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RELATION BETWEEN 3D AND 2D ELASTIC PROBLEMS THROUGH THE FINITE PRISM METHOD

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In this paper are presented the relations between the Finite Prism Method (FPM) and the well known 2D elastic problems (plane stress, plane strain and anti plane problem).

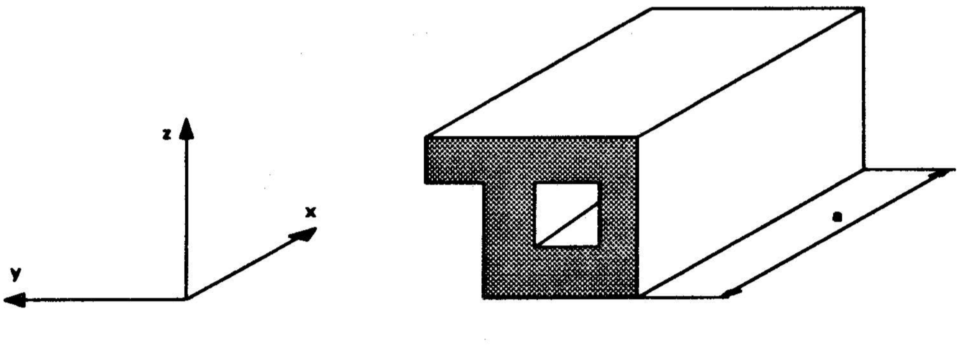

Fig. 1. Prismatic body.

Finite prism method

The FPM [1], [2] is a method for solving 3D elastic problems for prismatic bodies (Fig. 1) using separaton of variables. The displacements are presented as follows:

$$\left\{\begin{array}{c} u(y,z) \\ v(y,z) \\ w(y,z) \end{array}\right\} = \sum_{i=1}^{\infty} \frac{1}{N_i^2} \left[\begin{array}{ccc} \psi_i^*(x) & 0 & 0 \\ 0 & \psi_i(x) & 0 \\ 0 & 0 & \psi_i(x) \end{array}\right] \left\{\begin{array}{c} \tilde{u}(y,z) \\ \tilde{v}(y,z) \\ \tilde{w}(y,z) \end{array}\right\}, \tag{1}$$

$u(y,z), v(y,z), w(y,z)$ are the displacement functions in the cross section that are calculated using the Finite Element Method (FEM), N is normalize factor which is equal to the scalar product $\langle \psi_i / \psi_i \rangle$, and $\psi_i(x)$, $\psi_i^*(x)$, are trigonometric functions, $\psi_i^*(x) = |\psi_i/\mu|$ for which there exist 4 possibilities [2] (Fig. 2):

$$\begin{array}{llll} 1) & \psi_i^* = \cos\mu_i x, & \psi_i = \sin\mu_i x, & \mu_i = \frac{i\pi}{a} \\ 2) & \psi_i^* = \sin\mu_i x, & \psi_i = \cos\mu_i x, & \mu_i = \frac{(2i-1)\pi}{2a} \\ 3) & \psi_i^* = \sin\mu_i x, & \psi_i = \cos\mu_i x, & \mu_i = \frac{i\pi}{a} \\ 4) & \psi_i^* = \cos\mu_i x, & \psi_i = \sin\mu_i x, & \mu_i = \frac{(2i-1)\pi}{2a} \end{array} \tag{2}$$

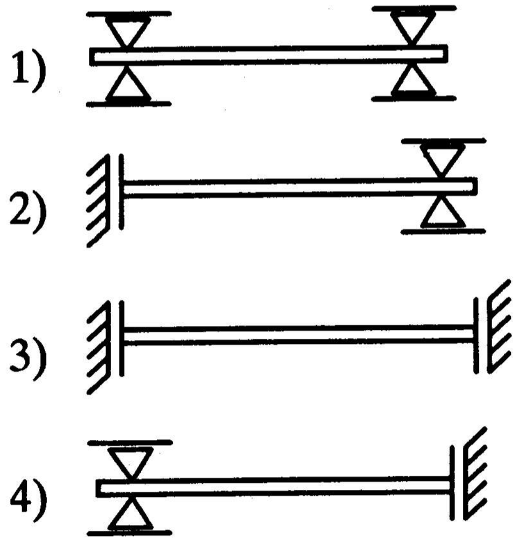

Fig. 2. Variants for ψ_i^*, ψ_i and μ_i.

In eq. (1) there is no zero term. By analogy with the solution for axial symmetric problems with non symmetrical loading [3], where also a semi analytical solution is used, it is clear that there is a zero term and this term performs the solution of the axial symmetric problem with axial symmetric load, which is a 2D problem. For the FPM it has to be considered the zero term for the different cases (2) and to be obtained the relation to the different 2D problems. Variants 2) and 4) have really no zero term, because ψ_i and ψ_i^* cannot be equal to zero. For Variants 1) and 3) in case of $i = 0$: $\mu_0 x = 0 \Longrightarrow \sin \mu_0 x = 0, \cos \mu_0 x = 1$.

In view of the fact that the solution by the FPM is obtained for the displacements, the continuity conditions are satisfied automatically. From the familiar basic equations for 3D elastic problems and from (1) it is obtained the following relation between the displacements and the strains:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \sum_{i=1}^{\infty} \frac{1}{N_i^2} \begin{bmatrix} \psi_i \mu_i \tilde{u}_i \\ \psi_i \frac{\partial \tilde{v}_i}{\partial y} \\ \psi_i \frac{\partial \tilde{w}_i}{\partial z} \\ \psi_i^* \left(\frac{\partial \tilde{u}_i}{\partial y} + \mu_i \tilde{v}_i \right) \\ \psi_i \left(\frac{\partial \tilde{v}_i}{\partial z} + \frac{\partial \tilde{w}_i}{\partial y} \right) \\ \psi_i^* \left(\mu_i \tilde{w}_i + \frac{\partial \tilde{u}_i}{\partial y} \right) \end{bmatrix} \tag{3}$$

For variant 1: $\psi_i = 0$, $\psi_i^* = 1$, and out of (1), (3) and using Hook's law we arrive at:

$$\begin{aligned} v = w = 0, \quad & \epsilon_x = \epsilon_y = \epsilon_z = \gamma_{yz} = 0, \\ & \sigma_x = \sigma_y = \sigma_z = \tau_{yz} = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} u(y,z) \neq 0, \quad & \gamma_{xy} \neq 0 \quad \gamma_{zx} \neq 0 \\ & \tau_{xy} \neq 0 \quad \tau_{zx} \neq 0. \end{aligned} \tag{4b}$$

This is exactly the formulation of the anti plane problem [4]. Typical applications of the anti plane problem are the solution for cracks of the third kind (the margines of the crack are moving parallel to each other and perpendicular to the plane of the crack), free torsion and sliding of cross sections of beams of arbitrary form.

For variant 3: $\psi_i = 1$, $\psi_i^* = 0$, and from (1), (3) and using Hook's law we arrive at:

$$\begin{aligned} u = 0, \quad & \epsilon_x = \gamma_{xy} = \gamma_{zx} = 0, \\ & \tau_{xy} = \tau_{zx} = 0, \end{aligned} \tag{5a}$$

$$\begin{aligned} v(y,z) \neq 0, \quad w(y,z) \neq 0, \quad & \epsilon_y(y,z) \neq 0, \quad \epsilon_z(y,z) \neq 0, \quad \gamma_{yz}(y,z) \neq 0, \\ & \sigma_y(y,z) \neq 0, \quad \sigma_z(y,z) \neq 0, \quad \tau_{yz}(y,z) \neq 0, \end{aligned} \tag{5b}$$

$$\sigma_x(y,z) = \nu(\sigma_y + \sigma_z). \tag{5c}$$

This is the formulation of the plane stress problem [4] (ν is Poisson's ratio). The reduction to the plain stress problem can be obtained by manipulation of the material properties, e.g. for the isotropic case:

$$\nu_1 = \frac{\nu}{\nu - 1}, \qquad E_1 = \frac{E(\nu-1)^2}{1-2\nu}, \tag{6}$$

E_1 and ν_1 are the material properties of the plane strain and E and ν are the material properties for the plane stress problem.

This reduction can be presented schematically (Fig. 3).

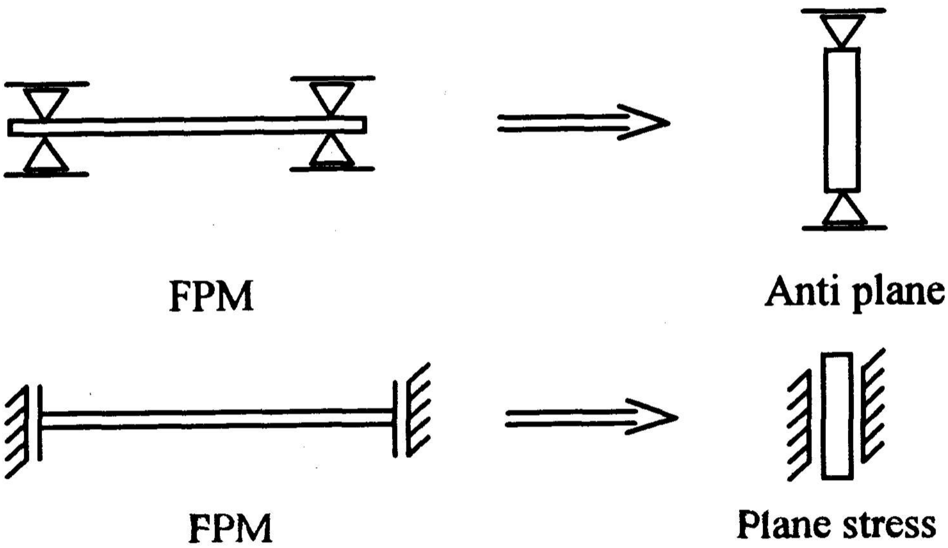

Fig. 3. Reduction from FPM to 2D problems.

By the solution in the plane $y-z$ with finite elements there are used quadrilateral 8-noded and triangular 6-noded isoparametrical elements. To obtain the solution for the zero term, the subroutines for the calculation of the element stiffness matrix and the element load, vectors are to be modified. There also have to be manipulated the zero displacements because the corresponding stiffness is also zero and the global stiffness matrix will be singular.

It is also interesting to note that by the solution with more terms of the infinite series (1) at the ends of the prismatic body by $x = 0$ and $x = a$ there are $\psi_i(0,a) = 0$, $\psi_i^*(0,a) = 1$ for variant 1 and $\psi_i(0,a) = 1$, $\psi_i^*(0,a) = 0$ for variant 3 respectively.

Very useful in this relation is variant 1, that approximately represents a beam on two supports and for $x = 0, a$ there can be obtained the solution of the problem for the distribution of the tangential stresses in the case of torsion and under a Q-force applied in different cross sections of beams [5].

Example 1: A square plate with a circular hole, tension in one direction

A plane problem for calculating the stress concentration using a theoretical solution for an endless plate $\alpha = \sigma_{max}/\sigma_{tension} = 3$. For radius of the hole 5 mm and square size 100 $\times$ 100 mm, mesh 4 $\times$ 4 elements which decrease in a geometrical progression with a ratio 0,3 towards the hole, there is obtained a stress concentration factor $\alpha = 3,172$.

Example 2: Torsion of a rectangular cross section

There is to be calculated the moment of inertia and the maximal tangential stress for $b = 10$ mm, $h = 20$ mm, $M_t = 50$ Nm and there is used a rectangular mesh 4 $\times$ 4 elements. The analytical solution gives $I_t = 10,458.10^4$ mm^4, the solution with the FPM gives $I_t = 10,4657.10^4$ mm^4 ; analytical $\tau_{max} = 101,6$ MPa, by the FPM $\tau_{max} = 104,3$ MPa.

These two simple but classical examples show that the errors are not big and it is possible to use the same programme for the solution of different 2D problems.

Conclusions

1. The relations between the Finite Prism Method (semianalytical solution of the 3D elastic problem at the specific conditions of the prismatic body) and 2D elastic problems – plane stress, plane strain and anti plane problem are presented.

2. The question about the zero term in the infinite series in case of variables separation is elucidated for the different cases (2).

3. Two simple examples show the variants to solve plane and anti plane problems using one programme for the FPM.

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