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General Numerical Programme for Solving Plan-Parallel Flow Problems

2. Introduction

A general, very simple but exact method, called Equivalent Electrodes Method (EEM), is developed in the paper. It is used for solving fluid flow problems. The method is based on the analogy between electrostatic and fluid flow problems and on proposed procedures for EEM applications in electrostatic field analysis [1-6]. Today, the EEM is very useful for numerical solution of Dirichlet's problems. Hence, the first good results obtained in electrostatics are for the evaluation of equivalent radius of uniform cylindrical conductors, having an arbitrary cross section [1]. The general application of EEM in the electrostatic field theory is presented in Refs. [2-6]. Using these investigations, general numerical programme for microstrip transmission lines analysis is developed [7]. The application of EEM is extended to the solution of magnetic field problems [8], to the theory of LF grounding problems [9] and to problems of heat flow [10].

The basic ideas of the EEM is as follows:

An arbitrary shaped electrode can be replaced by a finite system of equivalent electrodes. It enables that a large number of complicated problems be reduced to an equivalent problem. Depending on the problem geometry, flat of oval strips and spherical or thoroidal equivalent electrodes are commonly used.

A general computer programme, called SHYDRO [11], is developed on the basis of theoretical investigations. Illustrative examples show high convergence and accuracy of the results obtained for velocity distribution, velocity potential, and stream function.

3. Brief theoretical approach

Any analytical function

(1) $$\phi\,(z = x + jy) = \varphi\,(x,y) + j\,\theta\,(x,y),$$

where $j = (-1)^{1/2}$, satisfies the Cauchi-Riemann equations. It has been shown that curves $\phi(x,y)$ = const and $\theta(x,y)$ = const are orthogonal and, therefore, they can be used to define the mutually perpendicular stress lines and equipotentials in a field of force. In electrostatics, for example, $\phi(x,y)$ is the electric scalar potential, $\varphi(x,y)$ = const. are equipotentials, $\theta(x,y)$ = const are force lines and $\vec{E} = -\operatorname{grad}\varphi$ is the electric field strength. In the hydrodynamics of two-dimensional perfect fluids $\theta(x,y)$ is velocity potential, $\varphi(x,y)$ is stream function and $\vec{v} = -\operatorname{grad}\theta$ is velocity.

We use for simplicity, but with sufficient generalization, the EEM for solving a simple electrostatic problem. It involves an infinite uncharged cylindrical conductor with a cross section of arbitrary shape introduced into a uniform transversal electrostatic field, $\vec{E}_o = E_o x$ (Fig.1). The complementary hydrodynamic problem considers an infinite cylindrical body in a uniform transversal parallel flow with velocity $\vec{V}_o = V_o y$ (Fig.2). The electric scalar potential of the system in Fig.1 satisfies the Laplace equation with boundary condition φ = U at the electrode surface S, and $\varphi \to -E_o x$ away from it. Consider N parallel strips on the electrode surface (A_n, A_{n+1}, n = 1,2,...N), having the potential of the real electrode and a charge per unit length q_n'. So the total charge of the electrode is

(2) $$q' = \sum_{n=1}^{N} q_n = 0,$$

because the electrode is uncharged.

The present strips can be replaced by cylindrical conductors, having an equal radius a_{en} with respect to the strips,and equal potentials U and charge per unit of length q_n. The equivalent radius of the flat strip, having a width d, is $a_e = d/4$. For an oval strip, with an angular width α and radius a ($a\alpha$ is the strip width), the equivalent radius is $a_e = a \sin(\alpha/4)$. So the real electrode can be replaced by an equivalent cage structure. Approximately, the complex potential is

(3) $$\Phi = \Phi_o - E_o z - \sum_{n=1}^{N} g_n' \ln(z - z_n),$$

where Φ_o is constant, $g_n' = g_n'/2\pi\varepsilon$, n = 1,2,..N and ε is the electric permeability. $z_n = x_n + jy_n$ define the position of the strip middle point or the equivalent electrode axis. So, the electric scalar potential is

(4) $$\varphi = \operatorname{Re}\{\Phi\} = \varphi_o - E_o x - \sum_{n=1}^{N} g_n' \ln|\vec{r} - \vec{r}_n|$$

where $\varphi_o = Re\{\Phi_o\}$, $\vec{r}$ is the radius vector of the field point and $\vec{r}_n$ is the radius vector is the equivalent electrode middle point. When N is large, and the strip largest width is small, the logarithmic potential theory can be used and the unknown charges per unit length q'_n can be determined by solving the following system of linear equations:

$$U = \varphi_o - E_o x_m - \frac{1}{2} \sum_{n=1}^{N} g'_n \ln [(\vec{r}_n - \vec{r}_m)^2 + a_{en}^2 \delta_{nm}], \tag{5}$$

m = 1,2,...N

where δ is the Kronecker delta. We determine the unknown charges of the electrode elements by solving the system of linear equations (5). After that, the electrostatic value sought can be determined by a standard procedure. For instance, the strength of the electrostatic field at an arbitrary point is

$$\vec{E} = - \text{grad } \varphi, \tag{6}$$

except for the points at the electrode surface, where we use the normal component of the boundary condition:

$$E_{n,norm} = | d\Phi / dz | = \eta_n / \varepsilon, \tag{7}$$

where $\eta_n = q'_n / d_n$ is the surface charge density and d_n is the width of the strip.

The EEM is generally useful in electrostatics. The basic idea of the proposed theory is that an arbitrarily shaped electrode can be replaced by a finite system of equivalent electrodes. It enable that a large number complicated problems be reduced to equivalent ones. Depending on the problem geometry, flat or oval strips (for plan-parallel systems), spherical (for three-dimensional fields) or thoroidal ones for (for electrode systems with axial symmetry) are commonly used. Differently from the charge simulation method [12], when the fictitious sources are introduced into the electrode volume, the equivalent electrodes are located on the body surface.

It is very important to notice that it is not necessary to perform a numerical integration when the EEM is applied. The obtained electrostatic solution (3) is directly used in hydrodynamics. If we introduce in eqn. (3) $E_o = V_o$, the system velocity (Fig.3) takes the form

$$\vec{v} = v_x \hat{x} + v_y \hat{y} = - \text{grad } \theta, \tag{8}$$

where

$$\theta = \text{Im } \{\Phi\} = \theta_o - v_o y - \sum_{n=1}^{N} g'_n \text{ arctg } [(y - y_n)/(x - x_n)], \tag{9}$$

$$\theta_o = \text{Im } \{\Phi_o\},$$

(10) $$v_x = - \sum_{n=1}^{N} g'_n \ (y - y_n)/ \ [(x - x_n)^2 + (y - z_n)^2]$$

and

(11) $$v_y = v_o + \sum_{n=1}^{N} g'_n \ (x - x_n)/ \ [(x - x_n)^2 + (y - y_n)^2]$$

The velocity tangential component at the body surface is

(12) $$v_{tan} = | \, d \, \Phi / \, d \, z |.$$

4. Computer system SHYDRO

Using the present theoretical analysis and the electrostatically oriented computer programmes S1901, S835 and SPLOCA [4,5,7], we form the general programme SHYDRO for solving problems of plan-parallel fluid flow [11]. We consider plan-parallel systems with several cylindrical bodies. The number and the cross sections of the bodies are chosen arbitrarily. The number of the equivalent electrodes can be automatically selected or, for each electrode set, this can be done separately. We determine velocity potential, stream function or velocity distribution, if necessary.

5. Conclusions

The concept of equivalent electrodes is extended to the analysis of fluid flow problems. The general computer code SHYDRO is developed. The obtained numerical results are compared to the corresponding existing exact or approximate values and we can conclude that the EEM results have a very good accuracy and convergence with respect to the number of the equivalent electrodes. Differently from the existing numerical methods, the EEM application does not require a numerical integration.

4. References

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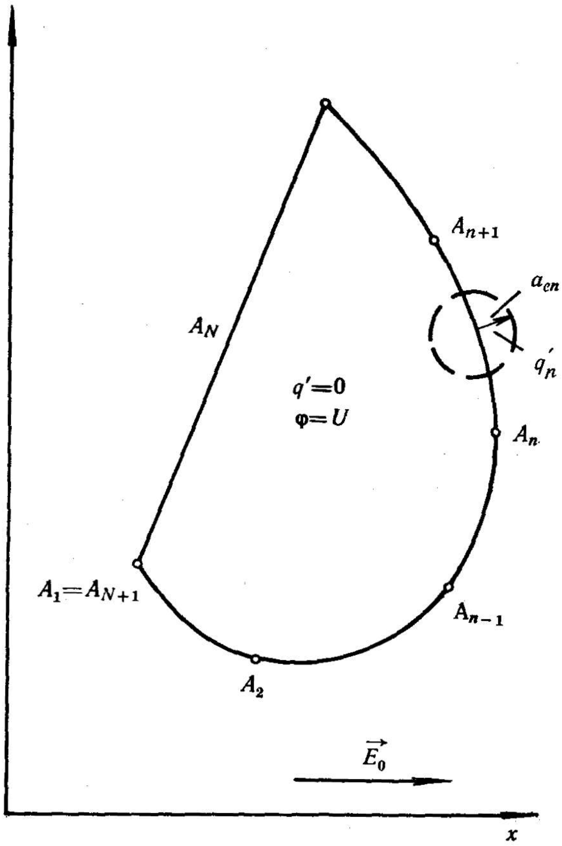

Fig.1 Infinite cylindrical conductor in uniform electrostatic field.

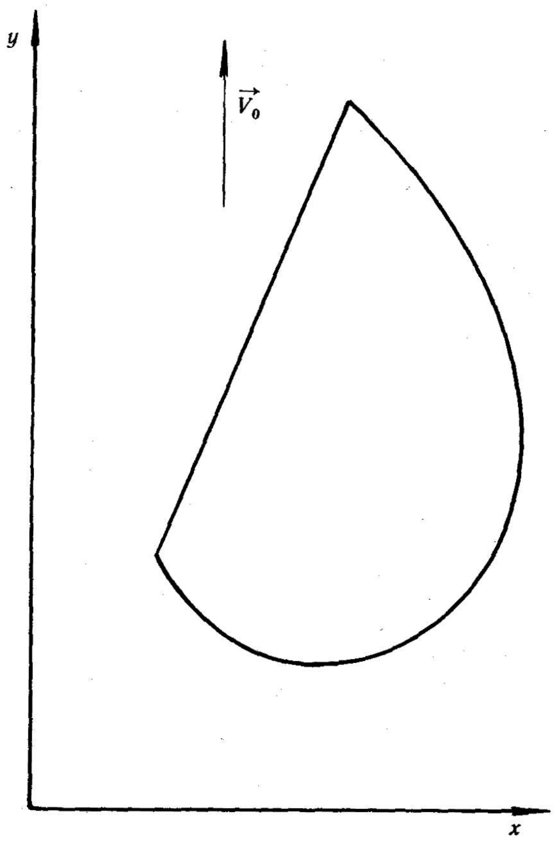

Fig.2 Infinite cylindrical body in uniform parallel flow.