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Mathematical and Numerical Modelling of a Casting Formation in Complicated Forms.

1. Introduction

A principal difficulty in the mathematical modelling of casting formation is the simultaneous treatment of hydrodynamical, thermomechanical and crystallization processes in moulds of complicated geometry. Sometimes this difficulty is enhanced further, when the shape is non-stationary, e.g. when the solidification begins during the filling of the mould, etc. Well known are the methods for the numerical solution of heat and mass transfer problems, in which a curvilinear boundary fitted coordinate system is constructed, [1,2]. The coordinate surfaces are obtained as equipotential solutions of an elliptic system of differential equations with boundary conditions of a Dirichlet's type imposed on them. In these methods the coordinate system is not adapted to the peculiarities of the physical process; besides, the method may become too cumbersome and unpractical, when the geometry of the region is non-stationary and the coordinate system has to be updated many times during the process. In [3,4] another method for treating of such problems was proposed - they are described and numerically solved in a Riemannian coordinate space, which is obtained by a metric mapping from the space of the real process. The mapping is constructed in such way, that a boundary fitted coordinate system is obtained, and the hydrodynamical equations, describing a laminar filling of the mould with a non compressible fluid, can be factorized under certain conditions over a family of coordinate surfaces S. These surfaces are congruent to the boundary surfaces of the region, and may have nonstationary metric on them. In this coordinate system the various problems of heat and mass transfer, related to the formation of the casting, could be treated simultaneously. In addition, the coordinate system is generated by algebraic methods, which makes more feasible the problem of it's updating in the case of non-stationary geometry.

2. Construction of the Riemannian space

Further the Greek indices $\alpha,\beta,...$ take values α, $\beta...=1,2$; the Latin indices $i,j,k,...=1,2,3$. Let us consider a medium, bounded by two surfaces S_1 and S_2 at a distance $h(x_\alpha,x_\beta)$, measured on the normal of S_1, where x_α and x_β are the curvilinear coordinates on S_1 (Fig.1). The metric space is constructed as a direct product of a one parameter family of surfaces $\hat{S}$ with a metric, congruent with the geometry of S_1 and S_2 and the field of the normal to the S_1 vectors. Let r_α are vectors in S_1, tangent to the coordinate curves x_α : $r_{\alpha_Q} = \frac{\partial R}{\partial x^\alpha}$, where R is the vector from an external coordinate system K_0 to the point Q on S_1. The metric on S_1 will be $g_{\alpha\beta} = (r_\alpha, r_\beta)$ where the scalar /product is defined by the external Euclidean metric in K_0. Then $N_Q = \frac{r_\alpha \times r_\beta}{g^{1/2}}\Big|_Q$ is the unit vector at the point Q on S_1. The components of the second quadratic form on the basic surface are $b_{\alpha\beta}=-(N,_\alpha,r^\beta)$. We parameterize the intermediate surfaces $\hat{S}$ by $x, 0\leq x\leq 1$. Then for two points A and B on $\hat{S}$, located over P and Q on S_1, is deduced:

$$R_A = R_P + xhN_P ; \qquad R_B = R_Q + xhN_Q$$

Let P and Q are located on the coordinate curve x. It follows that on $\hat{S}$:

$$\hat{r}_\alpha = r_\alpha + \lim_{P-Q} \frac{(h_P N_P - h_Q N_Q)}{||P-Q||} = r_\alpha - xhb_\alpha^c r_c + xN(\nabla h, r_\alpha)$$

The metric $G_{\alpha\beta}$ on $\hat{S}$ is derived :

$$G_{\alpha\beta} = (\hat{r}_\alpha, \hat{r}_\beta) = g_{\alpha\beta} - 2xhb_{\alpha\beta} + x^2h^2b_{\alpha\gamma}b^\gamma_\beta + x^2(\nabla h, r_\alpha)(\nabla h, r_\beta)$$

We choose x as the third coordinate. The metric of the Riemannian space becomes:

(1) $$G_{\alpha\beta} = (\hat{r}_\alpha, \hat{r}_\beta); \qquad G_{\alpha 3} = -xh(\nabla h, r_\alpha)/G^{1/2}_{\alpha\alpha} ; \qquad G_{33} = h^2$$

The equations of heat and mass transfer in metrics should be obtained in a covariant way in order to account for the effects of boundary curvature and nonstationarity. They are derived from the conservation laws of the energy-momentum tensor in space R

(2) mass $$(1/\sqrt{G})\partial_t(\sqrt{G}.\rho) + \nabla.(\rho v) = 0$$

momentum $$\rho(\partial_t v^i + v^k\nabla_k v^i + 2\Gamma^i_{tk}v^k) - \nabla_k t^{ik} = \rho F^i$$

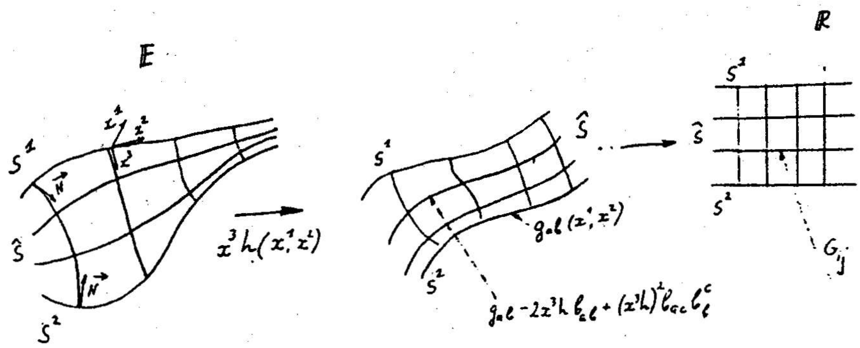

Fig. 1. Stages in the construction of the metric mapping to Riemannian space **R**.

3. Filling of a 3-D form

We shall describe the filling by a heavy uncompressible fluid of a 3-D form with small deviations from the cylindrical symmetry. The following restrictions are imposed on the geometry of the form: The size across the form $h(t,x)$ is smaller than the characteristic length scale l, $h/l=\varepsilon<<1$; The velocity at which the size across the form is changing, is smaller than the velocity of filling; There are no great horizontal parts of the form with length $l >> h$, where surface waves can emerge during the filling. With these conjectures accounted, it follows from the continuity equation in the space R, $1/\sqrt{G}\,\partial_t\sqrt{G} + \nabla_i v^i = 0$:

$$|v^3| < |\partial_t h| + \varepsilon\int_0^1 |u^a|\,dx^3 < |\partial_t h| + \varepsilon|u^a|$$

where u^a is the maximal velocity of the flow in a cross-section of the form. With the use of the small parameter ε, the velocity could be written down in the form of power series:

$$v^a = v_0^a + \varepsilon v_1^a + \varepsilon v_2^a + \dots$$

$$v^3 = \varepsilon v_1^3 + \dots$$

In this approximation the Navier-Stokes equations were obtained in [3,4,6] from the conservation equations (2) in the form:

$$\partial_t h + \nabla_a(hv^a) = 0$$

$$\rho(G^{ab}\partial_t(G_{bc}v^c) + v^b\nabla_b v^a) + \nabla^a p = \eta\Big({}^2\Delta v^a + \frac{1}{h^2}v^a_{,33} + \frac{1}{2}R^{(2)}v^a +$$

(3) $$G^{bc}v^a_{,c}\,h_{,b}/h - G^{bm}(h_{,m}/h)_{,b}v^a - h_{,r}\,h_{,b}G^{bm}v^a + \hat{G}^a_t\Big)$$

$$\hat{G}^k_t = -2\dot{G}^{kl}_{,l} + 2\dot{G}^{ls}\,G^{ik}_{,l}\,G_{is} + G^{ik}\,\dot{G}^{ms}\,G_{ms,i} - G^{ik}\,(\dot{G}/G)_{,i}$$

The term $\hat{G}^k_t$ describes the inertia effects, which appear when geometry changes with time; R is the scalar curvature on S.

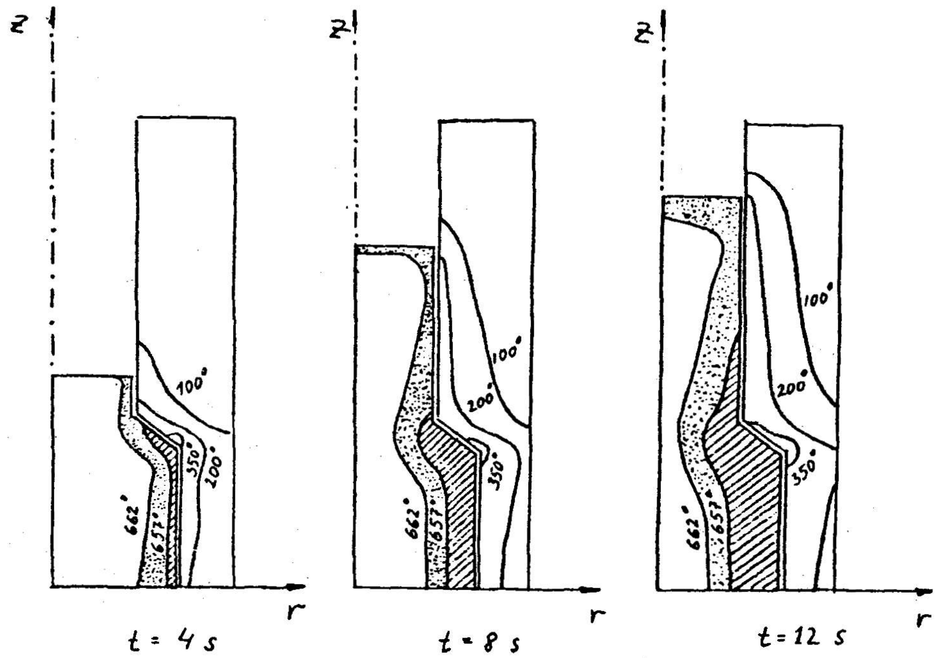

Fig. 2.

4. Filling of a mould with molten metal

We shall apply further this method to describe the filling of a mould by molten metal with simultaneous cooling and crystallization of the casting, including the case of nonstationary metrics too. Through the metric coefficients G, we shall take into account the changing geometry of the form when the crust is formed, and it's influence on the process of filling. The heat problem is solved in two regions - the cavity of the mould and the mould itself. For both regions we construct a mapping from the real physical space to the Riemannian space with deliberately designed metric as pointed out above. In the mould the heat conductivity equation is:

(4) $$c\rho\partial_t T = \lambda\Delta T$$

In the mould cavity the heat equation is considered also in two regions - the one with the frozen metal, where equation (4) is solved, and the one with the molten metal where the heat equation is:

(5) $$\rho(c + L\partial_T \kappa)\partial_t T = \lambda \Delta T - \rho c v.\nabla T$$

Here v(t,r) is the velocity obtained from (3). The components of the metric from (1) are computed with the help of the functions g(t,r) and h(t,r) , which on their turn are obtained as solutions of equation (5), because the growing crust creates a new geometry of the form influences the flow. The heat problem was described in [5,6].

The results from the numerical simulation of the cavity filling of an axially symmetric form, with simultaneous cooling and solidification, is shown in Fig.2. Shaded area presents the solidified metal and dotted area - the two phase region. These examples show that the proposed method can be used for a simultaneous description of a whole complex of processes connected with the formation of a casting in forms and moulds with complicated and non-stationary geometry.

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