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THE DIMER MODEL AS A MODEL OF SELF-ORGANIZED CRITICALITY*

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Introduction

There are several famous graph-theoretical problems, which turned out to be closely interrelated, such that the number of solutions they admit can be exactly determined by the Kirchhoff matrix theorem for electrical circuits. The corresponding enumeration problems for planar lattices of arbitrarily large size constitute the mathematical basis of exactly solvable models in statistical mechanics with relevance to the physics of dense polymer systems, monolayers of diatomic molecules adsorbed on a crystal surface, nonlinear dynamical systems which exhibit self-organized criticality, etc.

1. The spanning trees problem. A spanning tree on a graph G is a connected subgraph of G which has no cycles and contains all the vertices of G. In 1847 Kirchhoff has shown that the enumeration of all the distinct spanning trees on G is related to the problem of determining the effective resistance between two nodes of a resistor network. Let the vertices of G be labelled by v_i, $i = 1, 2, ...N$. The Kirchhoff matrix of G is called the $N \times N$ matrix C with elements: $C_{ii} = \deg(v_i)$, $i = 1, 2, ...N$; $C_{ij} = -1$ if v_i and v_j are neighbours in G; $C_{ij} = 0$ otherwise. All the minors of C are equal and their common value gives the number N_T of spanning trees on G.

2. Hamiltonian walks and circuits. A Hamiltonian walk (circuit) is an open (closed) path that visits once and only once all the vertices of a given graph G by

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following its edges. The name comes from the "tour around the world" game invented in 1857 by Sir William Hamilton: find a closed path visiting once and only once each of the 20 towns placed for simplicity at the vertices of a dodecahedron which represents the Earth; one is allowed to travel only along the edges of the dodecahedron. Actually, long before Hamilton, in 1759 Euler analyzed the puzzle of the knight's tour on a chessboard which is equivalent to finding a Hamiltonian circuit on the appropriate graph which represents the chessboard and the allowed moves of a knight on it. The exact number $N_H(M)$ of Hamiltonian walks on a finite $m \times n$ Manhattan lattice M (a square lattice with oriented edges: the alternating orientation of the horizontal and vertical edges resembles the traffic regulation of Manhattan) has been obtained in 1963 by Kasteleyn [1] for periodic boundary conditions, and in 1988 by Duplantier and David [2] for free boundary conditions.

The Hamiltonian walks have been considered as the extreme case of dense polymer configurations. It has been found that dense polymer systems exist in a new critical phase whose properties are quite different from those of dilute polymers [2].

3. Eulerian walks and circuits. An Eulerian walk (circuit) is an open (closed) path visiting once and only once all the edges of a given graph. They are named after Leonhard Euler, who first gave the (negative) solution to the problem of crossing each of Konigsberg's seven bridges on the Pregel river once and only once. The close relation to Hamiltonian walks and circuits is established through the notion of the covering graph (or line graph) G^c of an oriented graph G, wich is obtained by: (a) replacing each oriented edge of G by a vertex; (b) connecting two such vertices by an oriented edge if and only if the corresponding original edges of G were consecutive; the new edge orientation agrees with the orientation of the original edges. From this definition it follows that to each Eulerian walk on an oriented graph G there corresponds exactly one Hamiltonian walk on G^c, and vice versa. Therefore we have $N_H(G^c) = N_E(G)$, where $N_E(G)$ is the number of Eulerian walks on G. The determination of the latter is always possible by the theorem of van Aardenne-Ehrenfest and de Bruijn: the remarkable connection between the number of Eulerian walks starting at v_1 and terminating at v_2, and the number $N_T(v_2)$ of spanning trees with root v_2 is: $N_E = r_2 N_T(v_2) \prod_j (r_j - 1)!$, where r_j is the number of edges directed away from v_j. Actually, Kasteleyn [1] found also an unoriented lattice L related to M, such that $N_H(M) = N_T(L)$.

4. Perfect pairings (or matchings) of a graph – the dimer model. In 1961 Kasteleyn [3] solved exactly another nontrivial problem of lattice statistics: he found the number of the different sets of pairs of neighbouring vertices of a square lattice, such that each vertex enters into one and only one pair. The physical interpretation of the problem is known as a system of diatomic molecules (dimers) adsorbed on a crystal surface in close-packed configurations: each molecule covers two adjacent vertices of the square lattice and each vertex is occupied by one and only one atom.

The number of the dimer configurations on a square lattice, with m rows and n columns is (m = even):

$$Z_{mn} = \begin{cases} 2^{mn/2} \prod_{k=1}^{m/2} \prod_{l=1}^{n/2} \left[cos^2 \frac{k\pi}{m+1} + cos^2 \frac{l\pi}{n+1} \right], & n = \text{even} \\ 2^{m(n-1)/2} \prod_{k=1}^{m/2} \prod_{l=1}^{(n-1)/2} \left[cos^2 \frac{k\pi}{m+1} + cos^2 \frac{l\pi}{n+1} \right], & n = \text{odd} \end{cases}$$

Nagle [4] suggested an equivalent interpretation of the dimer model in terms of an anisotropic system of linear polymers on an associated triangular lattice, which can be thought of as a two-dimensional model of a biomembrane. The case of a finite graph has been treated in [5].

5. Abelian sandpile model. Recently, great and still increasing attention has been paid to the phenomenon of self-organized criticality (SOC) which provides a connection between nonlinear dynamics and the appearance of spatial and temporal self-similarity [6]. The concept of SOC has been used in describing turbulence, earthquakes, forest fires, the process of erosion, stability of ecological systems, coagulation, etc. A simple model of the dissipative evolution of a pile of sand, proposed in 1987 by Bak, Tang and Wiesenfeld, shows how a deterministic threshold (therefore, nonlinear) diffusion can naturally lead the system into a critical state with power-law decay of correlations in space and time. The Abelian sandpile models introduced by Dhar [7] were found to admit analytical treatment. In the simplest such model the height h_i of the sandpile at each vertex v_i, $i = 1,2,...,N$, of a finite square lattice L takes integer values $h_i \in \{1,2,3,4\}$. Adding sand at a randomly chosen vertex v_i leads to increasing h_i by unity. If the heihgt h_j at any place v_j exceeds the critical value $h_c = 4$, then four units of sand slide down from that place to each of its four neighboring places. The process continues untill a stable configuration is reached in which all the h_j are below the critical value. Then another grain of sand is added, and so on. The sand sliding upon addition of sand at v_i can be expressed as $h_j \to h_j - \Delta_{ij}$ for all $j = 1,2,...,N$, where $\Delta = \{\Delta_{ij}\}$ is the Laplacian matrix for the square lattice L. Away from the boundary ∂L of L, Δ has the elements $\Delta_{ii} = 4$; $\Delta_{ij} = -1$ if v_i and v_j are neighbours, and $\Delta_{ij} = 0$ otherwise. The standard boundary value problems are: the Dirichlet boundary conditions, when $\Delta_{ii} = 4$ for $i \in \partial L$, correspond to the case when the sand is allowed to leave the system through the boundary ∂L; the Neumann boundary conditions, when $\Delta = 3$ for $i \in \partial L$, if i is not a corner of L and $\Delta = 2$, if i is a corner, correspond to the case when the sand cannot leave the system through the boundary ∂L. In 1992 Majumdar and Dhar [8] found a one-to-one correspondence between the recurrent configurations

of the model under Dirichlet boundary conditions at all the four boundary sides of the lattice L and the spanning trees on a lattice of the same size whose boundary vertices are connected to an additional vertex in the external face.

Main results

In the present work we establish the equivalence of the dimer model on a square lattice L with free boundaries and the Abelian sandpile model on another square lattice L' with mixed Dirichlet-Neumann boundary conditions imposed on at least one pair of opposite boundary sides of L'. This is achieved by finding the equivalent spanning trees representation of the dimer model.

A graph $G = (V^G, E^G)$ is defined by its vertex set V^G and the set of edges E^G connecting some pairs of vertices. A vertex is of degree k, if it is connected by k edges to other vertices of G. A vertex v of a graph is said to be pendant if there is only one edge incident to v, and that edge is called pendant edge. If a subgraph G' of G contains all the edges of G that join two vertices of G', then G' is the subgraph induced by $V^{G'}$ and is denoted by $G(V^{G'})$. Here we will consider some special types of graphs wich are known as square lattices. Let $L = L(m,n)$ be a square lattice with free boundaries which consists of m rows and n columns. Its vertices will be denoted by a pair of integer indices, (i,j), $i = 1,2,..,m$; $j = 1,2,...,n$, where the first index is the row number and the second one is the column number of the vertex. Obviously, L admits a dimer configuration if at least one of the numbers m and n, say m, is even. A vertex of L with degree less than 4 is called a boundary vertex and with degree 2 is called a corner of L. We will need to distinguish between different subsets of vertices. A vertex (i,j) is called to have the $\alpha - \beta$ parity, if α and β are the parities of its row and column numbers, respectively. For example, if i is even and j is odd, the vertex (i,j) is of even-odd parity. Let $L = (V,E)$, then the subset of V which contains all the vertices with $\alpha - \beta$ parity is denoted by $V_{\alpha\beta}$. By $\bar{\alpha}$, $\bar{\beta}$ we denote the parities opposite to α and β, respectively.

With each square lattice L we assosiate a graph L' which is obtainable from L by the following three-step rule:

1. All the vertices of $\alpha - \beta$ parity are deleted from L. To delete a vertex from a graph G means to deleted it from the vertex set V^G and to delete from the edge set E^G all the edges incident to it. This step leads to the transformation of L into the graph, induced by the set $V \setminus V_{\alpha\beta}$, denoted as $L_1 = L(V \setminus V_{\alpha\beta}) = (V_1, E_1)$;

2. All the vertices of degree 2 in L_1, except the corners of L, are suppressed. A vertex of degree 2 is suppressed in G, if it is deleted from the set V^G and the two edges incident to it are identified. As a result, L_1 is transformed into L_2 which is denoted as $L_2 = (V_2, E_2)$.

3. All the pendant vertices of L_2 are merged into a single vertex, which we denote by $v_0 = (0,0)$. As a result we obtain the graph $L' = (V', E')$ associated with L.

The vertices, connected with v_0 in L' are placed on one, two or three adjacent boundary sides (columns and rows), depending on α, β and the parity of the size numbers m, n. One can write $L' = L(m,n) \cup B_i(m,n)$, where the vertex set of the boundary graph $B_i(m,n)$ consists of a vertex v_0 in the external face of $L(m,n)$ and the vertices belonging to i = 1, 2, 3 adjacent boundary sides $\partial_i L(m,n)$ of $L(m,n)$; v_0 is connected by one edge to each vertex of $\partial_i L(m,n)$ which is not a corner of $L(m,n)$ and by two edges, one horizontal and one vertical, to each vertex of $\partial_i L(m,n)$ which is a corner of $L(m,n)$, if such exists.

Theorem 1. *To every dimer configuration D on the lattice L there corresponds a spanning tree T on the assosiated graph L'.*

Proof. Consider the following class of paths on L_1, called for shortness d-paths, defined for any fixed pair α,β. A d-path starts from a vertex belonging to the set $V_{\alpha\beta}$, follows along the edge covered in L by a dimer, continues (in the same direction) along the adjacent edge to the nearest vertex from $V_{\alpha\beta}$, next follows the edge covered by a dimer in L, etc. Such paths cannot have cycles, because the corresponding cycle on L would encircle an odd number of vertices of L, which is forbidden by the dimer packing constraint [9]. Therefore, a d-path may terminate only with a pendant edge of L_1 which is covered by a dimer in L. The vertices and edges of all d-paths which terminate with a given boundary edge define a tree, to be called a d-tree. Obviously, to each dimer configuration D there corresponds a forest of d-trees, or a single d-tree, which contains all the vertices of the set $V_{\alpha\beta}$: each such vertex, being an end vertex of a dimer, is a starting point of a d-path. Under step (2) the d-forest naturally maps onto a forest on the graph L_2 and finally, under step (3), it becomes a single spanning tree T on L'.

Theorem 2. *To each spanning tree on $L' = L(m,n) \cup B_i(m,n)$ there corresponds a dimer configuration on:*

a) $L(2m, 2n-1)$, *if* $i = 1$ *and* $\partial_1 L(m,n)$ *is a boundary row, or* $L(2m-1, 2n)$, *if* $\partial_1 L(m,n)$ *is a boundary column;*

b) $L(2m, 2n)$, *if* $i = 2$;

c) $L(2m, 2n+1)$, *if* $i = 3$ *and* $\partial_3 L(m,n)$ *consists of one boundary row and two boundary columns, or* $L(2m+1, 2n)$, *if* $\partial_3 L(m,n)$ *consists of two boundary rows and one boundary column.* The proof of this theorem is given in [10].

The number of spanning trees on L', can be calculated by using the Kirchhoff theorem.

Corollary 1. *The number of close-packed dimer configurations on $L(2m, 2n)$ is equal to the number of the spanning trees on the graph $L' = L(m,n) \cup B_2(m,n)$.*

Choose $\alpha - \beta$ in step (1) to be odd-odd. Then, the vertex v_0 of the graph L' is connected to the vertices in the first row and the first column of L'. The matrix $A_{(0,0)}$, obtained by deleting the row and column corresponding to v_0 in the Kirchhoff matrix of L', is:

$$A_{(0,0)} = [4E_m - C_m] \otimes E_n - E_m \otimes C_n.$$

Here E_m is the unit matrix of order m, the matrix $C_m = \{c_{ij}\}, c_{ij} = \delta_{i,j-1} + \delta_{i-1,j} + \delta_{1,j}$, has the eigenvalues $\lambda_p = 2cos\frac{(2p-1)\pi}{2m+1}, p = 1,2,...,m$, and $\otimes$ denotes a direct product.

Therefore, the number of spanning trees on L' is

$$\begin{aligned} N_T = detA_{(0,0)} &= \prod_{p=1}^{m}\prod_{q=1}^{n}\left[(2 - 2cos\frac{(2p-1)\pi}{2m+1}) + (2 - 2cos\frac{(2q-1)\pi}{2n+1})\right] \\ &= 2^{2mn}\prod_{k=1}^{m}\prod_{l=1}^{n}\left[cos^2\frac{k\pi}{2m+1} + cos^2\frac{l\pi}{2n+1}\right] = Z_{2m,2n}. \end{aligned}$$

Corollary 2. *The number of the close-packed dimer configurations on $L(2m, 2n+1)$, is equal to the number of the spanning trees on the graph: (i) $L' = L(m,n) \cup B_3(m,n)$, or (ii) $L' = L(m,n+1) \cup B_1(m,n+1)$.*

(i) Choose $\alpha - \beta$ in step (1) to be odd-odd. In this case L' has m rows and n columns, the vertex v_0 is connected with the vertices in the first and the last columns and the first row of L'. The matrix $A_{(0,0)}$, obtained by deleting the row and column corresponding to v_0 in the Kirhhoff matrix, is:

$$A_{(0,0)} = [4E_m - B_m] \otimes E_n - E_m \otimes C_n,$$

where the matrix $B_m = \{b_{ij}\}, b_{ij} = \delta_{i,j-1} + \delta_{i-1,j}$, has the eigenvalues $\lambda_p = 2cos\frac{\pi p}{m+1}$, $p = 1,2,...,m$.

Therefore, the number of spanning trees on L' is:

$$\begin{aligned} N_T = detA_{(0,0)} &= \prod_{p=1}^{m}\prod_{q=1}^{n}\left[(2 - 2cos\frac{(2p-1)\pi}{2m+1}) + (2 - 2cos\frac{q\pi}{n+1})\right] \\ &= 2^{n(2n-1)}\prod_{k=1}^{m}\prod_{l=1}^{n}\left[cos^2\frac{k\pi}{2m+1} + cos^2\frac{l\pi}{2n+1}\right] = Z_{2m,2n+1}. \end{aligned}$$

(ii) Choose $\alpha - \beta$ in step (1) to be even-even. In this case L' has m rows and $n+1$ columns, the vertex v_0 is connected with the vertices in the last row of L'. The matrix $A_{(0,0)}$, obtained by deleting the row and column corresponding to v_0 in the Kirhhoff matrix, is:

$$A_{(0,0)} = [4E_m - A_m] \otimes E_{n+1} - E_m \otimes C_{n+1},$$

where the matrix $A_m = \{a_{ij}\}, a_{ij} = \delta_{i,j-1} + \delta_{i-1,j} + \delta_{1,j} + \delta_{n,j}$, has the eigenvalues $\lambda_p = 2cos\frac{\pi(p-1)}{m}, p = 1, 2, ..., m.$

Therefore, the number of spanning trees on L' is:

$$N_T = detA_{(0,0)} = \prod_{p=1}^{m}\prod_{q=1}^{n+1}\left[(2 - 2cos\frac{(2p-1)\pi}{2m+1}) + (2 - 2cos\frac{(q-1)\pi}{n+1})\right]$$

$$= 2^{n(2n-1)}\prod_{k=1}^{m}\prod_{l=1}^{n}\left[cos^2\frac{k\pi}{2m+1} + cos^2\frac{l\pi}{2n+1}\right] = Z_{2m,2n+1}.$$

Perspectives

The results obtained here make possible the application of the Monte Carlo algorithm for generation of dimer configurations [11] to numerical simulations on spanning trees and sandpile models. However, an explicit one-to-one mapping of the corresponding local observables is required. Due to the obviously nonlocal character of any such mapping, the efficiency of the dimer algorithm for the above purposes needs practical tests.

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