

NETLIKE REPRESENTATION OF THE THERMOELASTIC MEDIUM

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1. Geometric Structure of the Medium, Netlike Element, Netlike Medium

Let Ω mark the homocoherent area of Euclides' space that corresponds with the situation of the investigated elastic body in natural state. Let us divide the area into the countable number of polyhedrons at most W^i , $i=1, 2, \dots, N$. Assuming that those polyhedrons are small in relation to the area Ω and that their sum Ω' is a good approximation of the area in each polyhedron $W^{(i)}$ we can distinguish the point pattern

$$(1) \quad E^i = [e^j]^i, \quad i=1, 2, \dots, M$$

and in each of those sets E^j we can determine bilateral relation D^j

$$(2) \quad D^i \subset E^i \times E^i.$$

Orderly pair of sets $S^i = \langle E^i, D^i \rangle$ is called geometric structure of the netlike element. Introducing the notations

$$(3) \quad E^\Omega = \bigcup_{(i)} E^i, \quad D^\Omega = \bigcup_{(i)} D^i$$

we can determine geometric structure of the netlike model of the body as an orderly pair of sets $S^\Omega = \langle E^\Omega, D^\Omega \rangle$. The elements e^i of the set E_Ω are called the nodes and elements $d^{ij} = (e^i, e^j)$ of the set D^Ω — connectors of the netlike model if we relate physical parameters to the elements of the sets E^i and D^i of the structure S^i then the geometric structure along with the describing functions is called netlike element Z^i . The netlike element constitutes the local representation of the modelled medium. The set of those elements in the zone of the examined body is the body's netlike model. Representation which relates the element e^j being the translation of the element e^i of vector a to each of the elements e^j of the set E^i constitutes the translation of vector a

$$(4) \quad r^j = r^i + a$$

and it corresponds to the relation D , i. e.

$$(5) \quad ((e^j, e^k) \in D^i) \Leftrightarrow ((e^j, e^k) \in D^i),$$

where r is a running vector of a point of Euclides' space. If there are such three vectors a, b, c that for each pair of netlike elements of the body's model Z^i, Z^j it is possible to find there natural numbers N, M, P of such a kind that the structure of one of the

elements e. g. Z^j is the structure translation of the second element — Z^j of the vector (Na+Mb+Pc) then the body structure S^Ω is called regular. If the sets E^i of netlike elements include only all the polyhedron vertexes W^i of a given element then the structure S^Ω is called normal. Geometric structure representation of the netlike element which gives the same geometric structure is called geometric structure symmetry of the netlike element or medium geometric symmetry if the structure is regular. The regularity structure condition of the netlike model limits the number of possible symmetry structure groups to 32 [1, 2] — the number includes the symmetry groups corresponding to the open polyhedrons. For each node we can determine the sets ${}^iD \subset D^\Omega$ of the connectors

$$(6) \quad \{(d^{jk} \in {}^iD) \Leftrightarrow (j=i \vee k=i)\} \wedge \{(d^{jk} \notin {}^iD) \Leftrightarrow (j \neq i \wedge k \neq i)\}$$

it means that the set iD includes all the connectors including the node e^i and only such ones. A pair of sets ${}^iS = \langle {}^iE, {}^iD \rangle$ where ${}^iE = e^i$ is called a geometric structure of dual netlike element. The structure iS along with functions describing physical properties of its elements is called dual netlike element iZ . Further on — if anything else will not be introduced — we will be considering the medium of normal and regular structure dual element iZ , whose set iD has got the power that is equal to the power of sets D^j of element structure, called an inner element. If not, dual element iZ is called boundary one. The area iW of dual element iZ is called even polyhedron exact to a polyhedron translation W^i of netlike elements whose symmetry centre is node e^i . For a boundary element the area is a common part of the region described above and the region Ω' . Elastic material in netlike representation is called elastic netlike medium.

2. Deformation state of the netlike medium

2.1. Symmetric medium

Let us take into consideration the netlike elements Z^i . Let r^j determine a running vector of node e^j of that element before distortion, while vector $r^{j'}$ — after distortion. In the considered orthocartesian coordinates system the coordinates of those vectors for the nodes e^j and e^k are respectively

$$(7) \quad \begin{aligned} r^j &= \{x_i^j\}, & r^k &= \{x_i^j + x_i^{jk}\}, \\ r^{j'} &= \{\xi_i^j\}, & r^{k'} &= \{\xi_i^j + \xi_i^{jk}\}, \quad i=1, 2, 3. \end{aligned}$$

Coordinates of vector $r^{jk} = r^k - r^j$ are

$$(8) \quad r^{jk} = \{x_i^{jk}\},$$

while of vector $r^{j'k'}$ are

$$(9) \quad r^{j'k'} = \{\xi_i^{jk}\}.$$

The rise of second power of vector r^{jk} through the distortion of netlike element can be written as*)

$$(10) \quad r^{jk'^2} - r^{jk^2} = \xi_i^{jk} \xi_i^{jk} - x_i^{jk} x_i^{jk}, \quad i=1, 2, 3.$$

Let present the coordinates ξ_i^{jk} as the combination of coordinates x_i^{jk} ,

$$(11) \quad \xi_i^{jk} = \xi_{i,m} x_m^{jk}, \quad i, m=1, 2, 3,$$

where $\xi_{i,m}$ are numerical coefficients about which we can assume that $\xi_{i,m} = \xi_{m,i}$.

* Summing convention was used in the paper.

Because of the equations (11) the coefficients $\xi_{i,m}$ are not determined univocally by state of strain of one of the node pairs. Let us assume that there are 6 coefficients $\xi_{i,m}$ satisfying (11) for each node pair of a netlike element Z^i and also satisfying the condition of mutual explicitness of that representation

$$(12) \quad \det \xi_{i,m} = 0.$$

The connection (10) may be then presented as

$$(13) \quad r^{jk^2} - r^{jk^3} = (\xi_{i,m} \xi_{i,n} - \delta_{mn}) x_m^{jk} x_n^{jk} = 2e_{mn}^i x_m^{jk} x_n^{jk},$$

where δ_{mn} is Kronecker's delta.

The quantity on the left is invariant while translating and rotating the coordinate system.

It is evident that the right side of the equations has a property of invariance too. On that basis we can say that the object

$$(14) \quad e_{mn}^i = \frac{1}{2} (\xi_{i,m} \xi_{i,n} - \delta_{mn})$$

is the tensor of rank two. It is multipunctual tensor reforming to the node set E^i of netlike element [3]. We are going to call it the distortion tensor of element Z^i in which the tensor is determined. The state of distortion of the element that can be described by the distortion tensor is called homogeneous.

Let us mark the displacement vectors of nodes e^j and e^k respectively

$$(15) \quad \begin{aligned} u^j &= r^{j^1} - r^j, \\ u^k &= r^{k^1} - r^k. \end{aligned}$$

Subtracting the equation sides we get

$$(16) \quad u^{jk} = u_l^k - u_l^j = \xi_l^{jk} - x_l^{jk} = (\xi_{i,m} - \delta_{im}) x_m^{jk} = u_{i,m} x_m^{jk},$$

where

$$u_{i,m} = (\xi_{i,m} - \delta_{im})$$

have been marked.

The numbers $\xi_{i,m}$ are configurating and the numbers $u_{i,m}$ are dislocation numbers of the influence in Lagrange's description of the netlike medium [4]. On the basis of (16) we can write

$$(17) \quad \xi_{i,m} = u_{i,m} + \delta_{im}$$

and if substituting to the definition equation of tensor $e_{mn}^{(i)}$ we get

$$(18) \quad \begin{aligned} e_{mn}^i &= \frac{1}{2} (\xi_{i,m} \xi_{i,n} - \delta_{mn}) = \frac{1}{2} [(u_{i,m} + \delta_{im}) (u_{i,n} + \delta_{in}) - \delta_{mn}] \\ &= \frac{1}{2} (u_{i,m} u_{i,n} + u_{m,n} + u_{n,m}). \end{aligned}$$

In the case of homogeneous distortion of the element Z^i the dependence (18) describes value of components $e_{mn}^{(i)}$ of distortion tensor of element by the influence number displacement. The geometric object determined in that way is equivalent to Green's tensor which is determined in the points of continuous medium [4]. On the basis of (11) and (16) we can write the dependence between vector coordinates before and after distortion

$$(19') \quad \xi_n^{jk} = (u_{n,m} + \delta_{nm}) x_m^{jk} = \left[\frac{1}{2} (u_{j,m} u_{j,n} + u_{m,n} + u_{n,m}) + \frac{1}{2} (u_{n,m} - u_{j,m} u_{j,n} - u_{m,n}) + \delta_{mn} \right] x_m^{jk}$$

or

$$(19'') \quad \xi_n^{jk} = (e_{nm}^i + \gamma_{mn}^i + \delta_{mn}) x_m^{jk},$$

where the quantity

$$(20) \quad \gamma_{nm}^i = \frac{1}{2} (u_{n,m}^i - u_{j,m}^i u_{j,n}^i - u_{m,n}^i)$$

is tensor measure of vector turn determined by any node pair of element Z^i and it constitutes the equivalent of Lagrange's tensor turn in the symmetric continuous medium. It is also multipunctual tensor for the netlike element. Decomposition of tensors of netlike elements constitutes the tensor distortion area of the discrete medium. There is an analogous possibility of introducing Euler's description of distortion state of the netlike medium [4, 5].

2.2. Non-symmetric medium distortion—along with the medium node turn

Let us assume that in the set E^i of the nodes of netlike medium vector function $\varphi(r)$ is determined which in the natural state of medium takes zero values. Vector $\varphi(r)$ is the measure of node turn whose position is determined by vector r . Let the values of that function in the state of body distortion be respectively equal to

$$(21) \quad \varphi^j = \{\varphi_j^i\}, \quad \varphi^k = \{\varphi_i^j + \varphi_i^{jk}\}, \quad i = 1, 2, 3$$

in the nodes e^i and e^k of netlike element.

Vector of relative turn of nodes e^j and e^k is presented as follows:

$$(22) \quad \varphi^{jk} = \varphi^k - \varphi^j, \quad \varphi_i^{jk} = \varphi_{i,m} x_m^{jk},$$

where $\varphi_{i,m}$ are numerical coefficients. Connector angle of torsion can be presented by coefficients $\varphi_{j,m}$ in the following way

$$(23) \quad \varphi^{jk} = \varphi_{i,n} x_i^{jk} x_m^{jk}.$$

That quantity is invariant while translating and turning the coordinate system; so the object $\varphi_{i,m}$ is the tensor of second rank.

Let us assume that there is the coefficient system $\varphi_{i,m}$ satisfying the equality (23) for each node pair of netlike element Z^i . Multipunctual tensor determined in that case in the element set E^i

$$(24) \quad \kappa_{lm}^i = \varphi_{i,m}$$

is called torsional — flexural tensor of netlike element Z^i . There is a possibility of determining the pair of distortion tensors in the case of simultaneous displacements and node turns, there is a possibility of determining the pair of strain tensors

$$(25) \quad \begin{aligned} \kappa_{lm}^i &= \varphi_{i,m} \\ \gamma_{lm}^i &= u_{i,m} + \varepsilon_{imn} \varphi_n^P \end{aligned}$$

where φ^P is the turn vector of any nod e^P of the net element, ε_{imn} is Ricci's tensor.

In general case the tensors are not symmetric. The net element for which we can determine the co-ordinates of tensor κ^i and γ^i is said to be deformed assymmetrically.

2.3. The state of strain of the dual net elements

If we take the appropriate relations (11) as the basis of determining the co-ordinates of symmetrical tensor of deformation state ε or asymmetrical tensors α and γ of the asymmetrical medium and (16) and (24) for the set of connectors iD of the dual element iZ related to the nod e^i , the tensors determined in this way characterize the state of strain of the dual net element. They are related to the nod e^i . Let's take the following designations for them: ie , ${}^i\gamma$, ${}^i\alpha$. Properties of these objects are analogous to the ones of their equivalents determined for the net elements.

3. Identification of the parameters of the thermo-elastic net elements-constitutive equations of the net medium

Identification of the parameters of the physical net element depends on determination of the parameters set, their type and values for the elements of geometric structure of the net elements.

The form and parameters of the function of distortion energy of elastic elements, or work function for thermo-elastic elements or the form and parameters of constitutive relations describing those medium, were taken as the equivalence criteria of the physical characteristics determined in this way along with the empirical characteristics of the elastics material. The basis of determining the compared function of net element energy is the proper choice of the element's geometric structure and determining its physical parameters. While choosing the geometric structure we should consider the symmetry group of physical properties [4,6] characteristic for the investigated material. For the homogeneous medium — assuming the same characteristics of the physical net elements and dual ones as well as usual and regular structures — two forms of polyhedrons of net elements are possible — cuboid or parallelepiped. In the set of vertexes e^i of a chosen polyhedron W^i one should determine the relation D^i , i. e. the set of element connectors. The assumed geometric group of symmetry makes the division of the connectors' set into the equivalent choices because of that group.

Let's assume that the sequence of physical parameters has been related to the nodes and connectors of the net element

$$(26) \quad \begin{aligned} e^i &\rightarrow p_m^i, \quad m=1, 2, \dots, M, \\ d^{ij} &\rightarrow p_n^{ij}, \quad n=1, 2, \dots, N. \end{aligned}$$

The symmetry group of net element structure dividing the connectors and nodes into the sets of the same sequences of physical parameters is called the group of parametric symmetries of the element — it is the subgroup of geometrical symmetries. The symmetry group of physical properties, as the resultant of geometric and parametric symmetry groups includes the common subgroup according to Neuman's law [39]. So it is a supergroup of the group of parametric symmetries. Physical parameters are digital coefficients in the relations of strain energy of particular connectors with their longitudinal, flexural or torsional deformation characteristics. In the case of heat problems they characterize the relations describing heat movement and effects of thermal dilatation. On that basis it is possible to determine for example the density of deformation energy for the symmetric medium of the elastic element in the form

$$(27) \quad w^i = \frac{1}{2} C_{mnp}^i \varepsilon_{pm}^i \varepsilon_{rn}^i,$$

or in the form

$$(28) \quad u^i = \frac{1}{2} A_{mnp}^i \gamma_{pm}^i \gamma_{rn}^i$$

for the asymmetric medium.

Rigidity elements occurring in formulae of matrix are expressed by the physical parameters of the net elements connectors. Comparing defined co-ordinates of the rigidity tensors of the element being determined experimentally for a modelled elastic material leads to the conditions that are basic for choosing the values of the assumed physical parameters of the netlike medium.

4. The example of the parameters identification of the elastic net element for the symmetric medium

The structure of the element presented in Fig. 1 was taken for the identification.

The lengths of polyhedron edges d_1, d_2, d_3 were taken. The group of parametric symmetry equal to geometric element symmetry was considered. The connectors representing the symmetry classes were designated by numbers in the figure. Versor sets related to those connectors were marked. The (') marks denote the connectors of the same class but with different directions.

Connectors d^k have elastic parameters

$$(29) \quad \begin{aligned} d^k &\rightarrow \langle C^k, \tilde{G}^k, \bar{G}^k \rangle, & k=1, 2, 3, \\ d^k &\rightarrow C^k, & k=4, 5, 6. \end{aligned}$$

The homogeneous deformation state ε_j^i was assumed for the identification.

Longitudinal deformation of the connector and the value of its lengthening are formulated by

$$(30) \quad \varepsilon^{ik} = t_k^{ik} t_l^{ik} \varepsilon_{kl}^i \quad \Delta r^{jk} = r^{jk} \varepsilon^{ik}.$$

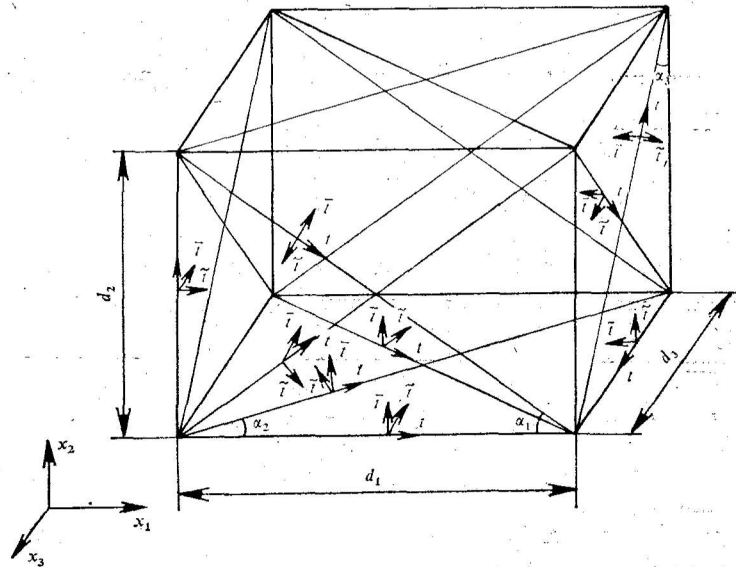


Fig. 1. Geometric structure of the net element of the symmetric medium

Component values of the connector's $d^{i,k}$ angle of rotation with directions of adequate versors are determined by the following relations

$$(31) \quad \tilde{g}^{ik} = t_k^{ik} t_l^{ik} \varepsilon_{kl}^i \quad \bar{g}^{ik} = t_k^{ik} \bar{t}_l^{ik} \varepsilon_{kl}^i$$

The force of mutual interacting of the nodes is determined by

$$(32) \quad F^{jk} = t_k^{jk} t_l^{jk} \varepsilon_{kl}^i r^{jk} C^{jk},$$

where "plus" mark denotes the force of mutual attraction, "minus" — repulsion. The force F^j interacting onto the node e^j and derived from the connector d^{jk} is

$$(33') \quad F^j = F^{jk} t^{jk},$$

while onto the node e^k is

$$(33'') \quad F^k = -F^{jk} t^{jk}.$$

The moment of mutual interacting of the nodes has got the component determined as follows

$$(34) \quad \tilde{M}^{jk} = 6t_k^{jk} \tilde{t}_l^{jk} \varepsilon_{kl}^i \tilde{G}^{jk}, \quad \bar{M}^{jk} = 6t_k^{jk} \bar{t}_l^{jk} \varepsilon_{kl}^i \bar{G}^{jk}$$

and adequate moments of interacting onto the nodes are

$$(35) \quad \begin{aligned} \tilde{M}^j &= \tilde{M}^{jk} \tilde{t}^{jk}, & \bar{M}^j &= \bar{M}^{jk} \bar{t}^{jk}, \\ \tilde{M}^k &= -\tilde{M}^{jk} \tilde{t}^{jk}, & \bar{M}^k &= -\bar{M}^{jk} \bar{t}^{jk}. \end{aligned}$$

Strain energy of the connector equals the sum of the longitudinal and flexural strain energies

$$(36) \quad U^{jk} = U_G^{jk} + \tilde{U}_G^{jk} + \bar{U}_G^{jk},$$

though we take only one component into consideration, of the type 4, 5, 6. Expressing the strain energy by means of components of the distortion tensor of the net element we get

$$(37) \quad U^{jk} = \frac{1}{2} A_{klmn}^{ik} \varepsilon_{kl}^i \varepsilon_{mn}^i,$$

where

$$(38) \quad \begin{aligned} A_{klmn}^{jk} &= r^{jk^2} C^{jk} t_k^{jk} t_l^{jk} t_m^{jk} t_n^{jk} \\ &+ 6t_k^{jk} t_m^{jk} (\tilde{G}^{jk} \tilde{t}_l^{jk} \tilde{t}_n^{jk} + \bar{G}^{jk} \bar{t}_l^{jk} \bar{t}_n^{jk}). \end{aligned}$$

Strain energy of the net element is determined in the following way

$$(39) \quad U^i = \sum_{jk} U^{jk}, \quad U^i = \frac{1}{2} A_{klmn}^i \varepsilon_{kl}^i \varepsilon_{mn}^i,$$

where

$$(40) \quad A_{klmn}^i = \sum_{jk} A_{klmn}^{jk}$$

while the strain energy density is

$$(41) \quad u^i = \frac{U^i}{V^i},$$

where V^i constitutes the polyhedron W^i capacity of the net element.

The constitutive equations for the net element can be formulated as follows:

$$(42) \quad \sigma_{kl}^i = \frac{\partial u^i}{\partial \varepsilon_{kl}^i}, \quad \sigma_{kl}^i = \varepsilon_{mn}^i a_{klmn}^i$$

The tensor a_{klmn}^i is called the rigidity tensor of the net medium. Tensor measurement of the stress σ_{kl}^i may be determined in the case when distortion tensor of the net element is determined. Quite similarly the stress tensor in the dual element can be defined. The basis to choose elastic parameter values of the net element is comparison of the co-ordinates of the rigidity of the net element to the values determined experimentally

$$(43) \quad a_{klmn}^i = a_{klmn}^{(\text{exper})}.$$

For the connectors of the element from the figure parameter values expressed by geometric parameters of the element d_1, d_2, d_3 and the constants of the very modelled medium λ, μ are

$$(44) \quad \begin{aligned} C^1 &= \frac{d_2 d_3}{d_1} \left[\mu + \frac{1}{2} \lambda \left(1 - \left[\frac{d_1}{d_3} \right]^2 - \left[\frac{d_1}{d_2} \right]^2 \right) \right], \\ C^2 &= \frac{d_1 d_3}{d_2} \left[\mu + \frac{1}{2} \lambda \left(1 - \left[\frac{d_2}{d_3} \right]^2 - \left[\frac{d_2}{d_1} \right]^2 \right) \right], \\ C^3 &= \frac{d_1 d_2}{d_3} \left[\mu + \frac{1}{2} \lambda \left(1 - \left[\frac{d_3}{d_1} \right]^2 - \left[\frac{d_3}{d_2} \right]^2 \right) \right], \\ C^4 &= \frac{1}{2} \lambda \frac{d_2}{\sin 2\alpha_2}, \quad C^5 = \frac{1}{2} \lambda \frac{d_3}{\sin 2\alpha_1}, \quad C^6 = \frac{1}{2} \lambda \frac{d_1}{\sin 2\alpha_3}, \\ C &= \frac{1}{12} d_1 d_2 d_3 (\mu - \lambda), \\ \text{tg } \alpha_1 &= \frac{d_2}{d_1}, \quad \text{tg } \alpha_2 = \frac{d_3}{d_1}, \quad \text{tg } \alpha_3 = \frac{d_3}{d_2}. \end{aligned}$$

It should be stressed that on the basis of the given symmetry groups of the geometric structure we could obtain a much richer group of the symmetry of the elastic properties.

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