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On the Permanent Rotations of a Gyrostat-Satellite

Considered are permanent rotations of an asymme ric gyrostat-satellite on a circular orbit: together with the orbital motion, the satellite rotates with an uniform angular velocity about an axis orientated arbitrarily, fixed in the body, passing through the mass centre and keeping its orientation along the normal to the orbital plane. This paper considers the problem of finding the motion of the rotors, guaranteeing the realization of such a permanent rotation (i. e. the control of the rotors).

Let the mass-centre of the satellite move on a circular orbit in a Newtonian field of forces. We consider the restricted problem, assuming that this movement is not affected by the satellite's motion about the mass-centre. Assume that the rotors are three and their spin axes are fixed in the body along the principal central axes of inertia.

Let $O_1\xi_i$ be an inertial coordinate frame with an origin in the gravitational centre O_1. Introduce also two other right hand coordinate trihedrals: OX_i — an orbital frame with an origin O in the mass-centre and with axes OX_1 — along the tangent to the orbit, OX_2 — along the normal to the orbital plane, OX_3 along the vector $\overrightarrow{O_1O}$ and Ox_i — a frame, fixed in the satellite, which axes coincide with the pricipal axes of inertia.

Following [1], the orientation of the satellite with respect to the orbital frame we shall determine by means of the angles α, β, γ (pitch, yaw and roll angles).

The direction cosines between the axes Ox_i and OX_i are defined by the formula

$$
\begin{aligned}
(1)\qquad & \alpha_i=\cos(X_1,\, x_i), && \beta_i=\cos(X_2,\, x_i), && \gamma_i=\cos(X_3,\, x_i),\\
& \alpha_1=c\alpha c\beta, && \beta_1=s\beta, && \gamma_1=-s\alpha c\beta,\\
& \alpha_2=s\alpha s\gamma-c\alpha s\beta c\gamma, && \beta_2=c\beta c\gamma, && \gamma_2=c\alpha s\gamma+s\alpha s\beta c\gamma,\\
& \alpha_3=s\alpha c\gamma+c\alpha s\beta s\gamma, && \beta_3=-c\beta s\gamma, && \gamma_3=c\alpha c\gamma-s\alpha s\beta s\gamma,
\end{aligned}
$$

$(s\alpha=\sin\alpha,\ c\alpha=\cos\alpha,\ \dots)$.

For the components $[\omega_1,\ \omega_2,\ \omega_3]^T$ of the absolute angular velocity $\bar{\omega}$ in the body-fixed frame we have the expressions

$$
(2)\qquad
\begin{aligned}
\omega_1&=(\omega_0+\dot{\alpha})\,\beta_1+\dot{\gamma},\\
\omega_2&=(\omega_0+\dot{\alpha})\,\beta_2+\dot{\beta}\sin\gamma,\\
\omega_3&=(\omega_0+\dot{\alpha})\,\beta_3+\dot{\beta}\cos\gamma,
\end{aligned}
$$

where ω_0 is the orbital velocity and dots denote derivatives with respect to the time t

Assume the following notations:

A_i — the principal moment of inertia of the gyrostat about the axis Ox_i;
J_i — the moment of inertia of the i-th rotor about its spin axis;
φ_i — the angle of rotation of the i-th rotor;
m_i — the torque acting on the i-th rotor;
k_i — the angular momentum of the i-th rotor about its spin axis $(k_i=J_i\dot{\varphi}_i)$; $(i=1,2,3)$.
With the assumption that there are not any other forces applied to the gyrostat than the gravitational ones, the dynamical equations for the motion of the satellite about its mass-centre can be written in the form

$$A_1\dot{\omega}_1+\dot{k}_1-(A_2-A_3)\,\omega_2\omega_3+k_3\omega_2-k_2\omega_3=-3\omega_0^2(A_2-A_3)\,\gamma_2\gamma_3, \quad [1,\ 2,\ 3]. \tag{3}$$

[1, 2, 3] denotes that the other two equations can be obtained by a cyclic permuta tion of the indexes.

The equations for the rotors are

$$\dot{k}_i+J_i\dot{\omega}_i=m_i \quad (i=1,\ 2,\ 3). \tag{4}$$

Consider an arbitrary permanent rotation of the satellite with angular velocity Ω about an axis fixed in the body, directed to the orbital plane-normal OX_2. In this motion the angles β and γ have constant values $\beta=\beta_0$, $\gamma=\gamma_0$, defining the orientation of the axis of rotation in the body-fixed frame, while the angle α is a linear function of the time $\alpha=\alpha_0+\Omega t$. Substituting $\alpha=\alpha_0+\Omega t$, $\beta=\beta_0$, $\gamma=\gamma_0$ into (1), (2) and (3) we obtain the system

$$\dot{k}_1=-(\omega_0+\Omega)[k_3\beta_2-k_2\beta_3]+(\omega_0+\Omega)^2(A_2-A_3)\,\beta_2\beta_3-3\omega_0^2\,(A_2-A_3)\,\gamma_2\gamma_3, \quad [1,\ 2,\ 3]$$

which can be written in the following matrix from

$$\dot{\mathbf{k}}=\mathbf{A}\mathbf{k}+[\mathbf{c}+\mathbf{f}\,(t)], \tag{5}$$

$$\mathbf{k}=[k_1,\ k_2,\ k_3]^T;\ \mathbf{c}=(\omega_0+\Omega)^2[(A_2-A_3)\,\beta_2\beta_3;\ (A_3-A_1)\,\beta_3\beta_1;\ (A_1-A_2)\,\beta_1\beta_2]^T$$

$$\mathbf{f}\,(t)=-3\omega_0^2\,[(A_2-A_3)\,\gamma_2\gamma_3;\ (A_3-A_1)\,\gamma_3\gamma_1;\ (A_1-A_2)\,\gamma_1\gamma_2]^T,$$

$$\mathbf{A}=\begin{bmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{bmatrix}.$$

The solution of (5) determines the motion of the rotors guaranteeing the realization of the considered permanent rotation.

The method of variation of constants applied to the above system gives the formula

$$\mathbf{k}\,(t)=\mathbf{K}\,(t)\,\mathbf{K}^{-1}\,(0)\,\mathbf{k}_0+\mathbf{K}\,(t)\int_0^t \mathbf{K}^{-1}\,(\tau)[\mathbf{c}+\mathbf{f}\,(\tau)]\,d\tau, \tag{6}$$

where $\mathbf{K}(t)$ is the fundamental matrix of the corresponding homogeneous system $\dot{\mathbf{k}}=\mathbf{A}\mathbf{k}$ and $\mathbf{k}_0=[k_1^0,\ k_2^0,\ k_3^0]^T$ are the initial values of the gyrostatic moments.

Using the well known procedure for finding a fundamental matrix of a linear homogeneous system with constant coefficients we obtain that $K(t)$ is equal to the transformation matrix from the coordinate frame OX_i to the body-fixed frame Ox_i, supposing that the angle α is the function $\alpha=(\omega_0+\Omega)t$, i. e.

$$K(t)=\begin{bmatrix}\hat{\alpha}_1 & \beta_1 & \hat{\gamma}_1\\ \hat{\alpha}_2 & \beta_2 & \hat{\gamma}_2\\ \hat{\alpha}_3 & \beta_3 & \hat{\gamma}_3\end{bmatrix},\qquad \begin{aligned}\hat{\alpha}_i&=\alpha_i\,|_{\alpha=(\omega_0+\Omega)\,t},\\ \hat{\gamma}_i&=\gamma_i\,|_{\alpha=(\omega_0+\Omega)\,t},\end{aligned}$$

(α_i and γ_i are given by (1)).
By virtue of this fact, the matrix $K^{-1}(t)$ is equal to the transposed $K^T(t)$.

Assume also the notations

$$\alpha_i^0=\alpha_i\,|_{\alpha=\alpha_0},\ \gamma_i^0=\gamma_i\,|_{\alpha=\alpha_0},\ \hat{\alpha}_i^0=\alpha_i\,|_{\alpha=0},\ \hat{\gamma}_i^0=\gamma_i\,|_{\alpha=0}\quad(t=0),$$

$$P=\sum_{i=1}^{3}A_i\beta_i\alpha_i,\quad \hat{P}=\sum_{i=1}^{3}A_i\beta_i\hat{\alpha}_i,\quad Q=\sum_{i=1}^{3}A_i\beta_i\gamma_i,\quad \hat{Q}=\sum_{i=1}^{3}A_i\beta_i\hat{\gamma}_i,$$

$$S=\sum_{i=1}^{3}A_i\gamma_i^2;\quad P^0=P\,|_{t=0};\quad Q^0=Q\,|_{t=0};\quad S^0=S\,|_{t=0}.$$

It is easy to verify the following identities

$$\begin{aligned}
(7)\qquad &\alpha_3\gamma_2-\alpha_2\gamma_3=\beta_1,\ [1,\ 2,\ 3],\\
&\hat{\alpha}_i=\cos(\omega_0t-\alpha_0)\,\alpha_i+\sin(\omega_0t-\alpha_0)\,\gamma_i,\\
&\hat{\gamma}_i=\cos(\omega_0t-\alpha_0)\,\gamma_i-\sin(\omega_0t-\alpha_0)\,\alpha_i,\\
&\alpha_i=\cos(\omega_0t-\alpha_0)\,\hat{\alpha}_i-\sin(\omega_0t-\alpha_0)\,\hat{\gamma}_i,\\
&\gamma_i=\sin(\omega_0t-\alpha_0)\,\hat{\alpha}_i+\cos(\omega_0t-\alpha_0)\,\hat{\gamma}_i,\\
&\hat{\alpha}_i\cos\alpha_0+\hat{\gamma}_i\sin\alpha_0=\alpha_i\cos\omega_0t+\gamma_i\sin\omega_0 t,\\
&\hat{\gamma}_i\cos\alpha_0-\hat{\alpha}_i\sin\alpha_0=\gamma_i\cos\omega_0t-\alpha_i\sin\omega_0t,\\
&\partial\gamma_i/\partial\alpha=-\alpha_i,\ \partial\alpha_i/\partial\alpha=\gamma_i,\\
&\cos(\omega_0t-\alpha_0)\,\hat{\alpha}_i^0-\sin(\omega_0t-\alpha_0)\,\hat{\gamma}_i^0=\alpha_i^0\cos\omega_0t-\gamma_i^0\sin\omega_0t,\\
&\sin(\omega_0t-\alpha_0)\hat{\alpha}_i^0+\cos(\omega_0t-\alpha_0)\,\hat{\gamma}_i^0=\alpha_i^0\sin\omega_0t+\gamma_i^0\cos\omega_0t,\\
&\hat{P}\hat{\alpha}_i+\hat{Q}\hat{\gamma}_i=P\alpha_i+Q\gamma_i,\\
&P^0\cos\Omega t+Q^0\sin\Omega t=P,\ P^0\sin\Omega t-Q^0\cos\Omega t=Q.
\end{aligned}$$

Calculating the integrals in (6), using the relations (7), for the solution of (5) we obtain

$$\begin{aligned}
(8)\qquad &k_i=k_i^{(h)}+k_i^{(p)},\\
&k_i^{(h)}=\alpha_i\sum_{s=1}^{3}(\alpha_s^0\cos\omega_0t-\gamma_s^0\sin\omega_0t)\,k_s^0+\beta_i\sum_{s=1}^{3}\beta_sk_s^0
\end{aligned}$$

$$+\gamma_i \sum_{s=1}^{3} (\alpha_s^0 \sin\omega_0 t + \gamma_s^0 \cos\omega_0 t)\, k_s^0,$$

$$k_i^{(p)} = -(\omega_0+\Omega)[(P-P^0\cos\omega_0 t + Q^0 \sin\omega_0 t)\,\alpha_i$$
$$+(Q-P^0\sin\omega_0 t - Q^0\cos\omega_0 t)\,\gamma_i]$$
$$+\frac{3\omega_0^2}{\omega_0^2-\Omega^2}[(-\Omega P+\Omega P^0\cos\omega_0 t+\omega_0 Q^0\sin\omega_0 t)\,\alpha_i$$
$$+(\omega_0 Q+\Omega P^0\sin\omega_0 t-\omega_0 Q^0\cos\omega_0 t)\,\gamma_i]-$$
$$-\frac{3\omega_0^2}{2\Omega}(S-S^0)\,\beta_i.$$

From (8) it follows that every permanent rotation about an axis, orientated arbitrarily in the body, can be realized by means of finite angular velocities of the rotors only if $\Omega \neq \pm\omega_0$.

Consider the special cases of rotations with $\Omega = \pm\omega_0$.

Let the rotational velocity is $\Omega = -\omega_0$. In this case the satellite keeps its orientation in the inertial space. The system (5) reduces to the matrix equation $\dot{\mathbf{k}} = \mathbf{f}(t)$. After its integration we find that the motions of the rotors guaranteeing these permanent rotations are defined by the expressions

$$(9) \qquad k_1 = \frac{3\omega_0^2}{4}(A_2-A_3)[2\omega_0\beta_2\beta_3 t + (\alpha_2\gamma_3+\alpha_3\gamma_2)-(\alpha_2^0\gamma_3^0+\alpha_3^0\gamma_2^0)]$$
$$[1,\ 2,\ 3].$$

Hence, if the axis of rotation is orientated arbitrarily in the body, the velocities of the rotors increase infinitely with time, because of the terms $2\omega_0\beta_i\beta_j t$. There are only two cases, where a permanent rotation with $\Omega = -\omega_0$ can be realized by means of finite angular velocities of the rotors:

1) If the satellite is asymmetric, the axis of rotation has to coincide with one of the inertia axes.

2) If the satellite has an axis of dynamical symmetry, the axis of rotation has to be orthogonal to this one (the axis of rotation is also a principal inertia axis). The case of absolute symmetry ($A_1=A_2=A_3$) is trivial and will not be considered.

Consider the second special kind of rotations with angular velocity $\Omega=\omega_0$. In order to find the motion of the rotors, we let Ω tend to ω_0 in (8). We obtain

$$(10) \qquad k_i^* = \lim_{\Omega\to\omega_0} k_i = F(t)+G(t),$$

$$F(t) = \alpha_i \sum_{s=1}^{3}(\alpha_s^0\cos\omega_0 t-\gamma_s^0\sin\omega_0 t)k_s^0 + \beta_i\sum_{s=1}^{3}\beta_s k_s^0$$
$$+\gamma_i\sum_{s=1}^{3}(\alpha_s^0\sin\omega_0 t+\gamma_s^0\cos\omega_0 t)\,k_s^0 - 2\omega_0[(P-P^0\cos\omega_0 t$$
$$+Q^0\sin\omega_0 t)\,\alpha_i+(Q-P^0\sin\omega_0 t-Q^0\cos\omega_0 t)\,\gamma_i]$$
$$-\frac{3}{2}\omega_0\sin\omega_0 t\,[\alpha_i Q^0+\gamma_i P^0]-\frac{3}{2}\omega_0\beta_i(S-S_0);$$

$$G(t)=-\frac{3}{2}\omega_0^2 t(\alpha_i Q+\gamma_i P).$$

Here, like in the case above, the presence of the term $G(t)$ makes impossible the realization of a permanent rotation about an axis orientated arbitrarily in the body, by means of finite velocities of the rotors. Only permanent rotations about an axis which is a principal inertia axis, can be realized by means of finite rotors' velocities.

Consider now the the equilibrium orientations of the gyrostat. They can be interpreted as permanent rotations with an angular velocity equal to zero ($\Omega=0$). Let Ω tend to zero in expressions (8). We obtain

(11)
$$\tilde{k}_i=\lim_{\Omega\to 0} k_i=-\left[P^0\omega_0+\frac{D}{\omega_0}\cos(\omega_0 t+\delta)\right]\alpha_i^0+(b_0 t+\lambda_0)\beta_i-\left[4\omega_0 Q^0+\frac{D}{\omega_0}\sin(\omega_0 t+\delta)\right]\gamma_i^0,$$

where

$$\frac{D}{\omega_0}\cos(\omega_0 t+\delta)=-P^0\cos\omega_0 t+4Q^0\sin\omega_0 t-\sum_{s=1}^{3}(\alpha_s^0\cos\omega_0 t-\gamma_s^0\sin\omega_0 t)k_s^0,$$

$$\frac{D}{\omega_0}\sin(\omega_0 t+\delta)=-P^0\sin\omega_0 t-4Q^0\cos\omega_0 t-\sum_{s=1}^{3}(\alpha_s^0\sin\omega_0 t+\gamma_s^0\cos\omega_0 t)k_s^0,$$

$$b_0=3\omega_0^2\sum_{s=1}^{3}A_s\alpha_s^0\gamma_s^0;\quad \lambda_0=\sum_{s=1}^{3}\beta_s k_s^0$$

($D=$const, $\delta=$const).

These formulas define the motion of the rotors, guaranteeing the realization of an arbitrary equilibrium orientation (α_0, β_0, γ_0) [1]. It's clear, that only the orientations, satisfying the condition

(12)
$$\sum_{s=1}^{3}A_s\alpha_s^0\gamma_s^0=0$$

can be realized by means of finite velocities of the rotors. The equation (12) defines the well known variety of the possible equilibrium orientations.

References

1. Anchev, A. Equilibrium Orientations of a Satellite with Rotors. Sofia, Bulg. Acad. Sci., 1982.

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