

## NUMERICAL PROCEDURE FOR IDENTIFICATION OF CONSTITUTIVE EQUATIONS BASED ON EXPERIMENTAL DATA\*

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**ABSTRACT.** In this research experimental data obtained by well-known methods of displacement measurement are considered, whereas a thin cell meshwork has been fixed to the surface of the observed loaded object (thin metal sheet, for instance) and a set of point displacements has been obtained. Based on such experimental evidence, an eight point numerical procedure is proposed in the paper to identify the constitutive relations for small time increments. The presented approach is also applicable in case of micro deformations and facilitates the derivation of nonlinear constitutive equations in stress/strain increments. Approximate error estimation of the procedure is additionally performed, and a test example is given.

**KEY WORDS:** Experimental data, numerical procedure, constitutive equations.

### 1. Statement of the problem

The majority of experimental methods in Solid Mechanics provide information about the displacement of some set of body points during a deformation process [1–4]. The main problem of these methods is how to develop a procedure of derivation constitutive relations between stress and strain. This problem becomes even more important when the constitutive equations are time dependent and nonlinear. In the present research, a numerical procedure is proposed to fulfil that task on the basis of some preliminary investigations [5].

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In order to simplify the problem, thin metal sheets loaded in their middle surfaces are taken into account. The model operates in plane stress. The basic assumptions are: (i) quasistatic deformation; (ii) adoption of continuum mechanics model; (iii) account for small displacements and corresponding stresses; (iv) negligible volumetric and inertia forces; (v) availability of sheet point displacements recorded during loading and control of sheet thickness variation; (vi) known loading information.

The sheet surface is meshed into small cells that vary in shape during deformation. The whole process is divided into small identical time intervals  $\Delta t$ . The process begins at time  $t_0$  and ends at time  $t_f = t_0 + m\Delta t$ . For each set of points  $M(p, t)$ ,  $p = 1, 2 \dots n$  and for each moment  $t = t_0 + k\Delta t$ ,  $k = 0, 1 \dots m$  the displacement vector  $\underline{u}(p, t)$  is measured, Fig. 1.

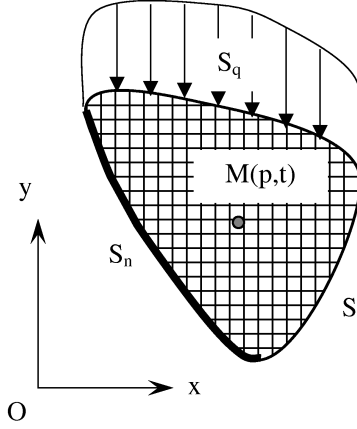


Fig. 1. Boundary conditions of sheet surface

The coordinate system  $(Oxy)$  is located in the middle surface of the sheet. The middle sheet boundary is defined as  $S = S_n \cup S_q \cup S_r$ , Fig. 1. At each moment  $t \in [t_0, t_f]$ , the corresponding boundary conditions  $S_n, S_q, S_r$  read: for points  $[x(s), y(s)] \in S_n$ , the displacement vector is  $\underline{u}(s, t) \equiv 0$ ; for points  $[x(q), y(q)] \in S_q$ , the stress vector is  $\underline{p} = \underline{q}(q, t)$ , where  $\underline{q}(q, t)$  are known functions; for points  $[x(r), y(r)] \in S_r$ , the stress vectors is  $\underline{p}(q, t) \equiv 0$ .

The main idea is to find, at each point  $M(p, t)$  and during every time interval  $\Delta t$ , the stress incremental tensor  $\underline{\Delta\sigma}(p, t)$ , the strain incremental tensor  $\underline{\Delta\varepsilon}(p, t)$  and their link  $\underline{\Delta\varepsilon}(p, t) = \underline{H}(\underline{\sigma}(p, t), t) \underline{\Delta\sigma}(p, t)$ , where  $\underline{H}$  is a nonlinear tensor function varying in time. The calculation procedure proposed is given in the paragraph that follows.

## 2. Eight points numerical procedure to calculate strain increments

Consider the inner point  $M(p, t)$ ,  $p = 1, 2 \dots n$  and eight of its neighbouring points  $N(j, t)$ ,  $j = 1, 2 \dots 8$  at a fixed moment  $t \in [t_0, t_f]$ , Fig. 2.

Since displacements  $[u_x(p, t), u_y(p, t)]$  and  $[u_x(j, t), u_y(j, t)]$  are known from experiments, the distances between point  $M(p, t)$  and  $N(j, t)$  can be calculated:

$$(1) \quad C(p, j, t) = \sqrt{[x(j, t) - x(p, t)]^2 + [y(j, t) - y(p, t)]^2}.$$

The corresponding strain increment for  $\Delta t = t_2 - t_1$  in the direction  $\underline{n}(p, j, t)$  according to assumption (iii) is as follows:

$$(2) \quad \Delta \varepsilon_n(p, j, t_2) = \frac{C(p, j, t_2) - C(p, j, t_1)}{C(p, j, t_1)}, \quad j = 1, 2 \dots 8,$$

and

$$(3) \quad \underline{n}(p, j, t_2) \equiv \left\{ \begin{array}{l} n_x(p, j, t_2) = \frac{x(j, t_2) - x(p, t_2)}{C(p, j, t_2)} \\ n_y(p, j, t_2) = \frac{y(j, t_2) - y(p, t_2)}{C(p, j, t_2)} \end{array} \right\}.$$

Eq. (2) expresses approximate substitution of the infinitesimal distance for a finite and small distance.

A relation between  $\Delta \varepsilon_n(p, j, t_2)$  and the components of the incremental strain tensor  $\underline{\Delta \varepsilon}(p, t_2)$  is introduced

$$(4) \quad \Delta \varepsilon_n(p, j, t_2) = \Delta \varepsilon_{xx}(p, t_2) [n_x(p, j, t_2)]^2 + \Delta \varepsilon_{yy}(p, t_2) [n_y(p, j, t_2)]^2 + 2\Delta \varepsilon_{xy}(p, t_2) [n_x(p, j, t_2) n_y(p, j, t_2)].$$

Eq. (4) sets forth eight equations for three unknown values  $\underline{\Delta \varepsilon}(p, t_2)$ .

Hence, a functional  $J$  is introduced as follows

$$(5) \quad J = \sum_{j=1}^8 \left\{ \begin{array}{l} \Delta \varepsilon_{xx}(p, t_2) [n_x(p, j, t_2)]^2 + \Delta \varepsilon_{yy}(p, t_2) [n_y(p, j, t_2)]^2 + \\ 2\Delta \varepsilon_{xy}(p, t_2) [n_x(p, j, t_2) n_y(p, j, t_2)] - \Delta \varepsilon_n(p, j, t_2) \end{array} \right\}^2.$$

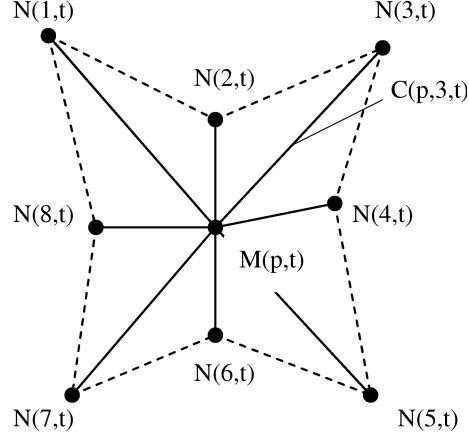


Fig. 2. Boundary conditions of sheet surface

Applying the method of minimum quadratic deviation, the corresponding system of equations reads:

$$\begin{aligned}
 \frac{\partial J}{\partial (\Delta \varepsilon_{xx})} &= 2 \sum_{j=1}^8 \left\{ \begin{array}{l} \Delta \varepsilon_{xx}(p, t_2) [n_x(p, j, t_2)]^2 + \\ \Delta \varepsilon_{yy}(p, t_2) [n_y(p, j, t_2)]^2 + \\ 2\Delta \varepsilon_{xy}(p, t_2) [n_x(p, j, t_2) n_y(p, j, t_2)] - \\ \Delta \varepsilon_n(p, j, t_2) \end{array} \right\} \times \\
 &\quad \times [n_x(p, j, t_2)]^2 = 0 \\
 \frac{\partial J}{\partial (\Delta \varepsilon_{yy})} &= 2 \sum_{j=1}^8 \{ \dots \} \cdot [n_y(p, j, t_2)]^2 = 0 \\
 \frac{\partial J}{\partial (\Delta \varepsilon_{xy})} &= 4 \sum_{j=1}^8 \{ \dots \} \cdot [n_x(p, j, t_2) n_y(p, j, t_2)] = 0
 \end{aligned}
 \tag{6}$$

Eqs (6) form a system of three equations for three values of  $\underline{\Delta \varepsilon}(p, t_2)$ . Solving this system, the strain incremental tensor in the middle surface of the sheet is obtained.

Since thin, sheet is assumed, the other strain components are  $\varepsilon_{xz} = 0$ ,  $\varepsilon_{yz} = 0$  and:

$$\Delta \varepsilon_{zz}(p, t_2) = \frac{h(p, t_2) - h(p, t_1)}{h(p, t_1)},
 \tag{7}$$

where  $h(p, t)$  is sheet thickness at point  $M(p, t)$  and moment  $t \in [t_0, t_f]$ . Note

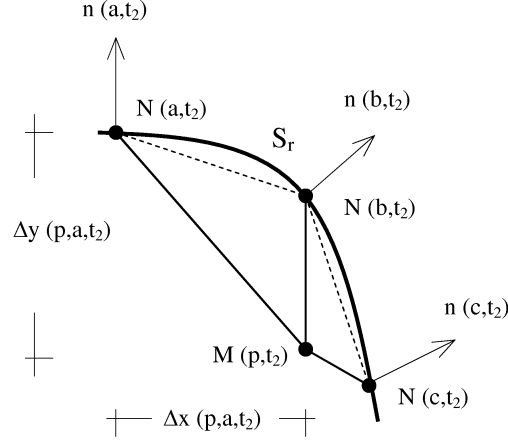


Fig. 3. Scheme of the location of the three points on  $S_r$

that sheet thickness is previously found from experiments.

### 3. Three point numerical procedure to calculate stresses

Next part of the work is to find the components of the stress incremental tensor. Since the model assumed operates in plane stress,  $\Delta\sigma_{xz} \equiv 0$ ,  $\Delta\sigma_{yz} \equiv 0$  and  $\Delta\sigma_{zz} \equiv 0$  for each point is valid  $M(p, t)$  and  $t \in [t_0, t_f]$ .

The procedure starts with a choice of three points belonging to the surface  $S_r$ :  $N(v, t_2)$ ,  $(v = a, b, c)$ , Fig. 3. These are the three nearest points to point  $M(p, t_2)$ .

The boundary curve is transformed into a polygon linking points  $N(v, t_2)$ ,  $(v = a, b, c)$ . On the basis of that polygon, the components of the unit normal vectors are found:  $\underline{n}(v, t_2) \equiv \begin{Bmatrix} n_x(v, t_2) \\ n_y(v, t_2) \end{Bmatrix}$ .

The boundary conditions on  $S_r$  read:

$$(8) \quad \begin{aligned} \Delta P_x(v, t_2) &= \Delta\sigma_{xx}(v, t_2) n_x(v, t_2) + \Delta\sigma_{xy} n_y(v, t_2) = 0 \\ \Delta P_y(v, t_2) &= \Delta\sigma_{yx}(v, t_2) n_x(v, t_2) + \Delta\sigma_{yy} n_y(v, t_2) = 0 \end{aligned}$$

Eq. (8) presents a system of six equations for nine unknown components  $\Delta\sigma_{xx}(v, t_2)$ ,  $\Delta\sigma_{yy}(v, t_2)$  and  $\Delta\sigma_{xy}(v, t_2)$ , where load components  $\underline{\Delta p}(v, t_2)$  applied to  $S_r$  are known, since  $S_r$  is free of loading.

The approximate equations of equilibrium (without volume forces) for point  $M(p, t_2)$  are added, substituting the derivatives for finite differences:

$$(9) \quad \frac{\Delta\sigma_{xx}(v, t_2) - \Delta\sigma_{xx}(p, t_2)}{\Delta x(v, p, t_2)} + \frac{\Delta\sigma_{xy}(v, t_2) - \Delta\sigma_{xy}(p, t_2)}{\Delta y(v, p, t_2)} = 0$$

$$\frac{\Delta\sigma_{yx}(v, t_2) - \Delta\sigma_{yx}(p, t_2)}{\Delta x(v, p, t_2)} + \frac{\Delta\sigma_{yy}(v, t_2) - \Delta\sigma_{yy}(p, t_2)}{\Delta y(v, p, t_2)} = 0$$

where

$$(10) \quad \Delta x(v, p, t_2) = x(v, t_2) - x(p, t_2)$$

$$\Delta y(v, p, t_2) = y(v, t_2) - y(p, t_2)$$

Eq. (9) presents a system of six equations for twelve functions  $\Delta\sigma_{xx}(v, t_2)$ ,  $\Delta\sigma_{yy}(v, t_2)$ ,  $\Delta\sigma_{xy}(v, t_2)$ ,  $\Delta\sigma_{xx}(p, t_2)$ ,  $\Delta\sigma_{yy}(p, t_2)$  and  $\Delta\sigma_{xy}(p, t_2)$ .

Both systems (8) and (9) form one system with twelve equations for twelve unknown functions. Solving this system, the components of the incremental stress tensor are found at the point  $M(p, t)$ .

#### 4. Approximate incremental constitutive equations

It is assumed that no changes of the constitutive functions take place within the interval  $\Delta t$ . If the material is isotropic, the constitutive functions in the sheet middle read:

$$(11) \quad \Delta\varepsilon_{xx}(p, t_2) = \frac{1}{E^s(p, t_2)}\Delta\sigma_{xx}(p, t_2) - \frac{v^s(p, t_2)}{E^s(p, t_2)}\Delta\sigma_{yy}(p, t_2)$$

$$\Delta\varepsilon_{yy}(p, t_2) = -\frac{v^s(p, t_2)}{E^s(p, t_2)}\Delta\sigma_{xx}(p, t_2) + \frac{1}{E^s(p, t_2)}\Delta\sigma_{yy}(p, t_2)$$

$$\Delta\varepsilon_{xy}(p, t_2) = \frac{1}{G^s(p, t_2)}\Delta\sigma_{xy}(p, t_2)$$

where  $E^s(p, t_2)$ ,  $v^s(p, t_2)$  and  $G^s(p, t_2)$  are the secant modules in the time interval  $\Delta t$ . It is possible to obtain their values from eqs. (11). In addition, the secant volume module  $k^s(p, t_2)$  can be found via the following equation:

$$(12) \quad \frac{1}{3}[\Delta\sigma_{xx}(p, t_2) + \Delta\sigma_{yy}(p, t_2)] = k^s(p, t_2) \Delta\theta(p, t_2),$$

where the volume change measure is as follows

$$(13) \quad \Delta\theta(p, t_2) = \Delta\varepsilon_{xx}(p, t_2) + \Delta\varepsilon_{yy}(p, t_2) + \Delta\varepsilon_{zz}(p, t_2).$$

Thus, knowing the secant modulus for all time intervals  $\Delta t$  within the time period  $(t_0, t_f)$ , we obtain the completed constitutive information.

### 5. Test example

The presented numerical procedure is applied considering a cantilever wall-bar sheet element of elastic isotropic material. The geometrical distances are non-dimensional. Uniform shear load  $q$  is applied to the end of the wall-bar. The material possesses elastic modulus  $E = 210$  GPa and Poisson coefficient  $\nu = 0.3$ . Process duration is  $t_f = 10$ sec and the starting time is at  $t_0 = 0$ . The time interval is  $\Delta t = 1$  sec. FE calculation is performed to find stress and strain increments. It is also found that  $E^s = E$  and  $\nu^s = \nu$ , and satisfactory agreement between the known and calculated characteristics is established.

### 6. Conclusions

The proposed numerical procedure increases the capability of experimental methods used to identify the constitutive relations of time dependent materials with nonlinear behaviour. The accuracy of the numerical calculations depends essentially on the mesh dimensions and the time interval  $\Delta t$ . Note, however, the experimental technique imposes limitations on the mesh dimensions, and the paper proposes an initial idea as how to solve the problem formulated.

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