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PECULIARITIES OF THE OPTIMAL DESIGN OF GIRDER STRUCTURES SUBJECTED TO BENDING LOADING

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1. Introduction

Analysis of problems, related to the optimal design of girder structures with arbitrary cross section is made in [1, 2]. The advantages of the variational approach are shown in regard to simplicity of the optimization procedures and possibility to reach the global optimum.

The peculiarities of this problem will be shown in the paper with different criterial functions and constraints. The conditions for the so called dual optimization problem will be discussed.

2. Optimization of rigidity (flexibility) at a given volume of the girder

The equations for describing the optimization problem are derived from the next expanded form of the total bending girder energy

$$
\begin{aligned}
(1)\quad \Pi^* \;=\; & \int_0^L [\frac{1}{2}EI(x)w''^2(x) - q(x)w(x)]dx - \frac{1}{2}\beta p\left(\int_0^L F(x)dx - V_0\right) - \\
& -\frac{1}{2}p\int_0^L \kappa(x)[g^2(x) - F(x) + \overline{F}(x)]dx + \frac{1}{2}\left(\frac{w_0'^2}{a_0} + \frac{w_L'^2}{a_L} + \frac{w_0^2}{A_0} + \frac{w_L^2}{A_L}\right),
\end{aligned}
$$

where

$$
(2)\qquad V_0 = \int_0^L F(x)dx
$$

is the girder volume, $I(x) = c_I F^p(x)$ is the moment of inertia of sectional area $F(x), \overline{F}(x)$ is geometric constraint; c_I, p are coefficients, depending on cross section form; $q(x)$ is lateral loading intensity; $g(x)$ is auxiliary function; $w(x)$ is deflection; a_0, a_L, A_0, A_L are flexibility coefficients at boundary sections; $\beta, \kappa(x)$ are Lagrange coefficients.

From the stationarity of Π^* with respect to $F(x)$ and $g(x)$, the following system of equations for optimal girder design can be written:

$$Ec_I F^{p-1} - \beta + \kappa(x) = 0, \tag{3}$$

$$g(x)\kappa(x) = 0. \tag{4}$$

Using symbols S_1 and S_2, the intervals along the girder length $0 \leq x \leq L$ in which $F(x) > \overline{F}(x)$ (then $g(x) \neq 0, \kappa(x) = 0$) and $F(x) \leq \overline{F}(x)$ (then $g(x) = 0$, $\kappa(x) \neq 0$) are respectively indicated. When joining (3) and (4), having in mind, that

$$M(x) = Ec_I F^p(x) w''(x) \tag{5}$$

the next formula can be obtained for the optimal cross section area

$$F(x) = \begin{cases} \left[\frac{M^2(x)}{\beta E c_I}\right]^{\frac{1}{p-1}} & x \in S_1, \\ \overline{F} & x \in S_2. \end{cases} \tag{6}$$

To define β, (6) is substituted in (2). The integral girder flexibility is equal to the double potential energy

$$R = \int_{S_1} \frac{M^2(x)dx}{Ec_I F^p} + \int_{S_2} \frac{M^2(x)dx}{E\overline{I}(x)} + \frac{w_0'^2}{a_0} + \frac{w_L'^2}{a_L} + \frac{w_0^2}{A_0} + \frac{w_L^2}{A_L}, \tag{7}$$

where $\overline{I}(x)$ is the moment of inertia of the constrained cross section area.

The decrease of flexibility after optimization is

$$\delta_R = \frac{R}{R_0}, \tag{8}$$

where the initial girder flexibility under natural boundary conditions for constant cross section is $\overline{F} = \alpha\frac{V_0}{L}, \alpha = 1$

$$R_0 = \frac{p^2 L^{p+3}}{3Ec_I V_0^p}. \tag{9}$$

Example: Cantilever with concentrated force on the free end with constant geometrical constraint $\overline{F} = \alpha\frac{V_0}{L}$ in the interval S_2, subjected to determination (Fig. 1).

The bending moment is

$$M(x) = P(l - x). \tag{10}$$

The next formula for optimal cross section area can be obtained

$$F(x) = \frac{(p+3)[V_0 - \overline{F}(L - S_1)](L - x)^{\frac{2}{p+1}}}{(p+1)[L^{\frac{p+3}{p+1}} - (L - S_1)^{\frac{p+3}{p+1}}]}, \quad x \in S_1 \tag{11}$$

Table 1

α	0,00	0,25	0,50	0,75	1,00
$\frac{S_1}{L}$	1,000	0,872	0,732	0,548	0,000

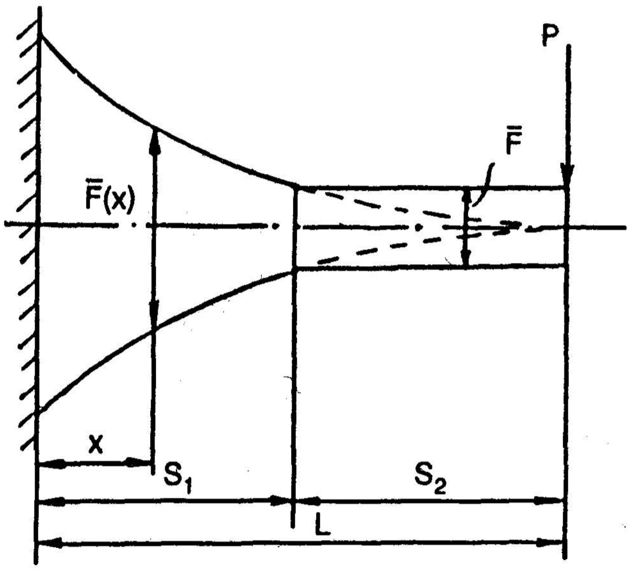

Fig. 1

The distance S_1, defined by means of (11) for different values of α, is given in Table 1.

The decreasing coefficient of flexibility for different α and p is given in Fig. 2.

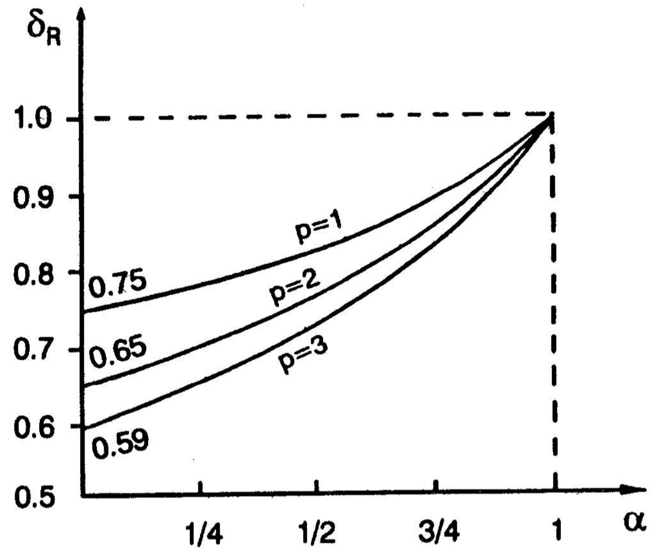

Fig. 2

Table 2

P	1	2	3
δ_ν	0,75	0,80	0,84

3. Optimization of the volume at given girder rigidity (flexibility)

As a result of variational operations related to the expanded form of the girder volume functional

$$
\begin{aligned}
V^* &= \int_0^L F(x)dx - \varepsilon\Bigg\{\int_0^L [\frac{1}{2}Ec_I F^p(x) w''^2(x) - q(x)w(x)]dx + \\
&+\frac{1}{2}\left(\frac{w_0'^2}{a_0} + \frac{w_L'^2}{a_L} + \frac{w_0^2}{A_0} + \frac{w_L^2}{A_L} + \frac{R_0}{2}\right)\Bigg\} - \\
&-\frac{1}{2}p\int_0^L \kappa(x)[g^2(x) - F(x) + \overline{F}(x)]dx
\end{aligned}
\tag{12}
$$

the next formula for the cross section area is obtained

$$
F(x) = \begin{cases} \left[\frac{p\varepsilon M^2(x)}{2Ec_I}\right]^{\frac{1}{p+1}} & x \in S_1, \\ \overline{F} & x \in S_2. \end{cases}
\tag{13}
$$

The Lagrange coefficient ε is defined by substitution of formula (15) in (7). The decrease of the volume after optimization is

$$
\delta_\nu = \frac{V}{V_0},
\tag{14}
$$

where $V_0 = F_C L$ is initial girder volume with constant cross section F_C and flexibility R_0. The value is defined by (7) with $R = R_0$ and $S_2 = 0$

$$F_C = \left(\frac{\int_0^L M^2(x)dx}{Ec_I R_0}\right)^{\frac{1}{p}}. \tag{15}$$

Example: For the same example, considered in §2, we can obtain the next formula for the optimal cross section area:

$$F(x) = \frac{(p+1)^{\frac{1}{p}} P^{\frac{2}{p}} \left[L^{\frac{p+3}{p+1}} - (L-S_1)^{\frac{p+3}{p+1}}\right]^{\frac{1}{p}} (l-x)^{\frac{2}{p+1}}}{(p+3)^{\frac{1}{p}} \left[R_0 - \frac{P^2(L-S_1)}{3EI}\right]^{\frac{1}{p}} (Ec_I)^{\frac{1}{p}}}. \tag{16}$$

The values of the decreasing coefficient of the girder volume as a result of optimization at $\alpha = 0$ are given in Table 2.

4. Dual optimization problem

The optimization scheme with respect to rigidity with a given volume of the girder V_0, developed in §2, can be used for determination of the optimal volume with a given flexibility R_0. A formula for this volume can be defined from the equality of the relative flexibilities of the optimal and basic structure.

$$V = \left(\frac{R}{R_0}\right)^{\frac{1}{p}} V_0 = \delta_R^{\frac{1}{p}} V_0. \tag{17}$$

For determination of cross section areas F_V, corresponding to the minimal girder volume, the cross section areas of girder with minimal rigidity F_R are multiplied with a coefficient $\delta_R^{\frac{1}{p}}$. It is not difficult to show for the example considered, that:

$$F_V = \delta_V F_R, \tag{18}$$

where

$$\delta_V = (\delta R)^{\frac{1}{p}}, \tag{19}$$

which is a proof for the dual nature of the optimization problem.

Dual optimization approach is applicable for problems without constraints. For problems with constraints, errors will arise, as well as the initially defined intervals S_1 and S_2 will be broken.

5. Comparison between variational approach and fully strained state

Let us consider a cantilever with equal section modules with permissible stress σ_0 and rectangular cross section $b \times h$ [3]. Next formula for the cross section area can be derived.

$$F(x) = bh = \sqrt{\frac{6bPl}{\sigma_0}}\sqrt{\frac{x}{L}}. \tag{20}$$

It is not difficult to prove, that the distribution of the girder volume in accordance with formula (20) leads to the global optimum. Last conclusion can be spread to all statically determined girders. It is shown in [2], that for the continuous girders, a fully strained state doesn't correspond to the global optimum.

6. Conclusion

– Expressions are derived for the optimal designing of girders in regard to the rigidity and the volume of the girder with geometric constraints.

– Dual optimization decision exists in unconstrained problems.

– The configuration of the girder at fully strained state coincides with this designed by variational approach only for statically determined girders.

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