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GENERAL MECHANICS

O. Christov

Stability of Certain Solutions in a Mechanical Model Generating the Rigid Body Problem

1. Introduction

Let us consider the following mechanical system. A particle of mass m, hanged up on a spring, is oscillating in a rigid body with a fixed point O, along a line that passes through the fixed point of the body.

This model is a direct generalization of the rigid body problem (for the history of the rigid body problem see [1]). It also falls into the class of systems of rigid bodies [2] in which one of the bodies is reduced to a point. Similar models are studied by Chernousko [3] and Akulenko [4]. The equations for our model are derived in [5] via Jourdain's variational principle in case of a Newtonian field. In [6], they are derived in Euler-Poisson variables. Then, in the moving frame (whose axes are usually along the principal inertia axes) the equations of motion around the fixed point are:

$$A\dot{\omega}_1 + (C-B)\omega_2\omega_3 = Mg(\gamma_2 z_G - \gamma_3 y_G) + mgr\gamma_2 - 2mr\dot{r}\omega_1 - mr^2\dot{\omega}_1 + mr^2\omega_2\omega_3$$

$$(1.1)\qquad B\dot{\omega}_2 + (A-C)\omega_3\omega_1 = Mg(\gamma_3 x_G - \gamma_1 z_G) - mgr\gamma_1 - 2mr\dot{r}\omega_2 - mr^2\dot{\omega}_2 - mr^2\omega_3\omega_1$$

$$C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = Mg(\gamma_1 y_G - \gamma_2 x_G),$$

$$(1.2)\qquad \ddot{r} + r(\sigma/m - \omega_1^2 - \omega_2^2) = -g\gamma_3,$$

$$\dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2,$$

$$(1.3)\qquad \dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3,$$

$$\dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1,$$

where ω_s are angular velocity components in the moving frame, γ_s (s = 1,2,3) -

the directory cosines of the vertical, x_G, y_G, z_G are the co-ordinates of the mass center of the body and r is the deviation of the oscillating particle from the fixed point. The particle is oscillating along one of the inertia axes. The body mass is denoted by M and the stiffness of the spring is denoted by G. The equations of motion (1.1) - (1.3) possess the following common first integrals:

$$2E = (A+mr^2)\omega_1^2 + (B+mr^2)\omega_2^2 + C\,\omega_3^2 + m\dot{r}^2 + \sigma r^2 + 2mgr\gamma_3 + 2Mg(x_G\gamma_1 + y_G\gamma_2 + z_G\gamma_3) = \text{const.}, \tag{1.4}$$

$$M_z = (A+mr^2)\omega_1\gamma_1 + (B+mr^2)\omega_2\gamma_2 + C\,\omega_3\gamma_3 = \text{const.}, \tag{1.5}$$

$$G = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \tag{1.6}$$

Many problems can be stated for this system. For example, the problem of integrability is considered in [6]. As corresponding additional integrals, sufficient for integrability in classical rigid body problem are obtained for special values of the parameters of the problem A, B, C, x_G, y_G, z_G. We introduce by analogy the cases of Euler, Lagrange and Kowalewski with the same values for the parameters. According to the structure of our model, for equations (1.1) - (1.3), there are two cases analogous to Euler's case in the classical problem. We call them Euler's I case (g = 0, without external forces) and Euler's II case ($x_G = y_G = z_G = 0$). Also we have Lagrange's case (A = B, $x_G = y_G = 0$) and Kowalewski's case (A = B = 2C, $y_G = z_G = 0$). The purpose of this paper is to study stability of certain solutions in the above four cases. The necessary and sufficient conditions for the stability of these solutions are obtained. The sufficient conditions for stability are obtained by constructing Lyapunov's function via Tchetaev's method (by the help of the first integrals). The necessary conditions are obtained studying the secular equations in order for their roots to be with non positive real parts. The necessary conditions mean that if they are not satisfied, unstability is observed. In fact, we have four problems and follow the above scenario for them (see also [7] for stability conditions in the classical rigid body problem). In order to take the systems of differential equations of the above cases in normal form, let us put $z_1 = r$, $z_2 = \dot{r}$.

The paper is organized as follows. Sections 2 and 3 deal with Euler's I case and Lagrange's case respectively. They are more convenient. However, the structure of the secular equations of Kowalewski's and Euler's II cases is more specific - it leads to degenerated Routh - Hurvitz problem. Section 4 is devoted to it. Then Kowalewski's and Euler's II cases are studied in sections 5 and 6 respectively.

2. Euler's I case (g = 0).

The dynamic equations of motion in Euler's I case are:

$$(A+mz_1^2)\dot{\omega}_1 + (C-B)\omega_2\omega_3 = -2mz_1z_2\omega_1 + mz_1^2\omega_2\omega_3$$

$$(2.1)\qquad (B+mz_1^2)\dot{\omega}_2 + (A-C)\omega_3\omega_1 = -2mz_1z_2\omega_2 - mz_1^2\omega_3\omega_1$$

$$C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = 0,$$

$$\dot{z}_1 = z_2$$

$$(2.2)\qquad \dot{z}_2 = z_1(\omega_1^2 + \omega_2^2 - \sigma/m)$$

They possess the following integrals of motion:

$$(2.3)\qquad U = (A+mz_1^2)\omega_1^2 + (B+mz_1^2)\omega_2^2 + C\,\omega_3^2 + mz_2^2 + \sigma z_1^2 = 2E = \text{const.}$$

$$(2.4)\qquad U_1 = (A+mz_1^2)^2\omega_1^2 + (B+mz_1^2)^2\omega_2^2 + C^2\omega_3^2 = \text{const.},$$

The equations of motion possess a solution of the kind:

$$(2.5)\qquad \omega_1 = \omega_2 = 0,\quad \omega_3 = \omega = \text{const.},\quad z_1 = z_2 = 0,$$

describing the uniform rotation of the body about the axes Ox_3. Let us investigate stability of solution (2.5) with respect to the variables ω_1, ω_2, ω_3, z_1, z_2. In order to do this, we introduce perturbations $\omega_3 = \xi + \omega$ and for the remaining variables we leave their previous notations.

A. Sufficient conditions.

The equations of the perturbed motion possess integrals:

$$(2.6)\qquad U = (A+mz_1^2)\omega_1^2 + (B+mz_1^2)\omega_2^2 + 2C\omega\xi + C\xi^2 + mz_2^2 + \sigma z_1^2 = \text{const}$$

$$(2.7)\qquad U_1 = (A+mz_1^2)^2\omega_1^2 + (B+mz_1^2)^2\omega_2^2 + 2C^2\omega\xi + C^2\xi^2 = \text{const.}$$

We shall find Lyapunov's function in the following form:

$$W = C\,U - U_1 + U^2.$$

The quadratic part of W is:

$$W_2 = A(C-A)\omega_1^2 + B(C-B)\omega_2^2 + C(mz_2^2 + \sigma z_1^2) + 4\,C^2\xi^2\omega^2,$$

which is positive definite when

$$(2.8)\qquad C > A,\ C > B.$$

According to Lyapunov's theorem for stability, these are the sufficient conditions for the stability of solution (2.5) with respect to the variables ω_1, ω_2, ω_3, z_1, z_2.

B. Necessary conditions

The equations of perturbed motion in linear approximation are:

$$A\dot{\omega}_1 + (C-B)\omega_2\omega = 0$$

$$B\dot{\omega}_2 + (A-C)\omega\omega_1 = 0$$

$$\dot{\xi} = 0$$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -\sigma/m\ z_1$$

Then, the secular equation of this system reads:

$$\lambda(\lambda^2 + \sigma/m)(AB\lambda^2 + (C - A)(C - B)\omega^2) = 0. \tag{2.9}$$

In order for this equation to have no roots with a positive real part, it is necessary that the following conditions are fulfilled:

$$C > A,\ C > B \quad \text{or}\ A > C,\ B > C\ . \tag{2.10}$$

So, we have proved the

THEOREM 1. (i) The necessary conditions for stability of solution (2.5) in Euler's I case are $C > A$, $C > B$ or $A > C$, $B > C$.

(ii) The sufficient conditions for stability of solution (2.5) with respect to ω_1, ω_2, ω_3, z_1, z_2 are $C > A$, $C > B$.

REMARK 1. The difference between necessary and sufficient conditions is due to the additional degree of freedom in our mechanical model.

3. Lagrange's case ($A = B$, $x_G = y_G = 0$).

The dynamic equations of motion in Lagrange's case are:

$$\begin{aligned}(A+mz_1^2)\dot{\omega}_1 + (C-A)\omega_2\omega_3 &= Mg\gamma_2 z_G + mgz_1\gamma_2 - 2mz_1z_2\omega_1 + mz_1^2\omega_2\omega_3\\ (A+mz_1^2)\dot{\omega}_1 + (A-C)\omega_3\omega_1 &= -Mg\gamma_1 z_G - mgz_1\gamma_1 - 2mz_1z_2\omega_2 - mz_1^2\omega_3\omega_1\\ C\dot{\omega}_3 &= 0,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_1(\omega_1^2 + \omega_2^2 - \sigma/m) - g\gamma_3\end{aligned} \tag{3.2}$$

$$\begin{aligned}\dot{\gamma}_1 &= \gamma_2\omega_3 - \gamma_3\omega_2,\\ \dot{\gamma}_2 &= \gamma_3\omega_1 - \gamma_1\omega_3,\\ \dot{\gamma}_3 &= \gamma_1\omega_2 - \gamma_2\omega_1.\end{aligned} \tag{3.3}$$

They possess the following integrals of motion:

$$U = (A+mz_1^2)(\omega_1^2 + \omega_2^2) + C\,\omega_3^2 + mz_2^2 + \sigma z_1^2 + 2mgz_1\gamma_3 + \tag{3.4}$$

$$2Mgz_G\gamma_3 = 2E = \text{const.},$$

(3.5) $$U_1 = (A+mz_1^2)(\omega_1\gamma_1 + \omega_2\gamma_2) + C\,\omega_3\gamma_3 = \text{const.},$$

(3.6) $$U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

(3.7) $$U_3 = \omega_3 = \omega = \text{const.}.$$

The equations of motion possess a solution of the kind:

(3.8) $$\omega_1 = \omega_2 = 0, \quad \omega_3 = \omega = \text{const.}, \quad z_1 = -\,gm/\sigma, \quad z_2 = 0,$$

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1.$$

In order to study the stability of this solution with respect to ω_s, γ_s, s=1,2,3, z_1, z_2, we introduce perturbations $\omega_3 = \omega + \xi$, $\gamma_3 = 1 + \eta$, $z_1 = z_1^0 + \zeta$ and for remaining variables we leave their previous notations. Let $\tilde{A} = A + m(z_1^0)^2$, $k = Mgz_G + mgz_1^0$.

A. Sufficient conditions.

The equations of the perturbed motion possess the integrals:

$$U = (A+m(z_1^0 + \zeta)^2)(\omega_1^2 + \omega_2^2) + C\xi^2 + 2C\omega\xi + mz_2^2 + \sigma\zeta^2 + 2k\eta +$$

$$2\sigma\zeta z_1^0 + 2mg\zeta\eta + 2mg\zeta = \text{const.},$$

$$U_1 = (A+m(z_1^0 + \zeta)^2)(\omega_1\gamma_1 + \omega_2\gamma_2) + C(\eta\omega + \xi) + C\xi\eta = \text{const.},$$

$$U_2 = \gamma_1^2 + \gamma_2^2 + 2\eta + \eta^2 = \text{const.},$$

$$U_3 = \xi = \text{const.}.$$

Consider the following Lyapunov's function:

$$W = U + 2\lambda U_1 + \mu U_2 + \delta U_3^2 + \upsilon U_3\,.$$

In order to eliminate the first order terms, we choose:

$$\upsilon = 2C(\omega + \lambda), \quad \mu = -\,(k + \lambda C\omega).$$

The quadratic part of W is

$$W_2 = \tilde{A}\omega_1^2 + 2\tilde{A}\lambda\omega_1\gamma_1 - (k + \lambda C\omega)\gamma_1^2 +$$

$$\tilde{A}\omega_2^2 + 2\tilde{A}\lambda\omega_2\gamma_2 - (k + \lambda C\omega)\gamma_2^2 +$$

$$\sigma\zeta^2 + 2mg\eta\zeta - (k + \lambda C\omega)\eta^2 + (C + \delta)\xi^2 + 2\lambda C\xi\eta + mz_2^2\,.$$

The corresponding quadratic forms are positive definite when

(3.9a) $$\tilde{A}\lambda^2 + C\lambda\omega + k < 0,$$

(3.9b)
$$C + \delta > 0,$$

(3.9c)
$$\lambda^2C^2 + (k + C\lambda\omega)(C + \delta) < 0,$$

(3.9d)
$$\sigma\,[\lambda^2C^2 + (k + C\lambda\omega)(C + \delta)] + (mg)^2(C + \delta) < 0.$$

Now, let $\delta = C(C - \tilde{A})/\tilde{A}$. Then conditions (3.9b) and (3.9c) are reduced to (3.9a). Having in mind notations for k and z_1^0 we see that (3.9d) is also reduced to (3.9a). But, we may choose λ to satisfy (3.9a) only when

(3.10)
$$C^2\omega^2 - 4\tilde{A}k > 0.$$

So, this is the sufficient condition for the stability of solution (3.8) with respect to ω_s, γ_s, $s=1,2,3$, z_1, z_2. The natural question is what happens when in (3.10) instead of inequality we have equality. In this case, the quadratic part of W : W_2 is positive, but not positive definite. Then considering the higher order terms, it is easy to be seen that W is positive definite and consequently we have stability according to Lyapunov's theorem.

B. Necessary conditions

The equations of the perturbed motion in linear approximation are:

(3.11)
$$\begin{aligned}
&\tilde{A}\,\dot{\omega}_1 + (C-\tilde{A})\omega_2\omega = k\gamma_2\\
&\tilde{A}\,\dot{\omega}_2 - (C-\tilde{A})\omega_2\omega = -\,k\gamma_1\\
&\dot{\xi} = 0\\
&\ddot{\zeta} = z_2\\
&\dot{z}_2 = -\,\sigma/m\,\zeta - g\eta\\
&\dot{\gamma}_1 = \gamma_2\omega - \omega_2,\\
&\dot{\gamma}_2 = \omega_1 - \gamma_1\omega,\\
&\dot{\eta} = 0.
\end{aligned}$$

The secular equation of this system reads:

$$\lambda^2(\lambda^2 + \sigma/m)\{[\tilde{A}\lambda^2 + (C - \tilde{A})\omega^2 - k]^2 + (2\tilde{A} - C)^2\omega^2\lambda^2\} = 0$$

The part of this equation which is in the figure brackets has the following roots:

$$\lambda_j = \frac{\pm (2\tilde{A} - C)\omega \pm \sqrt{4\tilde{A}k - C^2\omega^2}}{2\tilde{A}}, \quad j = 1,2,3,4.$$

In order for these roots to have no positive real part, it is necessary that

condition $C^2\omega^2 - 4\tilde{A}k \geq 0$ is fulfilled.
So, we have proved the following

THEOREM 2. The necessary and sufficient condition for the stability of solution (3.8) with respect to ω_s, γ_s, s = 1, 2,3, z_1, z_2 is $C^2\omega^2 - 4\tilde{A}k \geq 0$.

REMARK 2. This condition is a generalization of the classical condition of Mayevsky - Tchetaev.

4. On a degenerated Rauth - Hurvitz type problem.

Let us consider the following problem: when the polynomial

$$f(\lambda) = \lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3 \tag{4.1}$$

is stable i.e. when his roots are with negative real parts or are lying on an imaginary axis. Of course we are interested in the sufficient and necessary conditions warrant this. Let us also consider the polynomial

$$\tilde{f}(u) = u^3 + a_1u^2 + a_2u + a_3, \tag{4.2}$$

which is obtained from (4.1) after the change $\lambda^2 = u$.
First, it is obvious that $f(\lambda)$ can't have only roots with a negative real parts: if $u = \rho\exp(i\varphi)$ is a root of (4.2) with negative real part, then by Moivre's formula is seen that at least one of the corresponding λ is with a positive real part.
So, $f(\lambda)$ is stable when his roots are pure imaginary or the roots of (4.2) are negative, real and distinct.
PROPOSITION. (i) The necessary conditions (4.2) to have real, negative and distinct roots are $a_1 > 0$, $a_2 > 0$, $a_3 > 0$.
(ii) the sufficient conditions are: disc $(\tilde{f}(\lambda)) < 0$ and $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_1a_2 - a_3 > 0$.
The necessity is obvious. For the sufficiency, one should note that disc$(\tilde{f}(\lambda))<0$ gives that the roots are distinct and real and the remaining conditions are merely Hurvitz's conditions which gives that they are negative.

5. Kowalewski's case (A = B = 2C, $y_G = z_G = 0$).

The dynamic equations of motion in Kowalewski's case are:

$$\begin{aligned}
&(2C+mz_1^2)\dot{\omega}_1 - C\omega_2\omega_3 = mgz_1\gamma_2 - 2mz_1z_2\omega_1 + mz_1^2\omega_2\omega_3\\
&(2C+mz_1^2)\dot{\omega}_2 + C\omega_3\omega_1 = Mg\gamma_3x_G - mgz_1\gamma_1 - 2mz_1z_2\omega_2 - mz_1^2\omega_3\omega_1\\
&C\dot{\omega}_3 = -Mg\gamma_2x_G,\\
&\dot{z}_1 = z_2
\end{aligned} \tag{5.1}$$

(5.2) $$\dot{z}_2 = z_1(\omega_1^2 + \omega_2^2 - \sigma/m) - g\gamma_3$$

(5.3) $$\dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2, \quad \dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3, \quad \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1.$$

They possess the following integrals of motion:

(5.4) $$U = (2C+mz_1^2)(\omega_1^2 + \omega_2^2) + C\,\omega_3^2 + mz_2^2 + \sigma z_1^2 + 2mgz_1\gamma_3 + 2Mgx_G\gamma_1 = 2E = \text{const.},$$

(5.5) $$U_1 = (2C+mz_1^2)(\omega_1\gamma_1 + \omega_2\gamma_2) + C\,\omega_3\gamma_3 = \text{const.},$$

(5.6) $$U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

The equations of motion possess a solution of the kind:

(5.7) $$\omega_2 = \omega_3 = 0,\ \omega_1 = \omega = \text{const.},\ z_1 = z_2 = 0, \quad \gamma_2 = \gamma_3 = 0,\ \gamma_1 = 1.$$

In order to study stability of this solution with respect to ω_s, γ_s, s=1,2,3, z_1, z_2, we introduce perturbations $\omega_1 = \omega + \xi$, $\gamma_1 = 1 + \eta$, and for remaining variables we leave their previous notations. Without losing generality we set C = 1.

A. Sufficient conditions.

The equations of the perturbed motion possess the integrals:

$$U = (2+mz_1^2)(2\omega\xi + \xi^2 + \omega_2^2) + \omega_3^2 + mz_2^2 + (\sigma + m\omega^2)z_1^2 + 2mgz_1\gamma_3 + 2Mgx_G\eta = \text{const.},$$

$$U_1 = (2+mz_1^2)(\omega\eta + \xi + \xi\eta + \omega_2\gamma_2) + \omega_3\gamma_3 + mz_1^2\omega = \text{const.},$$

$$U_2 = 2\eta + \eta^2 + \gamma_2^2 + \gamma_3^2 = \text{const.}$$

Consider the following Lyapunov's function:

$$W = U + \lambda U_1 + \mu U_2.$$

In order to eliminate the first order terms we choose:

$$\lambda = -2\omega, \quad \mu = 2\omega^2 - Mgx_G.$$

Then, $W = W_2 + W^*$, where W^* denotes the terms of third or higher order terms and

$$W_2 = 2\xi^2 - 4\omega\xi\eta + (2\omega^2 - Mgx_G)\eta^2 +$$

$$2\omega_2^2 - 4\omega\omega_2\gamma_2 + (2\omega^2 - Mgx_G)\gamma_2^2 +$$
$$\omega_3^2 - 2\omega\omega_3\gamma_3 + (2\omega^2 - Mgx_G)\gamma_3^2 + (\sigma - m\omega^2)z_1^2 + 2mgz_1\gamma_3.$$

The quadratic form W_2 is positive definite when

(5.8a) $$x_G < 0,$$

(5.8b) $$(\omega^2 - Mgx_G)(\sigma - m\omega^2) - (mg)^2 > 0$$

Therefore, (5.8) are the sufficient conditions for the stability of solution (5.7).

B. Necessary conditions

Denote by $b = Mgx_G$ and $d = mg$. Then, the equations of the perturbed motion in linear approximation are:

(5.9)
$$\begin{aligned}
\dot{\xi} &= 0,\\
2\dot{\omega}_2 + \omega_3\omega &= b\gamma_3 - dz_1,\\
\dot{\omega}_3 &= - b\gamma_2,\\
\dot{z}_1 &= z_2\\
\dot{z}_2 &= z_1(\omega^2 - \sigma/m) - g\gamma_3,\\
\dot{\eta} &= 0,\\
\dot{\gamma}_2 &= \gamma_3\omega - \omega_3,\\
\dot{\gamma}_3 &= \omega_2 - \gamma_2\omega \text{ c}
\end{aligned}$$

The secular equation of this system reads:

$$\lambda^2(\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3) = 0,$$

where

$$a_1 = \sigma/m - 3b/2,$$
$$a_2 = b^2/2 + b\omega^2 - 3\sigma b/(2m) - dg/2 - \omega^4 + \sigma\omega^2/m,$$
$$a_3 = (bgd + b\omega^4 - b\omega^2\sigma/m + \sigma b^2/m - b^2\omega^2)/2 .$$

According to the preposition in section 4, we have to check the necessary conditions for stability i.e. $a_1 > 0$, $a_2 > 0$, $a_3 > 0$. Simple calculations give that these are the conditions

(5.10) $$b < 0,\ (\omega^2 - b)(\sigma - m\omega^2) - mgd > 0.$$

But, they are exactly the sufficient conditions (5.8).
Then, we have
THEOREM 3. The necessary and sufficient conditions for the stability of solution (5.7) with respect to

ω_s, γ_s, s =1,2,3, z_1, z_2 are: $x_G < 0$, $(\omega^2 - Mgx_G)(\sigma - m\omega^2) - (mg)^2 > 0$.

6. Euler's II case ($x_G = y_G = z_G = 0$).

This case is more complex, in a certain sense with respect to the other ones considered above, because we can't achieve positive definiteness directly through quadratic terms.

The dynamic equations of motion in Euler's II case are:

(6.1)
$$(A+mz_1^2)\dot{\omega}_1 + (C-B)\omega_2\omega_3 = + mgz_1\gamma_2 - 2mz_1z_2\omega_1 + mz_1^2\omega_2\omega_3$$
$$(B+mz_1^2)\dot{\omega}_2 + (A-C)\omega_3\omega_1 = - mgz_1\gamma_1 - 2mz_1z_2\omega_2 - mz_1^2\omega_3\omega_1$$
$$C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = 0,$$

(6.2)
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_1(\omega_1^2 + \omega_2^2 - \sigma/m) - g\gamma_3$$

(6.3)
$$\dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2,$$
$$\dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3,$$
$$\dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 .c$$

They possess the following integrals of motion:

(6.4)
$$U = (A+mz_1^2)\omega_1^2 + (B+mz_1^2)\omega_2^2 + C\,\omega_3^2 + mz_2^2 + \sigma z_1^2 + 2mgz_1\gamma_3 = 2E = \text{const.}$$

(6.5)
$$U_1 = (A+mz_1^2)\omega_1\gamma_1 + (B+mz_1^2)\omega_2\gamma_2 + C\,\omega_3\gamma_3 = \text{const.},$$

(6.6)
$$U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

The equations of motion possess a solution of the kind:

(6.7)
$$\omega_2 = \omega_3 = 0,\ \omega_1 = \omega = \text{const.},\ z_1 = z_2 = 0,$$
$$\gamma_2 = \gamma_3 = 0,\ \gamma_1 = 1.$$

In order to study stability of this solution with respect to ω_s, γ_s, s=1,2,3, z_1, z_2, we introduce perturbations $\omega_1 = \omega + \xi$, $\gamma_1 = 1 + \eta$, and for the remaining variables we leave their previous notations.

A. Sufficient conditions.

The equations of the perturbed motion possess the integrals:

$$U = (A+mz_1^2)(2\omega\xi + \xi^2) + (B+mz_1^2)\omega_2^2 + C\,\omega_3^2 + mz_2^2 + \sigma z_1^2 +$$

$$2mgz_1\gamma_3 = \text{const.}$$

$$U_1 = (A+mz_1^2)(\omega\eta + \xi + \xi\eta) + (B+mz_1^2)\omega_2\gamma_2 + C\,\omega_3\gamma_3 = \text{const.},$$

$$U_2 = 2\eta + \eta^2 + \gamma_2^2 + \gamma_3^2 = \text{const.}$$

Let us consider the following Lyapunov's function:

$$W = U + \lambda U_1 + \mu U_2.$$

In order to eliminate the first order terms we choose:

$$\lambda = -2\omega, \mu = A\omega^2.$$

Then, W takes the form $W = W_2 + W^*$, where

$$W_2 = A(\xi - \omega\eta)^2 + mz_2^2 + (\sigma - m\omega^2)z_1^2 + 2mgz_1\gamma_3 + C\omega_3^2 +$$

$$A\omega^2\gamma_3^2 - 2\omega C\omega_3\gamma_3 + B\omega_2^2 - 2\omega B\omega_2\gamma_2 + A\omega^2\gamma_2^2$$

and

$$W^* = mz_1^2(\xi^2 + \xi\eta - 2\omega^2\eta + \omega_2^2 + \omega_2\gamma_2).$$

It is obvious, that due to the first expression, W_2 can be at most positive, but not positive definite. It is positive when

(6.8) $\qquad A > C,\ A > B$ and $\omega^2(A - C)(\sigma - m\omega^2) - (mg)^2 > 0.$

W_2 is equal to zero at 0 and also at $(\xi = \omega\eta, 0,...0)$. In order to obtain positive definiteness we have to consider higher order terms. Let us consider this part in W^* which contains (ξ, η).

$$\tilde{W}^* = mz_1^2(\xi^2 + \xi n - 2\omega^2\eta).$$

Let $\xi = \omega\eta$.

$$\tilde{W}^*\big|_{\xi=\omega\eta} = mz_1^2(-2\omega^2\eta + ...).$$

But

$$\eta = \sqrt{1 - \gamma_3^2 - \gamma_2^2} - 1 = -(\gamma_2^2 + \gamma_3^2)/2 + ...$$

Therefore, W is positive definite when (6.8) are satisfied and these are also the sufficient conditions for the stability of solution (6.7).

B. Necessary conditions

The equations of the perturbed motion in linear approximation are:

$$
\begin{aligned}
&\dot{\xi} = 0,\\
&B\dot{\omega}_2 + (A-C)\omega_3\omega = -\ mgz_1,\\
&C\dot{\omega}_3 + (B-A)\omega\omega_2 = 0,\\
(6.9)\qquad &\dot{z}_1 = z_2,\\
&\dot{z}_2 = z_1(\omega^2 - \sigma/m) - g\gamma_3,\\
&\dot{\eta} = 0,\\
&\dot{\gamma}_2 = \gamma_3\omega - \omega_3,\\
&\dot{\gamma}_3 = \omega_2 - \gamma_2\omega .
\end{aligned}
$$

The secular equation of this system reads:

$$\lambda^2(\lambda^6 + a_1\lambda^4 + a_2\lambda^2 + a_3) = 0,$$

where

$$a_1 = \{(A - C)(A - B)m\omega^2 + BC\sigma\}/BCm,$$

$$a_2 = \{\sigma\omega^2(A - C)(A - B) + \omega^2BC(\sigma - m\omega^2) - C(mg)^2\}/BCm,$$

$$a_3 = \{\omega^2(A - B)[(A - C)\omega^2(\sigma - m\omega^2) - (mg)^2]\}/BCm.$$

According to the preposition of section 4, we have to check the necessary conditions for stability i.e. $a_1 > 0$, $a_2 > 0$, $a_3 > 0$. But simple calculations give that these are the conditions (6.8).

So, we have proved

THEOREM 4. The necessary and sufficient conditions for the stability of solution (6.7) with respect to ω_s, γ_s, $s = 1,2,3$, z_1, z_2 are $A > C$, $A > B$, $\omega^2(A - C)(\sigma - m\omega^2) - (mg)^2 > 0$.

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