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GENERAL MECHANICS

ON THE STABILITY OF LYAPUNOV'S PERIODIC SOLUTIONS OF A MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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1. Introduction

Let us consider a Hamiltonian system with n degrees of freedom. Then linearize the system in neighbourhood of equilibrium position. There exist, under certain conditions, special coordinates called normal modes in which the linearizing system decouples into n one degree of freedom systems, describing the motion of oscillators i.e. there exist n periodic solutions. The natural question arieses if there exist such periodic solutions for nonlinear system. One answer of this question is given by the theorems of Lyapunov and Weinstein [1] which claim that under certain conditions of nonresonance between eigenvalues of linearized system, there exist at least n periodic solutions on any fixed energy level. These periodic solutions are called Lyapunov's periodic solutions. The study of stability of such solutions can be done by several methods [4, 6]. Here we shall use one of the most efficient methods for investigating stability in nonlinear problems, namely KAM-theory methods [5, 6]. Let us consider the so called crosssection surface: fix the energy level and one of the variables in such a way that the closed trajectory intersects the obtained $2n-2$ dimensional hiperplane transversal. Then the phase curves, close to given, define the map on this $2n-2$ dimensional hiperplane onto itself with fixed point on the closed curve (it is called Poincare map). In case $n = 2$ we have a map in a plane to which we can apply Moser's theorem for existence of invariant curves. The sufficient conditions for existence of invariant curves around beginning (fixed point) will be sufficient conditions for the stability of the corresponding periodic trajectory.

Let us consider the following mechanical model. A particle, with mass m, attached to a spring, is oscillating in a rigid body with a fixed point, along a line which passes through the fixed point of the body. The problem for the movement of a rigid body with moving particles in it, is considered for instance in [2]. The equations of motion for the present model are derived in [3]. Consider the simplest case - without external forces, which is an analogue of Euler's top in classical rigid body problem. So, in a moving frame (which axes are usually along the principal inertia axes) the equations of motion around the fixed point are:

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = -2mr\dot{r}\omega_1 - mr^2\dot{\omega}_1 + mr^2\omega_2\omega_3,$$

$$B\dot{\omega}_2 + (A - C)\omega_1\omega_3 = -2mr\dot{r}\omega_2 - mr^2\dot{\omega}_2 - mr^2\omega_1\omega_3, \tag{1}$$

$$C\dot{\omega}_3 + (B - A)\omega_1\omega_2 = 0,$$

$$\ddot{r} + r(\sigma/m - \omega_1^2 - \omega_2^2) = 0, \tag{2}$$

where ω_s are angular velocity components in the moving frame and r is the deviation of the oscillating particle from the fixed point (particle oscillates along one of the principal axes). The equations (1), (2) possess the following integrals

$$H = \{(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + m[\dot{r}^2 + r^2(\omega_1^2 + \omega_2^2)] + \sigma r^2\}/2 = h, \tag{3}$$

$$M^2 = (A + mr^2)^2\omega_1^2 + (B + mr^2)^2\omega_2^2 + C^2\omega_3^2 = M_0^2. \tag{4}$$

When $A = B$ the above system is integrable and it's solutions are expressed via elliptic functions. The system (1), (2) is a Hamiltonian one. This is shown in [3] where a Poisson bracket is introduced on the fixed integral level of the integral (4). Thus, the system (1),(2) is effectively a two degrees of freedom system.

The system (1), (2) possesses the following one-parametric families of periodic solutions: First – denoted by $(PS1)$ with initial conditions

$$\omega_1 = \omega_2 = 0, r, \dot{r} \text{ - arbitrary}; \tag{5}$$

In order to get the second family if we put $r = \dot{r} = 0$, we obtain the Euler equations, which have different families of periodic solutions. We shall consider this family obtained when $A = B(PS2)$ and at initial conditions

$$r = \dot{r} = 0, \omega_1, \omega_2 \text{ - arbitrary.} \tag{6}$$

Now, denote the eigenvalues of the linearized system by

$$\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \tag{7}$$

where $\lambda_1 = i(\sigma/m)^{1/2}, \lambda_2 = i[(C-A)(C-B)\omega_3^2/(AB)]^{1/2}, i^2 = -1.$

Then, we shall state the main result:

Theorem 1. *Let the eigenvalues* (7) *be pure imaginary and there are no resonances between them. Then, if $B = A + \varepsilon$, where ε is sufficiently small,*

1) *the periodic solution* (5) *is orbitaly stable;*

2) *The periodic solution* (6) *is orbitaly stable for sufficiently small values of the energy.*

The proof will be done in section 3.

2. Moser's twist theorem

The Moser's theorem for existence of invariant curves is the base theorem of the KAM – theory in case of two degrees of freedom (see [5] for details). Consider the map

$$r_1 = r,$$
$$\theta_1 = \theta + a(r), \tag{8}$$

defined for $0 < a_0 \leq r \leq b_0$. This map obviously preserves the area and retains invariant all circles $r = c$ and rotates them about an angle a(c). Natural is the question, whether the map

$$r_1 = r + \varepsilon f,$$
$$\theta_1 = \theta + a(r) + \varepsilon g, \tag{9}$$

where f and g are certain functions, possesses invariant curves when ε is sufficiently small. The following theorem gives the sufficient conditions for this.

Theorem 2 (Moser [5]). *Let ε be sufficiently small and $a(r) \in C^l (l \geq 5)$, $|a'(r)| \geq \nu > 0$ in $A := \{a_0 \leq r \leq b_0\}$, f and $g \in C^l$. Then the map (9) of A into $\mathbf{R}^2$ possesses an invariant curve of the form*

$$r = c + u(\xi), \quad \theta = \xi + v(\xi) \tag{10}$$

in A, where $u(\xi)$ and $v(\xi)$ are sufficiently small, continuously differentiable, of period 2π and c is a constant $a_0 \leq c \leq b_0$. Moreover, the curve (10) *can be parameterized in such a way that the induced map of this curve is given by $\xi_1 = \xi + \omega$, where ω is incommensurable with 2π and satisfies infinitely many conditions $|\omega/2\pi - p/q| \geq \gamma q^{-\tau}$ with some positive γ, τ for all integers $q > 0$, p.*

In fact, it is easy to be seen that the above theorem gives the existence of infinitely invariant curves. We shall use this fact for investigating the stability of the fixed point of the map (9). When $\varepsilon = 0$, the fixed point of the map is surrounded by concentrate circles $r = c$. Now let $\varepsilon > 0$ be a sufficiently small positive number. Then according to Moser's theorem there exist infinitely many curves surrounding the fixed point. Therefore, the domain between two such curves is also invariant, from where the stability of the fixed point is obtained.

3. Proof

Let $B = A + \varepsilon$. In order to study stability we have to reduce the differential equations to maps in such a way that the periodic solutions are the fixed points for these maps and to apply Moser's theorem. It is seen from sec.2 that it is sufficient to consider the case $\varepsilon = 0$. Then, as we mentioned above, the system is integrable. We make some changes of the variables.

First, without loss of generality we assume $m = 1$. Then we put $z_1 = r$, $z_2 = \dot{r}$, $M_1 = (A+z_1^2)\omega_1$, $M_2 = (A+z_1^2)\omega_2$, $M_3 = C\omega_3 = M_{30}$. After that we put $I = M_1^2+M_2^2$, $\varphi = \text{arctg } (M_2/M_1)/(2M_{30})$, $\varphi \in [0, 2\pi]$. Then the equations of motion become

$$\dot{I} = 0 = -\partial H/\partial\varphi,$$

$$\dot{\varphi} = (1/(A+z^2) - 1/C)/2 = \partial H/\partial I,$$

$$\dot{z}_1 = z_2 = \partial H/\partial z_2, \tag{11}$$

$$\dot{z}_2 = -sz + zI/(A+z^2)^2 = -\partial H/\partial z_1, \quad \text{where}$$

$$H = I(1/(A+z^2) - 1/C))/2 + sz_1^2 + z_2^2/2 = h \tag{12}$$

with second obvious integral $I = f = \text{const}$.

Consider the periodic solution (5). We make isoenergetical reduction in the following way. Fix the energy h. Then from (12) we obtain

$$z_2 = -[2h - sz_1^2 - I(1/(A + z^2) - 1/C))]^{1/2} = -K(z_1, I, \varphi).$$

Note, that when the real motion takes place the expression in square brackets is positive. Then

$$dI/dz_1 = -\partial K/\partial\varphi = 0,$$

$$d\varphi/dz_1 = \partial K/\partial I = \tag{13}$$

$$-(1/(a + z^2) - 1/C)[2h - sz_1^2 - I(1/(A + z^2) - 1/C))]^{-1/2}/2,$$

i.e. we choose I, φ for the coordinates in the transversal plane to the periodic solution. Obviously $(PS1)$ is reduced to the singular point of the system (13). Integrating we get the map

$$I_1 = I,$$

$$\varphi_1 = \varphi_0 + a(I), \tag{14}$$

where

$$a(I) = -(1/2)\oint_{H=h}(1/(A + z^2) - 1/C)[2h - sz_1^2 - I(1/(A + z^2) - 1/C))]^{-1/2}dz_1.$$

Here integration is made over a closed path defined by $H = h$ and I sufficiently small. Then obviously

$$a'(I) = (1/2)\oint_{H=h}(1/(A+z^2)-1/C)^2[2h-sz_1^2-I(1/(A+z^2)-1/C))]^{-3/2}dz_1 \neq 0. \tag{15}$$

Then the map (14) satisfies Moser's condition and therefore the $(PS1)$ is orbitaly stable.

Now consider the second periodic solution $(PS2)$. In these variables it reads

$$z_1 = z_2 = 0, \quad I = f = const., \quad \varphi = (1/A - 1/C)t/2 + \varphi_0$$

with additional condition $f(1/A - 1/C) = 2h$. Again making isoenergetical reduction, but expressing now I and having in mind that the reduced system has an integral, since the original system is integrable, we may construct a map and Moser's condition takes us again to an integral of the kind (15). But, in this case, the closed curves along which we integrate exist only for small h. This completes the proof of theorem 1.

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