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In memory of JORDAN G. BRANKOV

SCALING BEHAVIOR OF CONFINED *O(n)* SYSTEMS INVOLVING LONG-RANGE INTERACTION

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ABSTRACT: To explore the finite-size scaling in confined systems involving an interaction with long-range tail one needs the development of suitable mathematical techniques. In the present review we consider the scaling behavior of the finite $O(n)$ model with long-range interaction that is widely used in the theory of classical and quantum phase transitions. We consider finite geometries subject to periodic boundary conditions. The present mathematical method may be used to compute different thermodynamic quantities such as: the free energy, the susceptibility, specific heat etc. Here, we present results for the susceptibility in different regions of the phase diagram. Furthermore, we investigate the effect of classical and quantum fluctuations and check various scaling hypotheses.

KEY WORDS: Critical phenomena, finite-size scaling, long-range interaction

1 INTRODUCTION

In many real systems the interparticle interaction with long-range tail is either a dipole-dipole or van der Waals interaction (for a review see Refs. [1,2] and references therein). Here we will be interested in isotropic ferromagnetic long-range interactions $J(\boldsymbol{x}_i, \boldsymbol{x}_j)$ decaying algebraically with the distance $x_{ij} = |\boldsymbol{x}_i - \boldsymbol{x}_j|$ between the spins $\boldsymbol{s}_i$ and $\boldsymbol{s}_j$ as $x_{ij}^{-d-\sigma}$, with d – the dimension of the system and $\sigma > 0$ – the decay exponent that sets up the extent of the interaction. The Hamiltonian of such a system reads

$$\mathcal{H} = -\frac{1}{2}\sum_{(i,j)}\left(J_{\rm sr}\delta_{x_{ij},1} + J_{\rm lr}x_{ij}^{-d-\sigma}\right)\boldsymbol{s}_i\cdot\boldsymbol{s}_j, \tag{1}$$

where $J_{\rm sr}$ and $J_{\rm lr}$ describe short- and long-range interactions, respectively. This model may be considered as bridging short-range models ($\sigma \to \infty$) and extended long-range models ($\sigma \to 0$) based on the Kac–type potentials i.e. potentials decreasing as $\lim_{\sigma\to 0}\sigma\exp(-\sigma x_{ij})$. Moreover, the condition $\sigma > 0$ is needed to avoid an ill-defined thermodynamic limit [3].

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Model (1) is extensively studied in the literature via different approaches and its critical behavior as a function of the temperature has attracted the attention of many scientists. It is known to exhibit a second order phase transition with a critical temperature T_c depending upon d, n and σ, provided $d > \sigma$. When $\sigma < 2$, the long-range term dominates and the critical properties of the system are σ dependent. By increasing σ, a crossover from long-range critical behavior to a short-range one with $\sigma \geq 2$ takes place. In this case the long-range interaction is *subleading* and the scaling behavior of any thermodynamic function may be expressed in terms of two scaling variables [4] with long-range interaction being a correction to scaling in the sense of renormalization group [5]. The exact value of σ, when the crossover from long-range behavior to its short-range counterpart occurs, has been the matter of a long-standing debate in the literature. For an early review of the literature see Ref. [6].

To explore the finite-size scaling behavior of model (1), one needs to compute the finite-size contributions to the bulk critical behavior emanating from the d-dimensional lattice sum

$$W^{\gamma}_{d,\sigma}(t,L) = \sum_{\boldsymbol{q}} \frac{1}{(t + \boldsymbol{q}^2 + b|\boldsymbol{q}|^{\sigma})^{\gamma}}, \quad \sigma, \gamma, b > 0. \tag{2}$$

Here, under periodic boundary conditions, the discrete vector $\boldsymbol{q}$ has components $q_i = 2\pi n_i/L\,(n_i = 0, \pm 1, \pm 2, \cdots)$, $i = 1, \cdots, d$. L is the linear size of the box confining the system and t is a parameter measuring the distance to the bulk critical point. The parameter γ allows the treatment of both the classical $(\gamma = 1)$ [7–14] and the quantum $\left(\gamma = \frac{1}{2}\right)$ [15–18] limits on an equal footing. Notice that, higher values of γ may appear in sums relevant to finite-size studies via field theoretic approach [19–22]. Moreover, expression (2) is tightly related to the investigation of critical properties of systems involving anisotropic interactions [23–27].

To assess the finite-size contribution of the d-dimensional sum (2) at large L to its limiting d-fold integral in the thermodynamic limit, that is infinite L, one reduces the summation to the corresponding one-dimensional effective problem. This is usually achieved with the aid of the Poisson summation formula. In this respect, the term $|\boldsymbol{q}|^{\sigma}$ in conjunction with the arbitrariness of the parameter γ brings about peculiar mathematical problems. Earlier attempts [7, 8, 19] to solve this problem led to distinct integral representations depending on the specific physical problem under investigation. An elegant solution [26] to analyze the finite-size effects in the general case $0 < \sigma \leq 2$ and arbitrary γ was found to involve the three-parametric Mittag-Leffler function [26, 28]

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \quad \mathrm{Re}(\alpha) > 0. \tag{3}$$

In the remainder of this article we will consider the finite-size scaling for different cases of σ and γ.

2 Scaling Behavior of Confined Classical Systems with Long Range Interaction

2.1 The model

In the vicinity of its critical point, Hamiltonian (1) is equivalent to the d-dimensional $O(n)$-symmetric φ^4 model

$$\mathcal{H}\{\varphi\} = \frac{1}{2T}\int_V d^d\boldsymbol{x}\left[(\nabla\varphi)^2 + b\left(\nabla^{\frac{\sigma}{2}}\varphi\right)^2 + r_0\varphi^2 + \frac{1}{2}u_0\varphi^4\right], \tag{4}$$

where φ is a short-hand notation for the space dependent n-component field $\varphi(x)$, $r_0 = r_{0c} + t_0$ ($t_0 \propto T - T_c$) and u_0 are model constants. V is the volume of the system. Here, we set $\hbar = k_B = 1$. Notice that the second term in the integrand in Eq. (4) transforms to $\boldsymbol{q}^\sigma|\varphi(\boldsymbol{q})|^2$ in the momentum space. The vector $|\mathbf{q}|$ has a cutoff $\Lambda \sim a^{-1}$ (a is the lattice spacing that tends to vanish in the continuum limit). Below we assume $L/a \to \infty$, $\xi/a \to \infty$ with $\xi/L = \text{const}$.

According to renormalization group theory [29], in the vicinity of the critical point, any thermodynamic quantity scales like [20–22]

$$X[t,b,L] = L^{\gamma_x/\nu} f_x\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right). \tag{5}$$

where $t = T - T_c$, γ_x and ν are the bulk critical exponents measuring the divergence of the bulk thermodynamic quantity X (*i.e.* $X \sim |t|^{-\gamma_x}$) and the correlation length (*i.e.* $\xi \sim |t|^{-\nu}$). In (5) the function $f_x(a,b)$ is a universal function of its arguments. Note that Eq. (5) generalizes to confined systems the scaling behavior of bulk systems [4].

2.2 Effective Hamiltonian and finite-size analysis

When Hamiltonian (4) is confined to a finite geometry one may decompose the field $\varphi(x)$ into its zero mode – being equivalent to the bulk magnetization – and nonzero modes (with $\boldsymbol{q} \neq 0$). The nonzero modes are treated perturbatively with the aid of a loop expansion. Thy are traced over to yield an effective Hamiltonian for the zero mode [30, 31]

$$\exp\left[-\mathcal{H}_{\text{eff}}\right] = \text{Tr}_{\phi_{q\neq 0}} \exp\left[-\mathcal{H}\left(\phi_{\boldsymbol{q}=0}, \phi_{\boldsymbol{q}\neq 0}\right)\right]. \tag{6}$$

For the bare Hamiltonian (4) with spatial integration over a volume of linear extent L in each dimension, we get [20–22]

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}L^d\left(R\phi^2 + \frac{1}{2}U\phi^4\right), \tag{7}$$

where the effective coupling constants are given by

$$R = r_0 + (n+2)u_0 L^{-d} \sum_{\boldsymbol{q}\neq 0} \frac{1}{r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma}, \tag{8a}$$

$$U = u_0 - (n+8)u_0^2 L^{-d} \sum_{\boldsymbol{q}\neq 0} \frac{1}{(r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma)^2}. \tag{8b}$$

Simple dimensional analysis shows that the effective coupling constants should have the following scaling forms [22]

$$R = L^{\eta-2} f_R\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right) \quad \text{and} \quad U = L^{d-4+2\eta} f_U\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right), \tag{9}$$

for $t \gtrsim 0$, where f_R and f_U are scaling functions inheriting the properties of the bulk critical point. They are analytic at $t = 0$, since only finite modes have been integrated out. After computing the functions f_R and f_U, we can evaluate the different thermodynamic quantities and the explicit forms of their associated scaling functions. A natural way to achieve this goal is to evaluate the thermal averages with respect to effective Hamiltonian (7). The thermal averages of the even moments of the field ϕ are given by

$$M_{2p} \equiv \langle \phi^{2p} \rangle_{\mathcal{H}_{\text{eff}}} = \left(\frac{1}{2}UL^d\right)^{-\frac{p}{2}} \frac{\Gamma\left[p + \frac{n}{2}\right]}{\Gamma\left[\frac{n}{2}\right]} \frac{D_{-p-\frac{n}{2}}\left(\frac{\mathfrak{z}}{\sqrt{2}}\right)}{D_{-\frac{n}{2}}\left(\frac{\mathfrak{z}}{\sqrt{2}}\right)}, \tag{10}$$

where $\Gamma(x)$ is the Γ-function, $D_p(\mathfrak{z})$ are the parabolic cylinder functions and $\mathfrak{z} = RL^{d/2}U^{-1/2}$ is a "characteristic" scaling variable.

For the purpose of this work we are interested in the behavior of the susceptibility given by

$$\chi = \frac{1}{n}\int_V d^dx \, \langle \varphi(\mathbf{x})\varphi(\mathbf{0})\rangle = L^{2-\eta} f_2(tL^{1/\nu}, bL^{2-\sigma-\eta}). \tag{11}$$

In the remainder of this Section we concentrate on the computation of the finite-size behavior of the coupling constants R and U of the effective Hamiltonian (7). We will consider separately the cases $0 < \sigma < 2$ and $\sigma \geq 2$, corresponding to *leading* and *subleading* interations, respectively. As a consequence we will deduce results for the characteristic variable $\mathfrak{z} = RU^{-1/2}L^{2-\eta-\varepsilon/2}$ and the susceptibility χ (11).

2.3 Leading long-range interaction ($0 < \sigma < 2$)

The short-range interaction may be regarded as a correction the to long-range one and thus neglected. The finite-size scaling is extracted with the aid of the identity [26]

$$\frac{1}{(1+y^\alpha)^\gamma} = \int_0^\infty dt e^{-yt} t^{\alpha\gamma-1} E^\gamma_{\alpha,\alpha\gamma}(-t^\alpha), \quad \text{Re}(\alpha), \quad \text{Re}(\gamma) > 0, \tag{12}$$

aiming to render the ensuing analytic calculations feasible. The aim of this approach is to reduce the d-dimensional sums in Eqs. (8) to a one-dimensional effective problem. Thus, for the renormalized theory [29] to the one-loop order, one gets [21]

$$R = t\left(1+\frac{n+2}{\varepsilon}\hat{u}\right) + (n+2)uL^{\sigma-d}\left[D^1_{d,\sigma}\left(tL^\sigma\right)^{\frac{d}{\sigma}-1} + F^1_{d,\sigma}\left(tL^\sigma\right)\right] \tag{13a}$$

and

$$U = u\left(1+\frac{n+8}{\varepsilon}\hat{u}\right) + (n+8)u^2L^{2\sigma-d}\left[\left(\frac{d}{\sigma}-1\right)D^1_{d,\sigma}\left(tL^\sigma\right)^{\frac{d}{\sigma}-2} - \frac{\partial F^1_{d,\sigma}\left(tL^\sigma\right)}{\partial(tL^\sigma)}\right], \tag{13b}$$

where

$$\hat{u} = uS_d^{-1}, \quad S_d = \frac{1}{2}(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right), \qquad D^\gamma_{d,\sigma} = S_d^{-1}\frac{1}{\sigma}\frac{\Gamma\left(\gamma-\frac{d}{\sigma}\right)\Gamma\left(\frac{d}{\sigma}\right)}{\Gamma(\gamma)}$$

and the functions

$$F^\gamma_{d,\sigma}(y) = \frac{1}{(2\pi)^{\gamma\sigma}}\int_0^\infty dx\, x^{\gamma\frac{\sigma}{2}-1}E^\gamma_{\frac{\sigma}{2},\gamma\frac{\sigma}{2}}\left(-\frac{y}{(2\pi)^\sigma}x^{\frac{\sigma}{2}}\right)\left[A^d(x) - 1 - \left(\frac{\pi}{x}\right)^{\frac{d}{2}}\right] \tag{14}$$

with

$$A(u) = \sum_{k=-\infty}^{\infty} e^{-k^2u} = \sqrt{\frac{\pi}{u}}A\left(\frac{\pi^2}{u}\right).$$

Let us note that $F^\gamma_{d,\sigma}(0)$ are related to the Madelung constants [32].

At the fixed point of the theory to the lowest order in ε, i.e. $\left(\hat{u}^* = \frac{\varepsilon}{n+8}\right)$, in the vicinity of the upper critical dimension ($d = 2\sigma - \varepsilon$), we get [20]

$$\mathfrak{z}^*(y) = \frac{\sqrt{n+8}}{\varepsilon}\left[y - \frac{\varepsilon}{2\sigma}y\left(1-\frac{n-4}{n+8}\ln y\right) + 2^{\sigma-1}\varepsilon\frac{n+2}{n+8}\Gamma(\sigma)F_{2\sigma,\sigma}(y) - \varepsilon 2^{\sigma-2}y\Gamma(\sigma)\frac{\partial}{\partial y}F^1_{2\sigma,\sigma}(y)\right].$$

Here, we introduced the scaling variable $y = tL^{1/\nu}$ with $\nu^{-1} = \sigma - \frac{n+2}{n+8}\varepsilon + O(\varepsilon)$, thus demonstrating that $\mathfrak{z}^*(y)$ verifies the finite-size scaling hypotheses and consequently all the thermodynamic functions do. At the critical point T_c, the dependence

$\mathfrak{z}^*(0) \sim \sqrt{\varepsilon}$ agrees with analytical results for the short-range interaction [30] and the numerical ones for the Ising model with long-range interaction [33].

From Eq. (11), we obtain the susceptibility in the vicinity of the critical point, i.e. $tL^\sigma \ll 1$. It reads

$$\chi = \frac{L^\sigma}{\sqrt{\varepsilon}} \frac{2}{\sqrt{S_{2\sigma}}} \frac{\sqrt{n+8}}{n} \frac{\Gamma\left(\frac{n+2}{4}\right)}{\Gamma\left(\frac{n}{4}\right)} + O(\varepsilon). \tag{15}$$

In the limit $n \to \infty$, the correlation length scales like $\xi \sim \varepsilon^{-1/2\sigma} L$, thus recovering a result obtained for the spherical model [10] and showing that this behavior is universal and is not a characteristic of the spherical limit i.e. $n \to \infty$.

In the disordered phase, away from the critical point, i.e. $tL^\sigma \gg 1$, from Eq. (11), we get

$$\chi = \chi_\infty \Bigg[1 - \frac{n+2}{n+8} \frac{\varepsilon}{\sigma} \ln y - 2^{\sigma-1} \frac{n+2}{n+8} \Gamma(\sigma) \varepsilon y^{-1} F_{2\sigma,\sigma}(y) - \frac{n+2}{n+8} S_{2\sigma} \varepsilon y^{-2} + O\left(\varepsilon^2\right) \Bigg], \tag{16}$$

in full agreement with the finite-size scaling hypothesis (11). Whence, the finite-size correction to the bulk susceptibility is dominated by the bulk critical behavior, with power-law correction in L [26]. This result generalizes to finite n the behavior of the susceptibility derived in the framework of the spherical model [8, 11].

The coupling constant R is related to the shift of the critical temperature of the bulk system. By setting $t = 0$ in (13a) and expanding around $d = 2\sigma$ ($\varepsilon = 2\sigma - d$), we obtain that the shift of T_c towards the so called pseudocritical point $T_c(L)$, where the rounding of the thermodynamic quantities takes place, is proportional to $L^{-\frac{1}{\nu}}$ [32, 34].

2.4 Subleading long-range interaction

In the case $d + \sigma < 6$, the short-range interaction in the presence of the nonanalytic term, $\boldsymbol{q}^\sigma$, a step towards the solution of the problem of finite-size scaling was proposed in Refs. [13, 19]. It is based upon the idea that the long-distance physics is essentially contained in the contributions associated to the effects of long-range fluctuations. That is, we will consider the leading behavior due to the small $\boldsymbol{q}$ contributions. Expanding the r.h.s of R in (8) around $\boldsymbol{q} = \boldsymbol{0}$, we obtain

$$R = r_0 + (n+2) u_0 \mathcal{S}_L(d, r_0, 2) - (n+2) u_0 b \left(1 + r_0 \frac{\partial}{\partial r_0}\right) \mathcal{S}_L(d, r_0, \sigma), \tag{17}$$

where

$$\mathcal{S}_L = \frac{1}{L^d}\sum_{\boldsymbol{q}\neq 0}\frac{|\boldsymbol{q}|^{\sigma-2}}{r+\boldsymbol{q}^2}. \tag{18}$$

To investigate the finite-size effects on the critical behavior of the bulk system we need to analyze the finite-size behavior of the function $\mathcal{S}_L(d,r,\sigma)$. This is performed by making use of the identity [13, 22]

$$\frac{q^{2p}}{r+q^2} = \int_0^\infty \exp[-(q^2+r)t]t^{-p}\gamma^*(-p,-rt)dt, \;\; p<1, \tag{19}$$

where $\gamma^*(a,x)$ is a single-valued analytic function of a and x, possessing no finite singularities [35],

$$\gamma^*(a,x) = e^{-x}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(a+n+1)}, \quad |x|<\infty. \tag{20}$$

Using dimensional regularization, at the fixed point, we obtain

$$\begin{aligned} RL^{2-\eta} = {} & y\left(1+\frac{\varepsilon}{2}\frac{n+2}{n+8}\ln y\right)+\frac{\varepsilon}{4}\frac{n+2}{n+8}bL^{2-\sigma}\frac{(2+\sigma)\pi}{\sin\left(\pi\frac{\sigma}{2}\right)}y^{\frac{\sigma}{2}}I^0_{\text{scaling}}(y,4) \\ & +\varepsilon\frac{n+2}{n+8}S_4-\varepsilon S_4\frac{n+2}{n+8}bL^{2-\sigma}\left(1+y\frac{\partial}{\partial y}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\,, \end{aligned} \tag{21}$$

where we have introduced the scaling variable $y = tL^{1/\nu}$ with $\nu^{-1} = 2-\frac{n+2}{n+8}\varepsilon + O\left(\varepsilon^2\right)$ and

$$I^p_{\text{scaling}}(x,d) = (4\pi)^{p-1}\int_0^\infty e^{-x\frac{u}{4\pi^2}}u^{-p}\gamma^*(-p,-\frac{x}{4\pi^2}u)\left[A^d(u)-\left(\frac{\pi}{u}\right)^{d/2}-1\right]du.$$

Equation (21) shows that R has the required scaling form predicted in Eq. (9). In the case of sort-range interaction, $b=0$, Eq. (21) reduce to a result obtained in Ref. [30].

The effective coupling constant U follows from the expression of R to obtain

$$\begin{aligned} UL^{\varepsilon} = {} & \frac{\varepsilon}{n+8}S_4\Big(1+\frac{\varepsilon}{2}(1+\ln y)\Big)+\frac{\varepsilon^2}{n+8}\frac{bL^{2-\sigma}}{8}S_4\frac{\sigma\pi(\sigma+2)}{\sin\left(\pi\frac{\sigma}{2}\right)}y^{\frac{\sigma}{2}-1} \\ & +\frac{\varepsilon^2}{n+8}S_4^2\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4)-\frac{\varepsilon^2}{n+8}S_4^2bL^{2-\sigma}\Big(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\Big)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4), \end{aligned} \tag{22}$$

at the fixed point, in agreement with the scaling relations (9). Eq. (22) generalizes the results of Ref. [30] to the case when the subleading long-range interaction is taken into account. Note that the effective coupling constant U has a finite limit at the critical point, i.e. in the limit $t \to 0$.

At the fixed point, we obtain (for $y \ll 1$)

$$(23)\quad \mathfrak{z}^*(y) = \sqrt{\frac{n+8}{\varepsilon S_4}}\Bigg[y - y\frac{\varepsilon}{4}\Big(1 - \frac{n-4}{n+8}\ln y\Big) + \frac{3n}{n+8}\frac{\varepsilon}{16}\frac{(2+\sigma)\pi}{\sin\left(\pi\frac{d}{\sigma}\right)}y^{\frac{\sigma}{2}} + \frac{n+2}{n+8}\varepsilon S_4\Big(I^0_{\text{scaling}}(y,4) - bL^{2-\sigma}\Big(1 + y\frac{\partial}{\partial y}\Big)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\Big) - \frac{1}{2}\varepsilon S_4 y\Big(\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4) - bL^{2-\sigma}\Big(2\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\Big)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\Big)\Bigg].$$

It is easy to check that at the bulk critical point, we obtain a generalization of the susceptibility obtained for the model with pure short range interaction [30]. Note that the scaling behavior of the expansion of $\mathfrak{z}^*(0)$ in terms of $\sqrt{\varepsilon}$ is preserved, while the coefficient is now a function the linear size L and the parameter σ controlling the long-range interaction i.e. $\mathfrak{z}^*(0)$ is a function of the scaling variable $bL^{2-\sigma}$, thus it shows that any thermodynamic function obeys the finite-size scaling hypothesis (5).

In the vicinity of the bulk critical temperature, the susceptibility is expressed by (15) with $\mathfrak{z}$ from (23). After some algebra, we find that χ has an expansion in powers of $\sqrt{\varepsilon}$. This is an extension to finite n, by means of perturbation method, of the result obtained for the mean spherical model [13].

Away from the critical point, using expressions (21) and (22), and with the aid of the asymptotic expansions of I^p_{scaling} [13], we get

$$(24)\qquad \chi = \chi_\infty\left[1 - \varepsilon\frac{n+2}{n+8}\frac{S_4}{y}bL^{2-\sigma}\left(C_{4,\frac{\sigma-2}{2}}y^{-2} - \frac{y^{\frac{\sigma}{2}}}{4S_4}\frac{\sigma+2}{\sin\frac{\pi\sigma}{2}}\right)\right],$$

where

$$(25)\qquad C_{d,p} = -\frac{(1+p)4^{1+p}}{\pi^{\frac{d}{2}}}\frac{\Gamma\left(1+p+\frac{d}{2}\right)}{\Gamma(-p)}\sum_{\boldsymbol{k}\neq 0}\frac{1}{k^{d+2(p+1)}}.$$

Expression (24) shows that the susceptibility has the scaling form given by the finite-size scaling hypothesis (11). In this regime the critical properties of the system are dominated by the bulk critical behavior, with finite-size corrections in powers of L.

In the case $d+\sigma = 6$, we must assume $d = 4-\varepsilon$ and $\sigma = 2+\varepsilon$. Thus, the $\boldsymbol{q}$ term in the spectrum of the system turns to be $\boldsymbol{q}^\sigma = \boldsymbol{q}^{2+\varepsilon}$. In all calculations we perform

ε-expansion only around $d = 4$ keeping the σ-dependent terms unchanged in (21) for $d + \sigma = 6$ to obtain that R and U depends upon $\ln L$ [22]. This dependence does not cancel when we compute the variable $\mathfrak{z}(y)$, but vanishes in the limit $y \to 0$. This peculiar behavior remains valid for the susceptibility as well.

3 Finite-Size Scaling and Quantum Critical Behavior

3.1 Quantum critical finite-size scaling

Unlike classical finite models, where the scaling can be achieved equally for all "spatial" dimensions, confined quantum models are anisotropic, in the sense that space and "imaginary-time", $L_\tau \sim T^{-z}$ directions are not necessarily equivalent. According to the finite-size scaling hypothesis [36] extended to quantum systems, the singular part of a physical quantity $A\,(g, L, T)$, with g the parameter driving the quantum phase transition, that is singular in the thermodynamic (bulk) limit at the quantum critical point g_c, scales like [16]

(26) $$A_s(g, T, L) = T^{-p/z} A_s\left(\delta g T^{-1/z\nu}, T b^z, T^{1/z} L^{-1}\right).$$

Here $\delta g = g - g_c$, p stands for the engineering dimension $d + z$ of the quantum system when the scaling function is associated to the singular part of the free energy. It is the normalized by ν, critical exponent γ_a of any other bulk physical quantity A at the quantum critical point.

The $O(n)$ system (4) in the limit $n \to \infty$ is exactly soluble. Thus, all thermodynamic quantities may be obtained in closed form. In this limit the critical behavior of this model is equivalent to that of the mean-spherical model (for a review see e.g. [37, 38]). The saddle point equation of the quantum counterpart of the $O(n)$ Hamiltonian (4) in the limit $n \to \infty$ reads [16]

(27) $$\phi = r_0 + u_0 \frac{T}{V} \sum_{\boldsymbol{q},m} \frac{1}{\phi + (2\pi m T)^2 + |\boldsymbol{q}|^\sigma},$$

where $2\pi m T$ (with $m = 0, \pm 1, \pm 2, \cdots$) are the Matsubara frequencies for bosonic systems. In the bulk limit Eq. (27) was used in Ref. [39–45] to explore the quantum critical behavior in the vicinity of the quantum critical point. It is known that the model exhibits a second order quantum phase transition with susceptibility $\chi = \phi^{-1}$ and correlation length $\xi = \phi^{-1/\sigma}$ [16]. For this model the dynamic critical exponent is $z = \frac{\sigma}{2}$.

3.2 Finite temperature effects on the quantum critical behavior

In the bulk limit corresponding to the geometry $\infty^d \times L_\tau^z$, Eq. (27) takes the scaling form, $\frac{\sigma}{2} < d < \frac{3\sigma}{2}$,

$$x_\tau = u_0 k_d \frac{y_\tau^{\frac{d}{z}-1}}{2\sigma\sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) \left[\Gamma\left(\frac{1}{2} - \frac{d}{\sigma}\right) + 4 \sum_{m=1}^{\infty} \frac{K_{\frac{d}{\sigma}-\frac{1}{2}}(m y_\tau)}{\left(\frac{1}{2} m y_\tau\right)^{\frac{d}{\sigma}-\frac{1}{2}}}\right]. \tag{28}$$

Here $y_\tau = \sqrt{\phi}/T$ and $x_\tau = T^{1-d/z}(r_{0c} - r_0)$. At $x_\tau = 0$, Eq (28) can be solved numerically [16,25]. It shows that y_0 depends upon the ratio $\frac{d}{\sigma}$ in a universal way for all values of $\sigma < 2$. For $d = \sigma$, Eq. (28) turns into

$$\xi^{-z} = 2T \operatorname{arcsinh}\left[\frac{1}{2}\exp\left(-\frac{\sigma}{2}\frac{x_\tau}{k_\sigma}\right)\right] \tag{29}$$

for the inverse correlation length. This result shows, among others, that thermal fluctuations destroy “quantum” long-order at any finite temperature.

3.3 Finite-size scaling at zero temperature:

At zero temperature, Eq. (27) becomes

$$\phi = r_0 + \frac{u_0}{2V} \sum_{\boldsymbol{q}} \frac{1}{\sqrt{\phi + |\boldsymbol{q}|^\sigma}}. \tag{30}$$

To investigate the finite-size scaling in the quantum limit one needs to solve the problem related to the term $|\boldsymbol{q}|^\sigma$ in the spectrum of the model. In other words the problem is how to linearize the summand in Eq. (30). The solution involving the particular case $E^{\frac{1}{2}}_{\alpha,\beta}(z)$ of the generalized Mittag-Leffler function (3) was proposed in Ref. [15]. Therefore, in the neighborhood of the quantum critical point, it is possible to write down Eq. (30) in the scaling form $\left(\frac{d}{\sigma} < d < \frac{3d}{\sigma}\right)$ [15, 16]

$$x_L = \frac{u_0}{2} D^{\frac{1}{2}}_{d,\sigma} y_L^{\frac{d}{\sigma}-\frac{1}{2}} + \frac{u_0}{2}\left[D^{\frac{1}{2}}_{d,\sigma} y^{\frac{d}{\sigma}-\frac{1}{2}} + F^{\frac{1}{2}}_{d,\sigma}(y)\right], \tag{31}$$

whose solution provides the behavior of the correlation length. Here the scaling variables are given by $x_L = (r_{0c} - r_0)L^{d-\frac{\sigma}{2}}$ and $y_L = L^\sigma \phi$,

Equation (31) shows that the correlation length is a universal function of the scaling variable x_L i.e. $\xi = L f_\xi(x_L)$. At the quantum critical point i.e. $x_L = 0$ the critical amplitude $f_\xi(0)$ depends upon the geometry of the system and the range of the interaction.

3.4 Interplay between quantum and finite-size effects

We consider a system confined to the finite geometry of the general form $L^d \times L_\tau^z$. This geometry accounts for the finite-size, as well as the finite temperature corrections to the zero temperature bulk limit. These are given by [17]
(32a)

$$\Upsilon(\phi, L, T) = \frac{\sqrt{2}L^{-d}}{(2\pi)^{\frac{d+1}{2}}} \sum_m \sum_{\boldsymbol{l}}{}' \int_0^\infty \frac{dz}{\sqrt{z}} \exp\left(-z\phi - \frac{T^2 m^2}{4z}\right) |\boldsymbol{l}|^{-d} \Phi_{\frac{d}{2}-1,\sigma}\left(\frac{z}{L^\sigma |\boldsymbol{l}|^\sigma}\right),$$

where

(32b) $$\Phi_{\nu,\sigma}(y) = \int_0^\infty dx x^{\nu+1} J_\nu(x) e^{-yx^\sigma},$$

with $J_\nu(x)$ the Bessel function.

The general equation obtained by combining (28), (31) and (32) may be expressed in a scaling form whose solution provides the behavior of the correlation length against a pair of scaling variables as

(33) $$\xi = L f_\xi\left(x_L, LT^{1/z}\right) = T^{-1/z} f_\tau\left(x_\tau, LT^{1/z}\right),$$

thus obeying the quantum finite-size scaling hypothesis (26). The explicit solution to the equation accounting for finite-size and finite-temperature effects is hard to be derived. Thus we will consider two limiting cases:

Low-temperature regime $LT^{1/z} \gg 1$: In this regime the finite temperature corrections dominate those originating from the confinement of the system. Let us mention that the case of short-range interaction ($\sigma = 2$) has been analyzed in detail [25] and it was found that the corrections fall off exponentially. Here for the case of long-range interactions we get *power-law corrections*. This seems to be a general characteristic for systems with long range interaction (for the case of the spherical model see Refs. [8, 11]).

Very low-temperature regime $LT^{1/z} \ll 1$: Here the finite temperature corrections fall-off exponentially. They are negligible in comparison to those coming from the finite-size effects.

4 Discussion

We have investigated the finite-size scaling properties in the $O(n)$-symmetric φ^4 model with long-range interaction potential decaying algebraically with the interparticle distance. To overcome the ensuing mathematical difficulties, we considered a method based upon Mittag-leffler-like functions to reduce the pertinent d-dimensional sums to the effective one-dimensional problem. These techniques allow the investigation to be simplified and express the results for various thermodynamic functions in terms of simple and known mathematical functions.

We have demonstrated that any thermodynamic quantity, such as the susceptibility, can be written in the scaling form (5). Notice that two scaling variables are needed to characterize in a proper way the finite-size scaling behavior of these quantities. We constructed an effective Hamiltonian, from the initial one, with new coupling constants R and U. These constants obey the scaling hypothesis (9). We found that the even moments of the field φ, related to the thermodynamics of the finite system, are scaling functions of the characteristic variable $\mathfrak{z} = RU^{-1/2}L^{2-\eta-\varepsilon/2}$.

We presented investigations of the finite-size scaling of the $O(\infty)$ φ^4-model with long-range interaction in the vicinity of its quantum critical point. We considered the model confined to the general geometry of the form $L^d \times L_\tau^z$, where L is the spatial size of the system, $L_\tau \sim T^{-1/z}$ and z is the dynamic critical exponent. The results are obtained by considering the temperature, which governs the crossover between the classical and the quantum critical behaviors as an additional temporal dimension.

The study of the general case when the system is confined to the general geometry $L^d \times L_\tau^z$, that is when the temperature as well as the sizes of the system are finite turns out to be a very difficult task because of the high anisotropy of the system due to the parameter σ. Nevertheless one can make interesting deductions in some limiting cases: (**i**) in the low-temperature regime ($LT^{1/z} \gg 1$), the finite-size corrections to the low-temperature behavior are *exponentially* small in the case of short-range interaction and are decreasing with a *power law* in the case of long-range interaction, (**ii**) in the very low-temperature regime ($LT^{1/z} \ll 1$), however, the finite temperature correction to the finite-size behavior are always *exponentially* small.

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