Journal of Theoretical and Applied Mechanics
Sofia, 1992, Year XXIII, No 2

N. Petrov
M. Mikrenska

An Effective Algorithm for Solving Convection - Diffusion Problem

I. Introduction

The problem to be solved in this paper is the transient convective diffusion boundary problem:

$$\frac{\partial}{\partial t} c(x,t) + v\nabla c(x,t) + \nabla k \nabla c(x,t) = f(x,t) \ , \ x \in V \tag{1}$$

$$c(x,t)\big|_{x \in \Gamma^1} = g(x,t)$$

$$\alpha \frac{dc(x,t)}{dn} + \beta c(x,t) = p(x,t), \ x \in \Gamma^2$$

$$\Gamma^1 \cup \Gamma^2 = \Gamma = \delta V$$

$$c(x,0) = c_0(x), \quad x \in V$$

where:

v - convectional velocity

k - diffusive coefficient.

Despite of the applied numerical method for solving of the above problem a non symmetric linear system of equations is obtained as a result. Commonly used methods for the solution of non symmetric linear systems of equations are based on Gauss elimination or Crout decomposition. They have a limited application in the solution of large 2D and 3D problems on any computer, due to their large storage requirements. Recently, several authors [7, 8] have turned to the generalized form of the conjugate gradient method (CG) to overcome these limitations. However the variants of CG developed for non symmetric systems [9] are less attractive than those developed for symmetric systems, since they require the storage of a certain number of previously iterated search directions [10]. Liesmann and Friend have recently proposed a time integration scheme that places the non symmetric component of the matrix at the old time level. Although this scheme makes possible the application of the preconditioned conjugate gradient method to systems of equations, it does not eliminate the above mentioned limitations.

This paper presents an alternative approach to the transient convective diffusion problem, that allows one to overcome the computational difficulties connected with the solution of a linear system with sparse nonsymmetric matrix coefficients, obtained from the Finite element solution of the problem.

Further consideration will be done under the assumption that the fluid is incompressible.

II. Finite element formulation

Consider the finite element analogue of the convection-diffusion problem under the assumption that a mass condensation has been carried out:

$$\frac{d}{dt}\{c\}+[A]\{c\}=\{F\}$$

where $A=A^s + A^a$.

The symmetric part of matrix A is the finite element analogue of the operator $\nabla k \nabla c$, the skew-symmetric part of A is the finite element analogue of $v\nabla c$.

As it is known A is a positive definite matrix.

III. Solution technique

The formal solution of (1) in the interval $[t_i, t_{i+1}]$ leads to:

$$c(t_{i+1}) = e^{-A(t_{i+1}-t_i)} c(t_i) + \int_{t_i}^{t_{i+1}} e^{-A(t_{i+1}-t)} F(t)dt \qquad (2)$$

After substitution

$$P_m = e^{-A(t_{i+1}-t_i)},$$

where P_m is the m-th partial sum of the series

$$\sum_{s=0}^{\infty} (-1)^s A^s (t_{i+1}-t_i)^s$$

we obtain

(3) $$\bar{c}_{i+1} = P_m \bar{c}_i + Q_m F_i h ,$$

where $h = t_{i+1} - t_i$,

F_i is supposed to be time independent in $[t_i, t_{i+1}]$.

$\bar{c}_i$ approximates $c_i = c(t_i)$, $Q_m = \sum_{j=0}^{m-1} (-1)^j \frac{(Ah)^j}{(j+1)!}$.

The following scheme is applied for the calculation of $\bar{c}_{i+1}$:

(4)
$$\gamma^{(k)} = -\frac{Ah}{k+1}\gamma^{(k-1)}$$

$$\bar{c}^{(k)}_{i+1} = \bar{c}^{(k-1)}_{i+1} + \gamma^{(k)} , \quad k=1,2,...,m$$

The initial values of γ and $\bar{c}_{i+1}$ are determined by the formulas:

$$\gamma^{(0)} = (F_i - A\bar{c}_i)h$$

$$\bar{c}^{(0)}_{i+1} = \bar{c}_i + \gamma^{(0)} , \quad \bar{c}_0 = c_0$$

The error of approximation is

$$\varepsilon = \| c_{i+1} - \bar{c}_{i+1} \| = O(h^{m+1}).$$

IV. Stability

In this section we shall derive a sufficient condition for stability of the proposed scheme.

After m consecutive substitution in /3/ we obtain

$$\bar{c}_{i+1} = \bar{c}^{(m)}_{i+1} = P_m^{i+1} c_0 + Q_m h (F_0 P_m^{i} + F_1 P_m^{i-1} + ... + F_{i-1} P_m^{1} + F_i)$$

Suppose that $m=2l+1$ is an odd number. Then

(5)
$$\| P_m \| = \| \sum_{j=0}^{2l+1} (-1)^j \frac{(Ah)^j}{j!} \| = \| \sum_{j=0}^{l} \frac{(Ah)^{2j}}{(2j)!} (E - \frac{Ah}{2j+1}) \| \le$$
$$\le \max_j \|(E - \frac{Ah}{2j+1})\| \sum_{j=0}^{l} \frac{\|Ah\|^{2j}}{(2j)!} = \max_j \|(E - \frac{Ah}{2j+1})\| \frac{1-\|Ah\|^{m+1}}{1-\|Ah\|^2}$$

It is not difficult to show that

(6)
$$\max_j \|(E - \frac{Ah}{2j+1})\| \le \sqrt{1+\lambda^2 h^2 + \frac{\mu h}{m}} \, .$$

λ is the spectral norm of A, $\mu>0$ is determined by the unequallity $((A^T+A)y,y) \ge \mu(y,y)$. A is a positive definite matrix.

From (5) and (6) it follows that unequallity (7) ensures the stability of the time-differencing scheme (3).

(7)
$$h \le \min \left[\frac{1}{\lambda}, \frac{\mu}{3m\lambda^2} \right]$$

V. Implementation

The basic operation in scheme (4) is the calculation of an (A.X) product. The original implementation, developed by the authors, is applied for a compact storage of sparse matrix A and matrix-vector multiplies. For more details see [6].

VI. Summary and conclusions

An alternative approach for the solution of convection-diffusion problem is proposed. A sufficient condition for the time-differencing scheme stability has been found and proved by the authors. Some of the most appealing features of the proposed technique are the following:

1) One can obtain the solution of the problem at the moment t_i as a result of several matrix-vector multiplications instead of as a result of a great number of operations using conventional methods for solving of the corresponding system of linear equations. This reduces considerably the CPU time.

2) The sparse matrix structure has been fully exploited so as to reduce memory requirements.

3) Especially for the 3D convective-diffusion problem the computational advantage of the proposed technique is undisputable because of the combining both of the simplicity of the differencing scheme and the effectiveness of the applied storage scheme [6] in this case.

References

1. C h a n g, A. Applications of sparse matrix methods in electrical power system analysis. — IBM Research Rept. RA1#11707, in: Sparse Matrix Proceedings, Yorktown Heights, N.Y., 1969.
2. G e o r g e, A., J. L i u. A minimal storage implementation of the minimum degree algorithm. — SIAM J. Numer. Anal., **17**, 1980, 282-299.
3. G e o r g e, A., J. L i u. Computer solution of large sparse positive definite systems. — Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1981.
4. G e o r g e, A., D. R. M c I n t y r e. On the application of the minimum degree algorithm to finite element systems. — SIAM J. Numer. Anal., **15**, 1978, 90-112.
5. P i s s a n e t z k y, S. — Sparse Matrix Technology, Academic Press Inc., London, 1984.
6. M i k r e n s k a, M. D., N. K. P e t r o v. A sparse matrix scheme for finite elements. — Finite Elements in Analysis and Design, **19**, 1992, 47-53.
7. O b e y s k a r e, U. R. B., M. B. A l l e n, R. E. E w i n g, J. H. G e o r g e. Application of conjugate gradient-like methods to a hyperbolic problem in porous media. — Int. J. Num. Meth. in Fluids, **7**, 1987, 551-555.
8. P i n i, G., G. G a m b o l a t i. 3-D Finite element transport models by upwind preconditioned conjugate gradients. — Comp. Methods in Water Resources, **2**, 1988, 35-43.
9. G r e s h o, P. M. Time integration and conjugate gradient methods for the incompressible Navier-Stokes equations. — Finite Elements in Water Resources, 1986, 3-27.
10. E l m a n, H. C. Iterative methods for large sparse nonsymmetric systems of linear equation. — Doctoral Dissertation, Yale University, 1982.
11. L e i s m a n, H. M., E. O. F r i e n d. A symmetric-matrix technique for the efficient solution of advection-dispersion problems with very large finite element grids. — Water Resources Research, 1989.