$ in all these topologies. Then by the von Neumann *bicommutant theorem* one gets that $\mathfrak{A}'' \subseteq \mathcal{L}(\mathscr{F})$ is a von Neumann subalgebra in $\mathcal{L}(\mathscr{F})$, which coincides with the closure $\overline{\mathfrak{A}}$. If in addition the set $\mathfrak{A}$ is *irreducible*, that is, its *commutant* $\mathfrak{A}'$ is trivial, namely, $\mathfrak{A}' := \{A \in \mathcal{L}(\mathscr{F}) : AB = BA, \forall\, B \in \mathfrak{A}\} = \mathbb{C}\,\mathbb{1}$, then this evidently yields that $\mathfrak{A}'' = \mathcal{L}(\mathscr{F})$. See, e.g., [12], Secs.3-5, or [13], Ch.2.4, for details

3.2 Evolution on the Weyl algebra

Comments to Note (a): Following Section 2.1 we define in the Fock space $\mathscr{F}$ symmetric operator

$$\Phi(\zeta) := \frac{1}{\sqrt{2}}\,(\overline{\zeta}\,b + \zeta\,b^*) \subset \Phi(\zeta)^*, \;\; \zeta \in \mathbb{C}, \tag{3.7}$$

on the dense domain $\mathrm{dom}(\Phi(\zeta)) = \mathrm{dom}(b) = \mathrm{dom}(b^*)$. Hence $\Phi(\zeta)$ is closable. Recall that the set of *finite-particle* vectors $\widehat{\mathscr{F}} \subset \mathscr{F}$ is defined as the *algebraic* direct sum $\widehat{\mathscr{F}} := \bigvee_{N\geq 0} \mathscr{F}_N$, $N \in \mathbb{N}_0$, that is, the linear hull (envelope) of N-particle subspaces $\mathscr{F}_N := \{\psi \in \mathscr{F}, \|\psi\|_{\mathscr{F}} = 1 : (\hat{n}\,\psi, \psi)_{\mathscr{F}} \leq N\}$.

Proposition 3.5. (1) *Operators* $\{\Phi(\zeta)\}_{\zeta\in\mathbb{C}}$ *are essentially self-adjoint on domain* $\widehat{\mathscr{F}}$.
(2) *If* $\lim_{k\to\infty} \zeta_k = \zeta$, *then* $\lim_{k\to\infty} \|\Phi(\zeta_k)\psi - \Phi(\zeta)\psi\| = 0$ *for all* $\psi \in \widehat{\mathscr{F}}$.

Proof. (1) Recall that by the Nelson *analytic vector* theorem [14],Thm.X.39, if there is a $D \subset \mathrm{dom}(\Phi(\zeta))$, which is invariant under $\Phi(\zeta)$ and which, in turn, contains a dense in $\mathscr{F}$ subset $D_a \subseteq D$ of analytic vectors for $\Phi(\zeta)$, then operator $\Phi(\zeta)$ is essentially self-adjoint on D. First we note that $\widehat{\mathscr{F}} \subset \mathrm{dom}(\Phi(\zeta))$ is evidently dense in $\mathscr{F}$ and that $\widehat{\mathscr{F}}$ is invariant under the action of any $\Phi(\zeta) : \widehat{\mathscr{F}} \to \widehat{\mathscr{F}}$.

Because of decomposition $\mathscr{F} = \bigoplus_{n\geq 0} \mathscr{F}^{(n)}$, where linear span $\mathscr{F}^{(n)} := \mathrm{span}\{(b^*)^n\Omega\}$ is the space of vectors $\psi^{(n)}$ from the n-particle sector, the Fock space is in turn the linear span of vectors $\psi_n := (0, \ldots, 0, \psi^{(n)}, 0, \ldots)$ with only one non-trivial component $(\psi_n)_{j=n} = \psi^{(n)}$. Since any vector from $\widehat{\mathscr{F}}$ belongs to some $\mathscr{F}_N$, it has *all*, but finitely many components, equal to zero. Hence, this vector is a finite sum of the one-component vectors ψ_n. Then to check the analyticity of $\psi \in D_a \subset \widehat{\mathscr{F}}$ it is enough to show only that vectors ψ_n are analytic for operators $\{\Phi(\zeta)\}_{\zeta\in\mathbb{C}}$.

Since for all $m \geq 0$ the vector $\psi_n \in \mathrm{dom}(\Phi(\zeta)^m)$, the straightforward estimates yield:

$$\|\Phi(\zeta)^m\psi_n\| \leq \sqrt{2}\sqrt{n+m}\,|\zeta|\,\|\Phi(\zeta)^{m-1}\psi_n\| \quad \text{and}$$
$$\|\Phi(\zeta)^m\psi_n\| \leq 2^{m/2}\sqrt{(n+m)!}\,|\zeta|^m\,\|\psi_n\|.$$

Therefore, the series

$$\sum_{m\geq 0} \frac{z^m}{m!}\,\|\Phi(\zeta)^m\psi_n\|\,,$$

converges for any $z \in \mathbb{C}$. Hence, by definition of analytic vector one gets that ψ_n is analytic for any of operators $\{\Phi(\zeta)\}_{\zeta\in\mathbb{C}}$, that proves the assertion (1).

(2) Let $\psi \in \widehat{\mathscr{F}}$. Then we get the estimate

$$\|\Phi(\zeta_k)\psi - \Phi(\zeta)\psi\| \leq \frac{1}{\sqrt{2}}(\|(\zeta_k-\zeta)\,b\,\psi\| + \|(\zeta_k-\zeta)\,b^*\psi\|) \leq \sqrt{2}\,|\zeta_k-\zeta|\,\|\sqrt{\hat{n}+1}\,\psi\|\,,$$

which yields for $\lim_{k\to\infty}\zeta_k = \zeta$ the limit in (2). □

Below we denote by $\Phi(\zeta)$ the *self-adjoint* closure of operator (3.7). Then for each $\zeta \in \mathbb{C}$ it generates on the Fock space $\mathscr{F}$ a strongly continuous one-parameter group of unitary operators: $t \mapsto \exp(it\Phi(\zeta))$, $t \in \mathbb{R}$, (the *Stone* theorem). These unitary operators define the family of the *Weyl operators*:

$$\mathscr{W}(\mathbb{C}) := \left\{W(\zeta) := e^{i\Phi(\zeta)}\right\}_{\zeta\in\mathbb{C}}\,. \tag{3.8}$$

The linear $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ of (3.8) generates a unital $*$-subalgebra of the von Neumann algebra of bounded operators $\mathcal{L}(\mathscr{F})$. The *operator-norm* closure of $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ generates on $\mathscr{F}$ the C^*-algebra of the Weyl *canonical commutation relations*, which we denote below by CCR($\mathscr{F}$).

Proposition 3.6. (1) *If* $\lim_{k\to\infty}\zeta_k = \zeta$, *then* $\lim_{k\to\infty}\|W(\zeta_k)\psi - W(\zeta)\psi\| = 0$ *for any* $\psi \in \mathscr{F}$, *i.e. the sequence* $\{W(\zeta_k)\}_{k\geq 1}$ *converges on* $\mathscr{F}$ *in the strong operator sense.*

(2) *The* $\mathrm{span}\{\mathscr{W}(\mathbb{C})\} \subset \mathcal{L}(\mathscr{F})$ *acts irreducibly on* $\mathscr{F}$.

Proof. (1) By Proposition 3.5 one has $\Phi(\zeta_k)\psi \to \Phi(\zeta)\psi$ for all $\psi \in \widehat{\mathscr{F}}$, which is a joint core for all the operators $\{\Phi(\zeta_k)\}_{k\geq 1}$ and $\Phi(\zeta)$. This implies that $\Phi(\zeta_k) \to \Phi(\zeta)$ converges in the strong resolvent sense. Consequently, the unitary groups $\{e^{i\,t\Phi(\zeta_k)}\}_{k\geq 1}$ converge to $e^{i\,t\Phi(\zeta)}$ on $\mathcal{L}(\mathscr{F})$ in the strong operator sense for all $t \in \mathbb{R}$. This proves the assertion (1).

(2) Recall that the family of operators $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ acts *irreducibly* on $\mathscr{F}$ if the only closed subspaces of $\mathscr{F}$, which are invariant under the action of $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ are the trivial subspaces: $\{0\}$ or $\mathscr{F}$. This property is equivalent to claim that family of the Weyl operators (3.8) is *irreducible*, which means that the commutant $\mathscr{W}(\mathbb{C})'$ is trivial in the sense: $\mathscr{W}(\mathbb{C})' = \mathbb{C}\mathbb{1}$. Now, suppose that there exists a nontrivial $B \in \mathcal{L}(\mathscr{F})$, which commutes with all Weyl operators $W(\zeta) \in \mathscr{W}(\mathbb{C})$. Then for any $\psi \in \mathrm{dom}(\Phi(\zeta))$ it follows that

$$\lim_{t\downarrow 0} \frac{e^{i\,t\Phi(\zeta)} - \mathbb{1}}{it}\, B\,\psi = \lim_{t\downarrow 0} B\, \frac{e^{i\,t\Phi(\zeta)} - \mathbb{1}}{it}\,\psi = B\Phi(\zeta)\,\psi\;,\; \zeta \in \mathbb{C}\,.$$

This yields: $B\,\psi \in \mathrm{dom}(\Phi(\zeta))$ and $\Phi(\zeta)B\,\psi = B\Phi(\zeta)\,\psi$. So, $B : \mathrm{dom}(\Phi(\zeta)) \to \mathrm{dom}(\Phi(\zeta))$. By definition of $\Phi(\zeta)$ this also means that B commutes with operators b and b^*. Then by definition of the cyclic vector Ω we get: $b\,B\,\Omega = B\,b\,\Omega = 0$ and so $B\,\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{C}$. To prove that in fact $B\,\psi = (\lambda\mathbb{1})\psi$ for any $\psi \in \mathscr{F}$, note that since

$$B\,(b^*)^m\,\Omega = (b^*)^m B\,\Omega = \lambda\,(b^*)^m\,\Omega\;,$$

one proves the assertion by cyclicity of Ω. □

Corollary 3.7. *Identifying* $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ *with the unital* $*$*-subalgebra* $\mathfrak{A}$ *in the comment* Note (a)*, we conclude that* $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ *is dense in* $\mathcal{A}_W := \mathscr{W}(\mathbb{C})''$ *in the weak***-topology, that is, the Weyl-von Neumann algebra* $\mathcal{A}_W$ *coincides with the closure of* $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ *in this topology. Moreover, since the Weyl family* $\mathscr{W}(\mathbb{C})$ *is irreducible, one gets* $\mathcal{A}_W = \mathcal{L}(\mathscr{F})$*. So, the closure of* $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ *in the weak***-topology coincides with* $\mathcal{L}(\mathscr{F})$*.*

Comments to Note (b): Our choice of a unital $*$-subalgebra $\mathfrak{A}$ (see Note (a)) is the linear span of the Weyl family (3.8). Besides the density of the $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ in $\mathcal{L}(\mathscr{F})$ (Corollary 3.7) the advantage of this choice is that the action of the semigroup $\{T_t^{\sigma\,*}\}_{t\geq 0}$ on $\mathscr{W}(\mathbb{C})$ is known *explicitly*, see [15], (A.32), (A.33). It follows from definition (3.4) and evolution equation (3.5) for bounded operator $A = W(\zeta)$.

(i) Let $0 \leq \sigma_+ < \sigma_-$. Then

$$T_t^{\sigma\,*}(W(\zeta)) = e^{-\Omega_\zeta^\sigma(t)}\, W(\zeta_\sigma(t))\;, \tag{3.9}$$

where

$$(3.10)\quad \Omega^{\sigma}_{\zeta}(t) := \frac{|\zeta|^2}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)t}\right\}, \quad \zeta_\sigma(t) := \zeta\, e^{i\,Et-(\sigma_- - \sigma_+)t/2}.$$

Let for $k \to \infty$ a sequence $\{\sigma^k_-\}_{k\geq 1}$ converges to σ_-. If we define $T^{\sigma(k)*}_t := T^{\sigma\,*}_t\,|_{\sigma_-=\sigma_{k-}}$, where $\Omega^{\sigma(k)}_{\zeta}(t) := \Omega^{\sigma}_{\zeta}(t)\,|_{\sigma_-=\sigma^k_-}$ and $\zeta_{\sigma(k)}(t) := \zeta_\sigma(t)\,|_{\sigma_-=\sigma^k_-}$, then Proposition 3.6(1) and (3.9), (3.10) yield the *strong* operator limit

$$(3.11)\qquad s\!-\!\lim_{k\to\infty} T^{\sigma(k)*}_t(W(\zeta)) = T^{\sigma\,*}_t(W(\zeta))\;,$$

on the Fock space $\mathscr{F}$ for any $W(\zeta) \in \mathscr{W}(\mathbb{C})$.

Remark 3.8. Since for $\zeta \neq 0$ the operator norm $\|W(\zeta) - \mathbb{1}\| = 2$, the evolution (3.10) is *not* continuous in the C^*-algebra topology, but it does in the weak*-topology on the von Neumann algebra $\mathcal{A}_W$. Hence, the pair $(\mathcal{A}_W, T^{\sigma\,*}_t)$ is, in fact, a W^*-dynamical system, [12, 13].

To proceed further with the point (i) we recall assertion about the operator product continuity in the trace-norm topology, see, for example, [16], Proposition 2.69, or [17], Proposition 2.78.

Proposition 3.9. *Let* $\{X_k\}_{k\geq 1} \subset \mathcal{L}(\mathscr{F})$ *and* $s\!-\!\lim_{k\to\infty} X_k = X$. *Let* $\{Y_k\}_{k\geq 1} \subset \mathfrak{C}_1(\mathscr{F})$ *be convergent sequence of trace-class operators*: $\|\cdot\|_1\!-\!\lim_{k\to\infty} Y_k = Y$. *Then* $X\,Y \in \mathfrak{C}_1(\mathscr{F})$ *and*

$$\|\cdot\|_1\!-\!\lim_{k\to\infty} X_k\,Y_k = X\,Y\;.$$

Therefore, duality relations (2.2), (3.1), together with (3.11) and Proposition 3.9 yield

$$(3.12)\qquad \begin{aligned}\lim_{k\to\infty}\langle T^{\sigma(k)}_t u\,|\,W(\zeta)\rangle &= \lim_{k\to\infty}\langle u\,|\,T^{\sigma(k)\,*}_t(W(\zeta))\rangle \\ &= \langle u\,|\,T^{\sigma\,*}_t(W(\zeta))\rangle = \langle T^{\sigma}_t u\,|\,W(\zeta)\rangle\,,\end{aligned}$$

for any $W(\zeta) \in \mathscr{W}(\mathbb{C})$ and all $u \in \mathfrak{C}_1(\mathscr{F})$. By linearity one can extend the convergence in (3.12) to the $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$. Then by irreducibility and weak*-density in $\mathcal{L}(\mathscr{F})$ of the $\mathrm{span}\{\mathscr{W}(\mathbb{C})\}$ (Corollary 3.7), we extend (3.12) to any operator $A \in \mathcal{L}(\mathscr{F}) \simeq \mathfrak{C}^*_1(\mathscr{F})$):

$$(3.13)\qquad \lim_{k\to\infty}\langle T^{\sigma(k)}_t u\,|\,A\rangle = \langle T^{\sigma}_t u\,|\,A\rangle\;.$$

Corollary 3.10. *By virtue of* (3.13) *the sequence of operators* $\{T_t^{\sigma(k)}\}_{k\geq 1}$ *converges on the space* $\mathfrak{C}_1(\mathscr{F})$ *to* T_t^{σ} *in the weak Banach topology* $\sigma(\mathfrak{C}_1(\mathscr{F}), \mathfrak{C}_1^*(\mathscr{F}))$*. Note that for* $A = \mathbb{1}$ *and for any rank-one projection operator* $u = P_{\varphi\,\psi} \in \mathfrak{C}_1(\mathscr{F})$ *defined by*

$$P_{\varphi\,\psi} : \phi \mapsto (\phi, \psi)_{\mathscr{F}}\, \varphi \quad \text{for} \quad \phi, \psi, \varphi \in \mathscr{F} \ , \tag{3.14}$$

one gets $\langle T_t^{\sigma(k)} P_{\varphi\,\psi} \,|\, \mathbb{1}\rangle = \mathrm{Tr}_{\mathscr{F}}(T_t^{\sigma(k)} P_{\varphi\,\psi}) = (T_t^{\sigma(k)} \varphi, \psi)_{\mathscr{F}}$*. Then, the limit* (3.13) *yields*

$$\lim_{k\to\infty} (T_t^{\sigma(k)} \varphi, \psi)_{\mathscr{F}} = (T_t^{\sigma(k)} \varphi, \psi)_{\mathscr{F}} \ .$$

Therefore, $\{T_t^{\sigma(k)}\}_{k\geq 1}$ *converges to* T_t^{σ} *on the space* $\mathscr{F}$ *in the weak operator topology.*

(ii) Let $0 < \sigma_+ < \sigma_-^k$ and $\lim_{k\to\infty} \sigma_-^k = \sigma_+$. Note that the strong limit (3.11) for any Weyl operator $W(\zeta) \in \mathscr{W}(\mathbb{C})$ exists

$$s\!-\!\lim_{k\to\infty} T_t^{\sigma(k)\,*}(W(\zeta)) = T_t^{\sigma_+\,*}(W(\zeta)) \ , \tag{3.15}$$

for operator $T_t^{\sigma_+\,*}(W(\zeta))$ with the evident limit values in (3.10):

$$\Omega_\zeta^{\sigma_+}(t) = \frac{|\zeta|^2}{2}\, \sigma_+ t \ \ , \ \ \zeta_{\sigma_+}(t) = \zeta\, e^{i\,Et} \ . \tag{3.16}$$

Then by the same line of reasoning as in (i) and in Corollary 3.10 we obtain the existence of the weak operator limit on $\mathscr{F}$:

$$w\!-\!\lim_{k\to\infty} T_t^{\sigma(k)} = T_t^{\sigma_+} \ , \tag{3.17}$$

for $\lim_{k\to\infty} \sigma_-^k = \sigma_+$.

Note, that for the *singular* case: $0 < \sigma_+ = \sigma_-$ the explicit solution of equations in [15], see (3.23), gives the same result as (3.16), that is:

$$T_t^{\sigma_+\,*}(W(\zeta)) = T_t^{\sigma\,*}(W(\zeta))\,|_{\sigma_+=\sigma_-} \ , \quad W(\zeta) \in \mathrm{span}\{\mathscr{W}(\mathbb{C})\} \ . \tag{3.18}$$

Therefore, (3.17) and (3.18) give for the weak operator limit

$$w\!-\!\lim_{k\to\infty} T_t^{\sigma^{(k)}} = T_t^{\sigma}\,|_{\sigma_-=\sigma_+} \ . \tag{3.19}$$

(iii) Let $0 \leq \sigma_- \leq \sigma_+$. Then explicit formulae (3.9), (3.10) yield the same properties of semigroups $\{T_t^{\sigma\,*}\}_{t\geq 0}$ and $\{T_t^{\sigma}\}_{t\geq 0}$ as those established in (i), (ii). The only difference is that, in contrast to (i), in the case (ii), (iii), these semigroups have *no* infinite-time limit ($t \to \infty$), for $\sigma_- = \sigma_+ > 0$, respectively in the strong-operator and in the weak Banach topologies.

Corollary 3.11. *By virtue of* (3.9),(3.10) *and of* (3.16), (3.18) *one gets that for any* $0 \leq \sigma_+$ *and* $0 \leq \sigma_-$ *the semigroup* $\{T_t^{\sigma\,*}\}_{t\geq 0}$ *is unity-preserving* (Markov) *on* $\mathcal{L}(\mathscr{F})$ *since*

$$T_t^{\sigma\,*}(W(\zeta = 0)) = T_t^{\sigma\,*}(\mathbb{1}) = \mathbb{1}\ . \tag{3.20}$$

Consequently, by duality (3.1) *the semigroup* $\{T_t^{\sigma}\}_{t\geq 0}$ *is trace-preserving* (Markov) *on* $\mathfrak{C}_1(\mathscr{F})$ *for all values of parameters* $\sigma_\pm \geq 0$*, including the critical cases* $0 < \sigma_+ = \sigma_-$ *and* $\sigma_+ > \sigma_-$*.*

We remark that the positivity-preserving C_0-semigroup $\{T_t^{\sigma}\}_{t\geq 0}$ (2.16) on $\mathfrak{C}_1(\mathscr{F})$ constructed in Section 2.2 by the particle-number cut-off regularisation is *minimal*, see Proposition 2.5(c). If $(\widehat{T}_t^{\sigma}\}_{t\geq 0}$ is a positivity-preserving C_0-semigroup with generator $\widehat{L}_\sigma$, which is another extension of $(H_\sigma - \mathcal{Q}_\sigma)$, then $T_t^{\sigma} \leq \widehat{T}_t^{\sigma}$ for all $t > 0$. This means that for any $\rho \in \Psi(\mathfrak{C}_1^{+})$ and all positive operators $A \in \mathcal{L}^{+}(\mathscr{F})$ one gets for all $t > 0$

$$\langle \rho\,|\, T_t^{\sigma\,*}(A)\rangle \leq \langle \rho\,|\, \widehat{T}_t^{\sigma\,*}(A)\rangle\ . \tag{3.21}$$

As a result this implies the minimality of the positivity-preserving dual semigroup: $T_t^{\sigma\,*}(A) \leq \widehat{T}_t^{\sigma\,*}(A)$ for $A \in \mathcal{L}^{+}(\mathscr{F})$.

3.3 From GKLD ansatz to Markov property

Here we return to the *core problem* formulated in the beginning of Section 2.3 in the framework of the general regularisation Proposition 2.5. Analysis of the open boson model in Section 2 and the proof of the Markov property of evolution in subsection 2.3 motivate the following *abstract* GKLD setting.

GKLD ansatz: (cf. [5], Chapter 3.5, Hypothesis **AA**)

(i) Operator G is the generator of contraction C_0-semigroup $\{U(t) = e^{-tG}\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$.

(ii) Linear operators $\{F_k\}_{k\geq 1}$ in $\mathcal{H}$ are such that domains: $\mathrm{dom}(F_k) \supset \mathrm{dom}(G)$, for all $k \geq 1$.

(iii) For any couple of vectors $x, y \in \mathrm{dom}(G)$ one has

$$(Gx, y)_{\mathcal{H}} + (x, Gy)_{\mathcal{H}} - \sum_{k\geq 1}(F_k x, F_k y)_{\mathcal{H}} = 0\ . \tag{3.22}$$

Then the mappings $S_{t\geq 0} : \rho \mapsto U(t)\,\rho\, U(t)^*$, define on the Banach space $\mathfrak{C}_1(\mathcal{H}) \ni \rho$ the corresponding to the GKLD *ansatz unperturbed* dynamical semigroup $\{S_t = e^{-tH}\}_{t\geq 0}$ with the densely defined in $\mathfrak{C}_1(\mathcal{H})$ generator

$$(H\,\rho) \supseteq (G\rho + \rho\, G^*)\ ,\ \rho \in \mathrm{dom}(H)\ . \tag{3.23}$$

The corresponding to the GKLD *ansatz* perturbation K (cf. (2.17), (3.22)) is the positivity-preserving operator defined on $\mathrm{dom}(H)$ by

$$K : \rho \mapsto \sum_{k\geq 1} F_k \rho F_k^* \,, \quad \rho \in \mathrm{dom}(H)\,. \tag{3.24}$$

Let $\mathcal{P}_{\mathrm{dom}(G)} := \{P_{x,y}(\cdot) : x, y \in \mathrm{dom}(G)\} \subset \mathfrak{C}_1(\mathcal{H})$ be set of *rank-one* linear operators $P_{x,y}\phi := (\phi, y)_{\mathcal{H}}\, x$, for $\phi \in \mathcal{H}$. Then $P_{x,y} \in \mathrm{dom}(H)$ and we can rewrite (3.22) as follows: $\mathrm{Tr}\,((H - K)P_{x,y}) = 0$. Since the linear span

$$\mathfrak{P} := \mathrm{span}\{\mathcal{P}_{\mathrm{dom}(G)}\} \subset \mathfrak{C}_1(\mathcal{H})\,, \tag{3.25}$$

is dense in $\mathrm{dom}(H)$ in the trace-norm topology $\|\cdot\|_1$, the condition (iii) is equivalent to

$$\mathrm{Tr}\,((H - K)\rho) = 0\,, \quad \rho \in \mathrm{dom}(H)\,, \tag{3.26}$$

which coincides with condition (2.17) in the *general* setting of Proposition 2.5. Note that the span $\mathfrak{P}$ (3.25) is also $\|\cdot\|_1$-dense in the whole Banach space $\mathfrak{C}_1(\mathcal{H})$, and hence, in $\Psi(\mathfrak{C}_1(\mathcal{H}))$.

Lemma 3.12. *The span $\mathfrak{P}$ is a core for generator H.*

Proof. Note that for rank-one operators one gets: $S_t P_{x,y} = P_{U(t)x,U(t)y} \in \mathfrak{C}_1(\mathcal{H})$. Since $U(t) : \mathrm{dom}(G) \to \mathrm{dom}(G)$, the couple of vectors $U(t)x, U(t)y \in \mathrm{dom}(G)$ if $x, y \in \mathrm{dom}(G)$, and hence $S_t P_{x,y} \in \mathcal{P}_{\mathrm{dom}(G)}$. Consequently, the span (3.25) is invariant under the semigroup $\{S_t\}_{t\geq 0}$, and so $\mathfrak{P}$ is a core for generator H. □

For simplicity we consider below the GKLD *ansatz* for the case of *finite* sums $\sum_{k\geq 1} F_k \rho F_k^*$ in (3.22). Extension to the infinite sum is straightforward under suitable conditions for its convergence.

Theorem 3.13. *Let for finite sum in (3.22) the operators G, F_k assure the GKLD ansatz. Then there exists a functional regularisation $(K_\alpha)_{\alpha\in J}$ of K such that:*
(a) *For all $\alpha \in J$ the operator $L_\alpha = (H - K_\alpha)$ is the generator of a positivity-preserving contraction C_0-semigroup $\{T_t^\alpha\}_{t\geq 0}$ on $\mathfrak{C}_1^{\mathrm{sa}}$.*
(b) *For $t > 0$, limits: $\lim_\alpha T_t^\alpha \rho = T_t \rho$, exist in $\mathfrak{C}_1^{\mathrm{sa}}$ for all $\rho \in \mathfrak{C}_1^{\mathrm{sa}}$ and define $T_t \colon \mathfrak{C}_1^{\mathrm{sa}} \to \mathfrak{C}_1^{\mathrm{sa}}$.*
(c) *The family $\{T_t = e^{-t\,L}\}_{t\geq 0}$ is the minimal positivity-preserving contraction C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$ for which the generator $L := (H - K)_{ext}$ is an extension of the operator $(H - K)$.*

Proof. The points (a) and (b) are corollary of regularisation Proposition 2.5. The assertion (c) that constructed in this way dynamical semigroup $\{T_t\}_{t\geq 0}$ has generator $L := (H-K)_{ext}$ and that the semigroup is minimal follows from Proposition 2.5(c). □

Remark 3.14. To construct for the perturbation K an example of regularisation family $\{K_\alpha\}_{\alpha\in J}$ one can follow a scheme of the particle-number cut-off regularisation in Section 2.2. For one-mode case this family reduces to the sequence $(K_N)_{N\in\mathbb{N}_0}$ (2.14) corresponding to increasing sequence of projections $(P_N)_{N\in\mathbb{N}_0}$. In the *two-mode* case the particle-number cut-off regularisation corresponds to projections $P_{N_1,N_2} := P_{N_1}\otimes P_{N_2} \colon \mathscr{F}\otimes\mathscr{F}\to\mathscr{F}\otimes\mathscr{F}$, $N_{1,2}\in\mathbb{N}_0$. As a consequence, the regularisation family $(K_{N_1,N_2})_{N_{1,2}\in\mathbb{N}_0}$ is not a sequence, but a *net*.

We notify that similar to the *general* setting (Proposition 2.5(a)-(c)) the GKLD *ansatz* itself is *not* sufficient for the *minimal* dynamical semigroup $\{T_t\}_{t\geq 0}$ be *trace-preserving*, see Theorem 3.13. However, in *contrast* to the general setting, in the framework of the GKLD *ansatz* the *converse* to the Proposition 2.5(d), is also true. For the proof we have to introduce a definition.

Definition 3.15. For each $u\in\mathcal{L}(\mathcal{H})$ we define in $\mathcal{H}\times\mathcal{H}$ the sesquilinear form

$$\mathfrak{L}(u)[x,y] := (u\,Gx,y)_{\mathcal{H}} + (u\,x,Gy)_{\mathcal{H}} - \sum_{k\geq 1}(u\,F_k x, F_k y)_{\mathcal{H}}\,, \tag{3.27}$$

with domain $\mathrm{dom}(G)\times\mathrm{dom}(G)$. Note that

$$\mathfrak{L}(u)[x,y] = \mathrm{Tr}\,(P_{x,y}(H-K)^* u) = \mathrm{Tr}\,((H-K)P_{x,y}u)\,.$$

On account of (3.23) and by definition of operators K (3.24) and L (Theorem 3.13 (c)), one checks by inspection that on the set $\mathcal{P}_{\mathrm{dom}(G)}$ of rank-one linear operators

$$LP_{x,y} = P_{Gx,y} + P_{x,Gy} - \sum_{k\geq 1} P_{F_k x, F_k y}\,, \tag{3.28}$$

for any $(x,y)\in\mathrm{dom}(G)\times\mathrm{dom}(G)$. Therefore, the linear span $\mathfrak{P}\subset\mathrm{dom}(L)$ and $L:\mathfrak{P}\to\mathfrak{P}$. This reveals that the span $\mathfrak{P}\subset\mathrm{dom}(H)$, (3.25), may be a core for generator L.

To bolster this conjecture we need the following proposition identifying a *core* of operator [10], Chapter 8.1.

Proposition 3.16. *Let X be a closed operator in a Banach space $\mathcal{B}$ and let λ be in the resolvent set of X. Then $\mathcal{D}\subset\mathrm{dom}(X)$ is a core of operator X if and only if the image of the map*: $\mathcal{D}\to(\lambda\mathbb{1}+X)\,\mathcal{D}$*, is dense in $\mathcal{B}$.*

Theorem 3.17. *Let operators G, F verify the GKLD ansatz. Then the following statements are equivalent*:
(a) *The minimal dynamical semigroup $\{T_t\}_{t\geq 0}$ constructed in Theorem 3.13 is trace-preserving.*
(b) *Domain $\mathrm{dom}(H)$ is a core for the generator $L := (H - K)_{ext}$.*
(c) *For any $\lambda > 0$ the characteristic equation $\mathfrak{L}_\lambda(u) := (\mathfrak{L}(u) \dot{+} \lambda\, u) = 0$ (in the sesquilinear form sense), has for $u \in \mathcal{L}(\mathcal{H})$ only trivial solution $u = 0$.*

Proof. The fact that (b) implies (a) is the statement of Proposition 2.5(d). The condition (2.17) in the general setting of Proposition 2.5, coincides with condition (iii) of the GKLD *ansatz*. Note that (b) $\Rightarrow$ (a) does *not* need any details concerning the structure of operators H and K.

Now, let (b) be true. Then by Lemma 3.12 and by Proposition 3.16 for any $\lambda > 0$, the set $(\lambda\mathbb{1} + L)\,\mathfrak{P}$ is dense in the Banach space $\mathfrak{C}_1(\mathcal{H})$. Next we use (i), (ii) of the GKLD *ansatz* to note that by (3.27), or by (3.28), for any $P_{x,y} \in \mathfrak{P}$ and $u \in \mathcal{L}(\mathcal{H})$ one gets

$$\mathfrak{L}_\lambda(u)[x, y] = ((\lambda\mathbb{1} + L^*)u\, x, y)_{\mathcal{H}} = \mathrm{Tr}\,((\lambda\mathbb{1} + L)P_{x,y}\, u)\ . \tag{3.29}$$

Therefore, if $\mathfrak{L}_\lambda(u)[x, y] = 0$ for any $P_{x,y} \in \mathfrak{P}$, then by (3.29)

$$\sup_{P_{x,y}\in\mathfrak{P}} |\mathrm{Tr}\,((\lambda\mathbb{1} + L)P_{x,y}\, u)| = 0\ , \tag{3.30}$$

which by density of $(\lambda\mathbb{1} + L)\,\mathfrak{P}$ yields $\|u\| = 0$. So, solution of equation $\mathfrak{L}_\lambda(u) = 0$ for any $\lambda > 0$ is trivial, that is, (b) $\Rightarrow$ (c).

Conversely, if (c) is true, then by (3.29) it is equivalent to the statement: $\mathrm{Tr}\,((\lambda\mathbb{1}+ L)P_{x,y}\, u) = 0$ for all $P_{x,y} \in \mathfrak{P}$, $\Rightarrow u = 0$. Note, that if $(\lambda\mathbb{1}+L)\,\mathfrak{P}$ is dense in Banach space $\mathfrak{C}_1(\mathcal{H})$, then this statement is consistent with (3.30). But if $(\lambda\mathbb{1} + L)\,\mathfrak{P}$ is *not* dense in $\mathfrak{C}_1(\mathcal{H})$, i.e., (b) is *not* true, then $\mathrm{Tr}\,((\lambda\mathbb{1} + L)P_{x,y}\, \widehat{u}) = 0$ for some bounded operator $\widehat{u} \neq 0$ and any $P_{x,y} \in \mathfrak{P}$. Consequently, by (3.29) one gets for this element $\mathfrak{L}_\lambda(\widehat{u}) = 0$, that contradicts to (c). Hence, (c) $\Rightarrow$ (b). As a result, (b) and (c) are equivalent.

Now, let (a) be true and let us prove that (a) $\Rightarrow$ (c). If one supposes that (c) does not hold, then for some $\lambda > 0$ there is a nontrivial solution $u_\lambda \in \mathcal{L}(\mathcal{H})$ of characteristic equation in (c), or in (3.29). This means that $L^* u_\lambda = -\lambda u_\lambda$, and thus for dual semigroup one gets $T_t^* u_\lambda = e^{t\,\lambda} u_\lambda$. To continue we note that

$$-2\|u_\lambda\|\,\mathbb{1} \leq (u_\lambda + u_\lambda^*) \leq 2\|u_\lambda\|\,\mathbb{1}\ , \tag{3.31}$$

$$-2\|u_\lambda\|\,\mathbb{1} \leq i\,(u_\lambda - u_\lambda^*) \leq 2\|u_\lambda\|\,\mathbb{1}\ . \tag{3.32}$$

Since by condition (a) the dual semigroup $\{T_t^*\}_{t\geq 0}$ is unity- and positivity-preserving, its application to (3.31) yields for all $t > 0$:

$$-2\|u_\lambda\|\,\mathbb{1} \leq (u_\lambda + u_\lambda^*)\, e^{t\,\lambda} \leq 2\|u_\lambda\|\,\mathbb{1}\,,$$

which implies that $u_\lambda + u_\lambda^* = 0$. By the similar argument applied to (3.32) one concludes that also $u_\lambda - u_\lambda^* = 0$. For that reason, the solution u_λ for any $\lambda > 0$ must be trivial, that is, (a) ⇒ (c). □

The line of reasoning in this proof is motivated by a detailed discussion presented by F.Fagnola in [5], Chapter 3.5.

Corollary 3.18. *Let unbounded operators H and K verify conditions of the GKLD ansatz. Then the converse to Proposition* 2.5(d)*, is true, that is, the following statements are equivalent:*
(a) *The minimal dynamical semigroup $\{T_t\}_{t\geq 0}$ constructed in Proposition* 2.5(a)-(c)*, is trace-preserving.*
(b) $\mathrm{dom}(H)$ *is a core for the generator $L := (H - K)_{ext}$.*

Corollary 3.19. *On account of Corollary* 3.11 *and Corollary* 3.18 *we infer* (*cf.* [2], *Theorem* 3.2*, and* [5]*, Chapter* 3.5) *the following statement*:
In the framework of the GKLD ansatz *the necessary and sufficient condition for the minimal dynamical semigroup with generator L, constructed in Theorem* 3.13 *by regularisation à la Kato, be trace-preserving* (Markov) *is that domain* $\mathrm{dom}(H)$ *of the unperturbed semigroup must be a* core *of operator L.*

As a consequence, Corollary 3.11 and Corollaries 3.18, 3.19 claim that a necessary and sufficient condition insuring that the *minimal* dynamical semigroup with generator L, constructed in Proposition 2.5(a)-(c) by regularisation à la Kato and condition (2.17), is Markovian (*trace-preserving*) is that domain $\mathrm{dom}(H)$ of unperturbed semigroup is a *core* of operator L.

Corollary 3.20. *The open boson model in Section* 2 *satisfies the GKLD ansatz if operator G coincides with h_σ* (2.10) *and operator K* (3.24) *corresponds to $\mathcal{Q}_\sigma$* (2.7)*, cf. also* (3.4)*. Seeing that by Corollary* 3.11 *the semigroup $\{T_t^\sigma\}_{t\geq 0}$ is trace-preserving for all values of parameters $\sigma_\pm \geq 0$, one concludes that* $\mathrm{dom}(H) = \mathrm{core}(L)$ *for all $\sigma_\pm \geq 0$, including the* critical *regime*: $\sigma_+ = \sigma_- > 0$*. This means that the problem formulated in Remark* 2.9 *has a* complete *affirmative solution in the framework of the GKLD ansatz.*

In the next section we present more details about the GKLD-evolution of the open resonator model (2.8).

4 Gaussian Quasi-Free Evolution

4.1 GKLD evolution of quasi-free states

Although dynamical semigroup $\{T_t^{\sigma}\}_{t\geq 0}$ rests *Markovian* for any $\sigma_{\pm} \geq 0$ (Corollary 3.11 and Corollary 3.20), the *critical* case $\sigma_- = \sigma_+ > 0$ is in a certain sense singular. To scrutinise the nature of this *critical regime* we study both finite- and infinite-time evolutions corresponding to the mapping $\varrho \mapsto T_t^{\sigma} \varrho$, on the set of *density matrices* $\mathcal{M}_1 := \{\rho \in \mathfrak{C}_1^{\mathrm{sa}}(\mathscr{F}) : \|\rho\|_1 = 1\}$ in the limit $\sigma_- \downarrow \sigma_+ > 0$. For making our analysis explicit we need more details about the the GKLD evolution of the open resonator (2.1), (2.8), as well as, about appropriate set of initial conditions selected from $\mathcal{M}_1$, or, in general, from the set of *states* $\{\omega : \mathcal{A}_W \to \mathbb{C}\}$ on the CCR($\mathscr{F}$)-algebra $\mathcal{A}_W$.

(a) To this aim we start by observation that semigroup $\{T_t^*\}_{t\geq 0}$ on $\mathcal{A}_W$ (cf. Corollary 3.11) is a unity-preserving *quasi-free* dynamical semigroup since it has the *canonical* form (see [18, 19]):

$$T_t^*(W(\zeta)) := \Psi_t(\zeta)\, W(\Gamma_t(\zeta))\,,\ \zeta \in \mathbb{C}\,, \tag{4.1}$$

where $\Psi_{t=0}(\zeta) = 1$, $\Gamma_{t=0} = 1$ and $\Psi_t(\zeta = 0) = 1$, $\Gamma_t(\zeta = 0) = 0$. Indeed, on account of (3.9) and (3.10) one has:

$$\begin{aligned} &T_t^{\sigma\,*}(W(\zeta)) = \Psi_t(\zeta)\, W(\Gamma_t(\zeta))\,, \quad \Psi_t(\zeta) := e^{-\Omega_\zeta^{\sigma}(t)}\,, \quad \Gamma_t(\zeta) := \zeta_\sigma(t)\,, \\ &\Omega_\zeta^{\sigma}(t) = \frac{|\zeta|^2}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)\,t}\right\}, \ \ \zeta_\sigma(t) = \zeta\, e^{i\,Et - (\sigma_- - \sigma_+)\,t/2}. \end{aligned} \tag{4.2}$$

Remark 4.1. Let a *normal* state $\omega_\varrho : \mathcal{A}_W \to \mathbb{C}$, be defined by density matrix $\varrho \in \mathcal{M}_1$: $\omega_\varrho(W(\zeta)) = \langle \varrho\,|W(\zeta)\,\rangle$ for $W(\zeta) \in \mathcal{A}_W$. Suppose that this state is *regular*, that is, for any $\zeta \in \mathbb{C}$ the function: $\mathbb{R} \ni \lambda \mapsto \omega_\varrho(W(\lambda\,\zeta))$ is continuous. Then owing to (4.2) for $\sigma_- > \sigma_+ \geq 0$ we obtain

$$\begin{aligned} \omega_\infty^{\sigma}(W(\zeta)) &:= \lim_{t\to\infty} \omega_\varrho(T_t^{\sigma\,*}(W(\zeta))) = \lim_{t\to\infty} e^{-\Omega_\zeta^{\sigma}(t)}\, \omega_\varrho(W(\zeta_\sigma(t)) \\ &= \exp\{-\frac{1}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\}\,, \quad W(\zeta) \in \mathcal{A}_W\,, \end{aligned} \tag{4.3}$$

since $\lim_{t\to\infty} \omega_\varrho(W(\zeta_\sigma(t)) = \omega_\varrho(\mathbb{1})$. Note that stationary state $\omega_\infty(\cdot)$ does not depend on the initial regular normal state ω_ϱ. By duality (2.2)

$$\omega_\infty^{\sigma}(W(\zeta)) = \lim_{t\to\infty} \langle T_t^{\sigma}\,\varrho\,|W(\zeta)\,\rangle = \omega_{\varrho_\infty^{\sigma}}(W(\zeta))\,. \tag{4.4}$$

Hence, the stationary state $\omega^{\sigma}_{\infty}(\cdot)$ is *normal* with density matrix

$$\varrho^{\sigma}_{\infty} := w - \lim_{t\to\infty} T^{\sigma}_{t}\,\varrho\,, \quad \forall \varrho \in \mathcal{M}_1 \subset \mathfrak{C}_1(\mathscr{F})\,, \tag{4.5}$$

which is the weak Banach limit on $\mathfrak{C}_1(\mathscr{F})$.

Remark 4.2. In fact, the arguments in (4.3) show that the quasi-free GKLD evolution of the open resonator (2.1), (2.8) transforms (in the limit $t \to \infty$) *any* initial regular (and *not necessarily* normal) state ω into the limit state ω^{σ}_{∞}, which, as we shall see, is a *quasi-free* state (4.19), but also a normal Gibbs state (4.20) with temperature defined by the environmental *reservoir* with parameters $\sigma_{\pm}$. This motivates our study of the case when the *quasi-free* states are selected as initial states. Then one can follow the quasi-free GKLD evolution of the open resonator in details at any instant.

(b) We recall that the *state* $\omega_{r,s}(\cdot)$ on the CCR($\mathscr{F}$)-algebra $\mathcal{A}_W$ over Hilbert space $\mathfrak{H}$ is called *quasi-free*, if its *characteristic function* (see, e.g., [12], p.146 and p.214) has the form:

$$\omega_{r,s}(W(f)) := \exp\{i\,r(f) - \frac{1}{2}\,s(f,f)\}\,, \quad f \in \mathfrak{H}\,. \tag{4.6}$$

Here $r(\cdot)$ is a linear functional on $\mathfrak{H}$, whereas $s(\cdot,\cdot)$ is a non-negative (closable) *sesquilinear* form on $\mathfrak{H} \times \mathfrak{H}$, that verifies condition

$$\frac{1}{4}\,|\mathrm{Im}(f,g)|^2 \leq s(f,f)\,s(g,g)\,, \quad f,g \in \mathfrak{H}\,,$$

to ensure the *positivity* of the quasi-free state: $\omega_{r,s}((\Phi(f)+i\Phi(g))^*(\Phi(f)+i\Phi(g))) \geq 0$.

By the *Araki-Segal* theorem ([12], p.146) the states defined by (4.6) are *regular* (and *analytic*), verifying for $\Phi(f) := (b(f) + b^*(f))/\sqrt{2}$ equations:

$$r(f) = \omega_{r,s}(\Phi(f)) \quad \text{and} \quad s(f,f) = \omega_{r,s}(\Phi(f)^2) - \omega_{r,s}(\Phi(f))^2\,. \tag{4.7}$$

We also remember that any *normal* state ω on $\mathcal{A}_W$ is defined by a density matrix $\varrho \in \mathcal{M}_1$ and duality (2.2):

$$\omega_{\varrho}(\cdot) := \langle \varrho \,|\, \cdot \,\rangle\,, \quad \text{on } \mathcal{A}_W\,. \tag{4.8}$$

If the state (4.6) is *gauge-invariant*: $\omega_{r,s}(W(f)) = \omega_{r,s}(W(e^{i\varphi} f))$ for $\varphi \in \mathbb{R}$, then $r(\cdot) = 0$. We denote these states by $\omega_s(\cdot) := \omega_{r=0,s}(\cdot)$ and similar to subsection 3.1 consider the one-dimensional Hilbert space $\mathfrak{H} = \mathbb{C}$.

Note that by definitions (4.1) and (4.6) the quasi-free semigroup maps the quasi-free state $\omega_{r,s}(\cdot)$ into the states:

(4.9) $$\omega_{r,s}(T_t^*(W(f))) = \Psi_t(f)\omega_{r,s}(W(\Gamma_t(f))) = \Psi_t(f)\omega_{r_t,s_t}(W(f))\,, \quad t > 0\,,$$

where $r_t(f) := r(\Gamma_t(f))$ and $s_t(f,f) := s(\Gamma_t(f),\Gamma_t(f))$ for $r(f) := r_{t=0}(f)$ and $s(f,f) := s_{t=0}(f,f)$. In general, the states $\{\omega_{r,s}(T_t^*(\cdot))\}_{t>0}$ defined by the quasi-free *evolution* (4.9) are *not* quasi-free because of the factor $\Psi_t(f)$.

(c) A sufficient condition on the evolution, which provides the *invariance* of quasi-free states under mapping (4.9) is a restriction to the *Gaussian* quasi-free *evolution*, [15, 20], A.5. It is defined by conditions:
(1) Semigroup $\{\Gamma_t\}_{t\geq t}$ has a particular form

$$\Gamma_t f := \exp\{i\,t\,\widehat{h} - \frac{t}{2}\,(\Sigma_- - \Sigma_+)\}\,f\,, \quad f \in \mathfrak{H}\,,$$

where $\widehat{h} = \widehat{h}^*$ is a self-adjoint operator and $\Sigma_\mp$ are bounded positive operators in $\mathfrak{H}$ such that $\Sigma_- \geq \Sigma_+ \geq 0$.
(2) Derivative $\partial_t\Psi_{+0}$ is defined by a sesquilinear form:

$$\lim_{t\to+0}\partial_t\Psi_t(f) = -\frac{1}{4}\,(R\,f,f)_{\mathfrak{H}}\,, \quad f \in \mathfrak{H}\,,$$

which is determined by a positive bounded operator $R \geq \Sigma_-$.

It turns out that the GKLD open resonator evolution, (2.1), (2.8), satisfies these restrictions on $\mathfrak{H} = \mathbb{C}$ (cf. (4.2)) since: $\widehat{h} = E$, $\Sigma_\mp = \sigma_\mp$, $R = (\sigma_- + \sigma_+)$ and

(4.10) $$\Gamma_t(\zeta) := \zeta_\sigma(t) = e^{i\,Et-(\sigma_- - \sigma_+)\,t/2}\,\zeta\,, \quad t \geq 0\,, \quad \zeta \in \mathbb{C}\,, \quad \sigma_- \geq \sigma_+ \geq 0\,.$$

Then on account of (4.2), (4.6) and (4.9),(4.10) the gauge-invariant states

(4.11) $$\omega_{\widetilde{s}_t}(\cdot) := \omega_s(T_t^{\sigma*}(\cdot))\,, \quad t \geq 0\,,$$

on $\mathcal{A}_W$ are quasi-free for the time-dependent sesquilinear form

(4.12) $$\widetilde{s}_t(\zeta,\zeta) := s(\zeta_\sigma(t),\zeta_\sigma(t)) + \frac{1}{2}\,|\zeta|^2\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\,(1 - e^{-(\sigma_- - \sigma_+)\,t})\,, \quad \sigma_- > \sigma_+\,.$$

As a consequence, (4.12) yields that the states (4.11) rest quasi-free:

$$\omega_\infty^\sigma(W(\zeta)) = \lim_{t\to\infty}\omega_{\widetilde{s}_t}(W(\zeta))) = \exp\{-\frac{1}{4}\,\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\}\,,$$

including the infinite-time limit, cf. (4.3).

4.2 GKLD evolution of the Gibbs states

After these preliminaries we first study the GKLD evolution of open resonator (2.1), (2.8) when at $t = 0$ the initial state ω_β is a gauge-invariant normal quasi-free Gibbs state for temperature β^{-1}. Then it is defined by (4.8) and density matrix

$$\varrho_\beta := (1 - e^{-\beta E})\, e^{-\beta\, E\, b^* b}\,, \quad \varrho_\beta \in \mathcal{M}_1 \subset \mathfrak{C}_1^{\rm sa}(\mathscr{F})\,. \tag{4.13}$$

Owing to (4.7) and (4.8), we can calculate the characteristic function (4.6) of the normal state corresponding to (4.13). This yields that the state ω_β is gauge-invariant: $r_\beta(\zeta) = 0$, and quasi-free with sesquilinear form

$$s_\beta(\zeta, \zeta) = \frac{1}{2}\, |\zeta|^2\, \frac{e^{\beta E} + 1}{e^{\beta E} - 1}\,. \tag{4.14}$$

On account of (4.12) and (4.14), as initial form at $t = 0$, we obtain the time-dependent sesquilinear form for the GKLD evolution of the open resonator

$$\widetilde{s}_t(\zeta, \zeta) = s_\beta(\zeta_\sigma(t), \zeta_\sigma(t)) + \frac{1}{2}\, |\zeta|^2\, \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\, (1 - e^{-(\sigma_- - \sigma_+)\, t})\,, \quad t \geq 0\,, \tag{4.15}$$

for initial *thermal* density matrix (4.13). Consequently, the corresponding to (4.15) quasi-free states of the open resonator are at any moment $t \geq 0$ gauge-invariant normal Gibbs states for the temperature $\beta(t)^{-1}$, which is given by equation $\widetilde{s}_t(\zeta, \zeta) = s_{\beta(t)}(\zeta, \zeta)$. On account of this equation and (4.14), (4.15), we obtain an explicit formula for time dependence of the inverse temperature:

$$\beta(t) = \frac{1}{E}\, \ln \left[\frac{F(t) + 1}{F(t) - 1}\right]\,, \quad t \geq 0\,, \tag{4.16}$$

where

$$F(t) = \frac{e^{\beta\, E} + 1}{e^{\beta\, E} - 1}\, e^{-(\sigma_- - \sigma_+)\, t} + \frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\, (1 - e^{-(\sigma_- - \sigma_+)\, t})\,. \tag{4.17}$$

For the *regular* case: $0 \leq \sigma_+ < \sigma_-$, by virtue of (4.11)–(4.17) and duality (3.1), or (4.8), we infer that :

(a) For $\omega_s = \omega_{s_\beta}$ equation $\omega_{\widetilde{s}_t}(W(\zeta)) = \langle T_t^\sigma \varrho_\beta \,|\, W(\zeta)\rangle$, (4.11), implies for $\zeta = 0$ that dynamical semigroup $\{T_t^\sigma\}_{t\geq 0}$, which preserves *quasi-free* state along the orbit $t \in [0, \infty)$, is also *trace-preserving*, i.e., Markovian.

(b) Moreover, $\{T_t^\sigma\}_{t\geq 0}$ also preserves the *Gibbs property* of the initial quasi-free state with $\widetilde{s}_{t=0}(\zeta, \zeta)$ (4.15) since sesquilinear form for the characteristic function satisfies equation: $\widetilde{s}_t(\zeta, \zeta) = s_{\beta(t)}(\zeta, \zeta)$, where $\beta(t)^{-1}$ is a time-dependent temperature defined by (4.16).

(c) By (4.11), for $\omega_s = \omega_{s_\beta}$, and by (3.9) we obtain

(4.18) $$\omega_{\widetilde{s}_t}(W(\zeta)) = \langle \varrho_\beta \,|\, T_t^{\sigma *}(W(\zeta))\rangle = \langle \varrho_\beta \,|\, e^{-\,\Omega_\zeta^\sigma(t)}\, W(\zeta_\sigma(t))\rangle \,,$$

where continuous functions: $t \mapsto \Omega_\zeta^\sigma(t)$ and $t \mapsto \zeta_\sigma(t)$ are defined for $t \geq 0$ by (3.10). Note that by Proposition 3.6 the function

$$\zeta \mapsto W(\zeta) := e^{i\Phi(\zeta)}\,, \quad \zeta \in \mathbb{C}\,, \quad W(\zeta) \in \mathcal{A}_W\,,$$

is continuous in the strong operator topology on $\mathcal{L}(\mathscr{F})$. Then given that $\varrho_\beta \in \mathfrak{C}_1(\mathscr{F})$ and hence, $\varrho_\beta\, T_t^{\sigma *}(W(\zeta)) \in \mathfrak{C}_1(\mathscr{F})$, we obtain thanks to (4.18) and Proposition 3.9 the limit:

(4.19) $$\omega_\infty^\sigma(W(\zeta)) := \omega_{\widetilde{s}_\infty}(W(\zeta)) = \lim_{t\to\infty} \langle \varrho_\beta | T_t^{\sigma *}(W(\zeta))\rangle = \exp\Big[-\frac{|\zeta|^2}{4}\frac{\sigma_- + \sigma_+}{\sigma_- - \sigma_+}\Big],$$

for any $W(\zeta) \in \mathcal{A}_W$ and $\zeta \in \mathbb{C}$.

(d) The unique limiting state (4.19) is the *stationary* (steady) state for the quasi-free GKLD evolution of the open resonator, (2.1), (2.8). Because of explicit formulae (4.19) and (4.6), (4.13), (4.14) we deduce that this state is *gauge-invariant*, *quasi-free* and *normal* for the *Gibbs* density matrix

(4.20) $$\varrho_{\beta(\infty)} := (1 - e^{-\beta(\infty)E})\, e^{-\beta(\infty)\, E\, b^* b} \in \mathfrak{C}_1^{\rm sa}(\mathscr{F})\,, \quad \beta(\infty) = \frac{1}{E}\,\ln\frac{\sigma_-}{\sigma_+}\,,$$

on the Fock space $\mathscr{F}$. Equation (4.16) implies that the limit $\beta(\infty) = \lim_{t\to\infty}\beta(t)$ is entirely determined by the *pumping in* and *leaking out* parameters of the resonator-environment contact.

(e) For that reason, if the *pumping in* parameter $\sigma_+ = 0$, then

(4.21) $$s_{\beta_{\sigma_-}(t)}(\zeta,\zeta) = \frac{1}{2}\,|\zeta|^2\,\left[1 + \frac{2\,e^{-\,t\,\sigma_-}}{e^{\beta E} - 1}\right]\,,$$

where the value of inverse temperature

(4.22) $$\beta_{\sigma_-}(t) = \frac{1}{E}\,\ln\,\left[1 + e^{t\,\sigma_-}(e^{\beta\, E} - 1)\right]\,.$$

Then in the limit $t \to \infty$ we obtain a stationary state, which for any $W(\zeta) \in \mathcal{A}_W$ has characteristic function:

$$\omega_{\beta=\infty}(W(\zeta)) = \lim_{t\to\infty} \omega_{\beta_{\sigma_-}(t)}(W(\zeta)) = \exp\{-\frac{1}{2}|\zeta|^2\} = (e_0, W(\zeta)\, e_0)_{\mathscr{F}}\,, \quad \zeta \in \mathbb{C}\,, \tag{4.23}$$

and corresponds to the ground state of resonator (*cyclic* vector) $e_0 \in \mathscr{F}$, cf. subsection 2.1. As a result the stationary limiting Gibbs state (4.23) of the resonator has *zero* temperature: $\beta_{\sigma_-}(\infty) = \lim_{t\to\infty} \beta_{\sigma_-}(t) = \infty$, cf. (4.22). This has an evident interpretation. Because of the dominating leaking, the initial state ω_β relaxes along the Gibbs states Markovian orbit for $s_{\beta_{\sigma_-}(t)}$ (cf. (a),(b)) to the zero-temperature *ground state*.

(f) Besides the temperature we use for characterisation of the state the *mean value* of bosons $\omega(\hat{n})$ in the open resonator, where $\hat{n} = b^*b$, cf.(2.1). By reason of evolution equation (3.5) one *formally* obtains

$$\partial_t T_t^{\sigma *}(\hat{n}) = -T_t^{\sigma *}(L_\sigma^*(\hat{n})) = -(\sigma_- - \sigma_+) T_t^{\sigma *}(\hat{n}) + \sigma_+, \quad T_t^{\sigma *}(\hat{n})\,|_{t=0} = \hat{n}. \tag{4.24}$$

Note that expression (4.24) is *formal* because operator $\hat{n} \notin \mathrm{dom}(L_\sigma^*)$. If we denote $\hat{n}(t) := T_t^{\sigma\,*}(\hat{n})$, the solution of (4.24) for $\sigma_- \neq \sigma_+$ is

$$\hat{n}(t) = e^{-(\sigma_- - \sigma_+)\,t}\, b^*b + \frac{\sigma_+}{\sigma_- - \sigma_+}\left\{1 - e^{-(\sigma_- - \sigma_+)\,t}\right\}\,. \tag{4.25}$$

Let initial state ω be regular. Then by Remark 4.2 the infinite-time limit stationary state ω_∞^σ for the case: $0 \leq \sigma_+ < \sigma_-$, is quasi-free (4.19) and normal Gibbs state (4.20) for temperature defined by parameters $\sigma_\pm$. If in addition the initial state is such that $\omega(\hat{n}) < \infty$, then application of this assertion to (4.25) yields a finite stationary *mean value* of bosons in the open resonator:

$$\omega_\infty^\sigma(\hat{n}) = \lim_{t\to\infty} \omega(\hat{n}(t)) = \frac{1}{e^{\beta(\infty)\,E} - 1}\,, \quad \beta(\infty) = \frac{1}{E}\ln\frac{\sigma_-}{\sigma_+}\,. \tag{4.26}$$

We remind that in general the initial regular state ω is neither normal, nor Gibbs.

4.3 Evolution in a critical regime

A regime complementing to the *regular* case ($\sigma_- > \sigma_+ \geq 0$) corresponds to conditions: $\sigma_- = \sigma_+ > 0$. We define it as the limit $\sigma_- \downarrow \sigma_+$ and we call it the *critical* regime.

(a) Following the arguments in subsection 3.2, *Comments* to Note (b)(ii), we infer that for any Weyl operator $W(\zeta) \in \mathscr{W}(\mathbb{C})$ there exits the strong operator limit in $\mathscr{F}$

(cf. (3.15)):

$$(4.27)\qquad \begin{aligned} &T_t^{\sigma_+ *}(W(\zeta)) := s-\lim_{\sigma_-\downarrow\sigma_+} T_t^{\sigma *}(W(\zeta)) = e^{-\Omega_\zeta^{\sigma_+}(t)}\, W(\zeta_{\sigma_+}(t))\,, \\ &\Omega_\zeta^{\sigma_+}(t) := \frac{1}{2}\,|\zeta|^2\,\sigma_+\,t\,,\;\; \zeta_{\sigma_+}(t) := e^{i\,Et}\,\zeta\,, \qquad \zeta\in\mathbb{C}\,. \end{aligned}$$

The operator family $\{T_t^{\sigma_+ *}\}_{t\geq 0}$ is a semigroup because

$$T_\tau^{\sigma_+ *}\,(T_t^{\sigma_+ *}\,(W(\zeta))) = T_{\tau+t}^{\sigma_+ *}(W(\zeta))\,, \quad W(\zeta)\in\mathcal{A}_W\,.$$

Since by (4.27) it is unity-preserving: $T_t^{\sigma_+ *}(W(\zeta))|_{\zeta=0} = \mathbb{1}$, by duality the dynamical semigroup $\{T_t^{\sigma_+}\}_{t\geq 0}$ is trace-preserving.

(b) On account of (4.27), semigroup $\{T_t^{\sigma_+ *}\}_{t\geq 0}$ has *canonical* form (4.1). Consequently, it is a *quasi-free* semigroup. Owing to (4.10) we infer that (4.27) is clearly a *Gaussian* quasi-free semigroup, that is, it *preserves* quasi-free states, cf. subsection 4.1(c). As a consequence, by (4.12) and (4.27) we obtain for the corresponding sesquilinear form

$$(4.28)\qquad \widetilde{s}_t^{\sigma_+}(\zeta,\zeta) := s(\zeta,\zeta) + |\zeta|^2\,\sigma_+ t\,, \quad \zeta\in\mathbb{C}\,.$$

(c) Following the arguments in subsection 4.2 one infers *mutatis mutandis* that the Markov dynamical semigroup $\{T_t^{\sigma_+}\}_{t\geq 0}$ preserves also the *Gibbs property* of the initial Gibbs quasi-free state (4.13). As a consequence, the sesquilinear form (4.28): $\widetilde{s}_t^{\sigma_+}(\zeta,\zeta) = s_{\beta_{\sigma_+}(t)}(\zeta,\zeta)$, is again entirely defined by a time-dependent inverse temperature $\beta_{\sigma_+}(t)$, see (4.16) for $F(t) = F_{\sigma_+}(t)$, where

$$(4.29)\qquad F_{\sigma_+}(t) = \frac{e^{\beta\,E}+1}{e^{\beta\,E}-1} \;+\; 2\,\sigma_+\,t\,.$$

(d) On account of (4.16) for (4.29), in the *critical regime* $\sigma_- = \sigma_+ > 0$, the *temperature* $(\beta_{\sigma_+}(t))^{-1}$ of the state $\omega_{\beta_{\sigma_+}(t)}$ corresponding to the form $s_{\beta_{\sigma_+}(t)}(\zeta,\zeta)$, is monotonously increasing over time and it gets *infinite* value in the infinite-time limit.

State $\omega_{\beta_{\sigma_+}(t=\infty)}$ is called the "chaotic Gibbs state" [21], Section 5.3.1. It is characterised by sesquilinear form $s_{\beta_{\sigma_+}(t=\infty)}(\zeta,\zeta)$ with an *infinite* jump at $\zeta=0$. This means that the state is not *regular*, that is, function: $\lambda\mapsto\omega_{\beta_{\sigma_+}(t=\infty)}(W(\lambda\,\zeta))$, is not continuous for all $\zeta\in\mathbb{C}$, cf. Remark 4.1. In a round-about way this singularity can be expressed via infinite-time limits of characteristic function:

$$(4.30)\qquad \begin{aligned} &\lim_{t\to\infty}\omega(T_t^{\sigma_+ *}(W(\zeta))) = \lim_{t\to\infty}\omega(e^{-|\zeta|^2\,\sigma_+\,t/2}\,(W(e^{i\,Et}\,\zeta))) = 0\,,\;\; \zeta\neq 0\,, \\ &\lim_{t\to\infty}\omega(T_t^{\sigma_+ *}(W(\zeta=0))) = 1\,, \end{aligned}$$

cf. (4.27), for any regular state ω.

(e) By virtue of (4.24) and (4.25) we obtain in the critical regime: $\sigma_- \downarrow \sigma_+ > 0$, a (*formal*) equation for time evolution of the number operator for bosons in the open resonator:

$$\hat{n}(t) = T_t^{\sigma_+\,*}(\hat{n}) := \lim_{\sigma_- \to \sigma_+} T_t^{\sigma\,*}(\hat{n}) = \hat{n} + \sigma_+ \, t \,. \tag{4.31}$$

Following the same line of reasoning as in subsection 4.3(f) one gets

$$\omega_{\beta_{\sigma_+}(t=\infty)} = \lim_{t\to\infty} \omega(\hat{n}(t)) = \lim_{t\to\infty} (\omega(\hat{n}) + \sigma_+ \, t) = \infty \,, \tag{4.32}$$

for any initial regular state such that $\omega(\hat{n}) < \infty$. The infinite number of bosons in (4.32) corresponds to the established in (d) infinite temperature of the open resonator .

(f) Because of formulae (4.2) the case: $\sigma_+ > \sigma_- \geq 0$ (a ”supercritical” regime), corresponds to the *exponentially* fast growing of temperature $(\beta(t))^{-1}$ over time, see subsection 4.2, (4.16). The same rate one also gets for increasing of the number of bosons in the open resonator, see (4.25). The rest of this ”supercritical” regime can be treated for any finite time *mutatis mutandis* the regular case from subsections 4.1 and 4.2.

Indeed, by virtue of (3.9) and (3.10) we obtain:

$$\begin{aligned} &T_t^{\sigma\,*}(W(\zeta)) = \Psi_t(\zeta)\, W(\Gamma_t(\zeta)) \,, \quad \Psi_t(\zeta) := e^{-\,\Omega_\zeta^\sigma(t)} \,, \quad \Gamma_t(\zeta) := \zeta_\sigma(t) \,, \\ &\Omega_\zeta^\sigma(t) = \frac{|\zeta|^2}{4}\, \frac{\sigma_- + \sigma_+}{\sigma_+ - \sigma_-} \left\{ e^{(\sigma_+ - \sigma_-)\,t} - 1 \right\} , \quad \zeta_\sigma(t) = \zeta \, e^{i\,Et + (\sigma_+ - \sigma_-)\,t/2} \,. \end{aligned} \tag{4.33}$$

Then similarly to (4.30) the singularity makes an appearence in the limit $t \to \infty$:

$$\begin{aligned} &\lim_{t\to\infty} \omega(T_t^{\sigma_+\,*}(W(\zeta))) = \lim_{t\to\infty} \omega(e^{-\Omega_\zeta^\sigma(t)}\, W(\zeta_\sigma(t))) = 0 \,, \;\; \zeta \neq 0 \,, \\ &\lim_{t\to\infty} \omega(T_t^{\sigma_+\,*}(W(\zeta = 0))) = 1 \,, \end{aligned} \tag{4.34}$$

for any initial regular state ω.

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