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A solution of the Navier-Stokes differential equations

K. Shulev

An exact solution of the initial-boundary value problem for the complete Navier-Stokes differential equations, expressed in an explicit form, is derived. The problem corresponds to a viscous fluid flow induced by transverse motion of a long circular cylinder with a given initial speed.

Formulation of the problem

The complete Navier-Stokes equations of the form

$$\frac{\partial \bar{u}}{\partial t}+\operatorname{rot}\bar{u}\times\bar{u}+\operatorname{grad}\frac{u^2}{2}=-\frac{1}{\rho}\operatorname{grad}p-\nu\operatorname{rot}(\operatorname{rot}\bar{u}),\quad \operatorname{div}\bar{u}=0, \tag{1}$$

in standard notations are considered. The problem is to find a solution of system (1) for $\bar{u}$ and p, subject to boundary conditions at the initial instant $t=0$.

$$\bar{u}_s=\bar{U},\quad \bar{u}_\infty=\bar{0},\quad p_\infty=p_0, \tag{2}$$

where $\bar{U}$ is the initial speed, s is the cylinder surface, p_0 is the pressure of resting fluid.

Solution

The cylinder motion is supposed to be along x direction, as the cylinder axis remains parallel to direction z of an absolute frame of reference. Suppose the flow to be in plane, in the cross plane to the cylinder axis xy. The components of $\operatorname{rot}\bar{u}=\bar{\omega}(\omega_r, \omega_\theta, \omega_z)$ in cylindrical co-ordinates are

$$\omega_r=\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z},\quad \omega_\theta=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r},\quad \omega_z=\frac{1}{r}\left(\frac{\partial r u_\theta}{\partial r}-\frac{\partial u_r}{\partial\theta}\right),$$

so that the components of $\operatorname{rot}(\operatorname{rot}\bar{u})=\operatorname{rot}\bar{\omega}=\bar{\Omega}(\Omega_r, \Omega_\theta, \Omega_z)$ take the form

$$\Omega_r=\frac{1}{r}\frac{\partial\omega_z}{\partial\theta}-\frac{\partial\omega_\theta}{\partial z},\quad \Omega_\theta=\frac{\partial\omega_r}{\partial z}-\frac{\partial\omega_z}{\partial r},\quad \Omega_z=\frac{1}{r}\left(\frac{\partial r\omega_\theta}{\partial r}-\frac{\partial\omega_r}{\partial\theta}\right).$$

In case of polar co-ordinates in the plane xy then $u_z=0$, also $\frac{\partial(\circ)}{\partial z}=0$, consequently $\omega_r=0$, $\omega_\theta=0$. It follows in polar co-ordinates, that

$$\Omega_r=\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r}\left(\frac{\partial ru_\theta}{\partial r}-\frac{\partial u_r}{\partial\theta}\right),\ \Omega_\theta=-\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial ru_\theta}{\partial r}-\frac{\partial u_r}{\partial\theta}\right),\ \Omega_z=0.$$

The system (1) in polar co-ordinates takes the form

$$\begin{aligned}
&\frac{\partial u_r}{\partial t}+\frac{1}{2}\frac{\partial}{\partial r}(u_r^2+u_\theta^2)+\frac{u_\theta}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\nu}{r}\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)\\
(3)\quad&\frac{\partial u_\theta}{\partial t}+\frac{1}{2r}\frac{\partial}{\partial\theta}(u_r^2+u_\theta^2)-\frac{u_r}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}-\nu\frac{\partial}{\partial r}\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)\\
&\frac{\partial ru_r}{\partial r}+\frac{\partial u_\theta}{\partial\theta}=0.
\end{aligned}$$

Applying rot in both sides of the first equation (1) it follows the system for the function $\bar{u}$ only

$$(4)\qquad \frac{\partial}{\partial t}\,\text{rot}\,\bar{u}+\text{rot}\,(\text{rot}\,\bar{u}\times\bar{u})=-\nu\,\text{rot}\,\text{rot}\,(\text{rot}\,\bar{u}),\quad \text{div}\,\bar{u}=0.$$

Such an elimination of pressure term is equivalent to cross differentiation of the first equation (3) with respect to θ, multiplying the second equation of (3) by r and differentiation with respect to r, by subtracting both equations. It follows the system

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)+\frac{\partial}{\partial\theta}\left[\frac{u_\theta}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)\right]+\frac{\partial}{\partial r}\left[u_r\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)\right]\\
(5)\quad&=\frac{\nu}{r}\frac{\partial^2}{\partial\theta^2}\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)+\nu\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)\right]\\
&\frac{\partial ru_r}{\partial r}+\frac{\partial u_\theta}{\partial\theta}=0
\end{aligned}$$

corresponding to system (4). The problem is then reduced to find a solution of system (5) subject to the following boundary conditions, at the initial instant $t=0$, when the cylinder axis coincides with axis z

$$6)\qquad u_r|_{r=a}=U\cos\theta,\quad u_\theta|_{r=0}=-U\sin\theta,\quad u_r|_{r=\infty}=0,\quad u_\theta|_{r=\infty}=0,$$

as $U=|U|$, a is the cylinder radius.

Introducing the stream function $\Phi\ (r,\theta,t)$

$$\Phi(r,\theta,t)=\psi(r,\theta)\exp\left(-\frac{\nu}{a^2}t\right)$$

through substitutions

$$u_r=\frac{1}{r}\frac{\partial\Phi}{\partial\theta};\quad u_\theta=-\frac{\partial\Phi}{\partial r},$$

it follows that

$$(7)\qquad u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\exp\left(-\frac{\nu}{a^2}t\right),\quad u_\theta=-\frac{\partial\psi}{\partial r}\exp\left(-\frac{\nu}{a^2}t\right).$$

Obviously the $\frac{\nu}{a^2}t$ is dimensionless variable, since the dimension of ν is $[\nu]=\frac{L^2}{T}$.

Making use of (7) we obtain

$$(8)\qquad \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-\frac{\partial ru_\theta}{\partial r}\right)=\Delta\psi\exp\left(-\frac{\nu}{a^2}t\right)$$

where

(9) $$\Delta\psi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi$$

is Laplace operator in polar co-ordinates. Substituting (7) and (8) into system (5) it follows the equation

(10) $$-\frac{\nu}{a^2}\exp\left(-\frac{\nu}{a^2}t\right)r\Delta\psi+\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\Delta\psi}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial\Delta\psi}{\partial\theta}\right)\left[\exp\left(-\frac{\nu}{a^2}t\right)\right]^2$$
$$=\nu\left(\frac{1}{r}\frac{\partial^2\Delta\psi}{\partial\theta^2}+\frac{\partial}{\partial r}r\frac{\partial\Delta\psi}{\partial r}\right)\exp\left(-\frac{\nu}{a^2}t\right).$$

On the other hand we have

$$\frac{\partial}{\partial r}r\frac{\partial\Delta\psi}{\partial r}+\frac{1}{r}\frac{\partial^2\Delta\psi}{\partial\theta^2}=r\frac{\partial^2\Delta\psi}{\partial r^2}+\frac{\partial\Delta\psi}{\partial r}+\frac{1}{r}\frac{\partial^2\Delta\psi}{\partial\theta^2}=r\left(\frac{\partial^2\Delta\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta\psi}{\partial\theta^2}\right)=r\Delta\Delta\psi,$$

so that the equation (10), after reduction $\exp\left(-\frac{\nu}{a^2}t\right)$, takes the form

(11) $$\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\Delta\psi}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial\Delta\psi}{\partial\theta}\right)\exp\left(-\frac{\nu}{a^2}t\right)=\nu r\Delta\left(\Delta\psi+\frac{\psi}{a^2}\right).$$

It is readily shown that

(12) $$\Delta\psi=-\frac{\psi}{a^2}$$

satisfy the equation (11), and this being so, a solution of equation (11) may be deriv ed from the solutions of equation (12). It follows from (7) at the initial instant $t=0$ that

(13) $$u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta},\quad u_\theta=-\frac{\partial\psi}{\partial r}.$$

In such a way the stated initial-boundary value problem is reduced to the boundary problem in finding a solution of equation (12), subject to boundary conditions (6) through substitutions (13). This problem already has been solved in [1] or [2], which is, in the same time, a solution of the Euler differential equations subjected to no-slip boundary condition at the cylinder surface. In this case the stream function has the form

$$\psi(r,\theta)=Uaf\left(\frac{r}{a}\right)\sin\theta,$$

where $f\left(\frac{r}{a}\right)=f(\xi)$ is determined as solution of Bessel equation

$$f''+\frac{1}{\xi}f'+\left(1-\frac{1}{\xi^2}\right)f=0.$$

The general solution is

$$f\left(\frac{r}{a}\right)=AJ_1\left(\frac{r}{a}\right)+BN_1\left(\frac{r}{a}\right),$$

where $J_1\left(\frac{r}{a}\right)$ and $N_1\left(\frac{r}{a}\right)$ are standard Bessel functions. The constants of integration A and B are to be determined from the first two boundary conditions (6) which leads to the system

(14) $$\begin{aligned}AJ_1(1)+BN_1(1)&=1\\AJ_1'(1)+BN_1'(1)&=1\end{aligned}$$

with the solution

$$A=\frac{\pi}{2}[N_1'(1)-N_1(1)],\quad B=-\frac{\pi}{2}[J_1'(1)-J_1(1)].$$

As it is easily verified, the last two conditions of (6) are satisfied too, because the functions $J_1(\xi)$, $J_1'(\xi)$, $N_1(\xi)$ and $N_1'(\xi)$ tend to 0 as $\xi\to\infty$. The solution then for u and u for the initial instant $t=0$ is:

(15)
$$u_r(r,\ \theta)=\frac{U\pi a}{2r}\left[A\,J_1\left(\frac{r}{a}\right)+BN_1\left(\frac{r}{a}\right)\right]\cos\theta$$
$$u_\theta(r,\ \theta)=-\frac{U\pi}{2}\left[AJ_1'\left(\frac{r}{a}\right)+BN_1'\left(\frac{r}{a}\right)\right]\sin\theta.$$

Then the solution of the stated initial boundary-value problem for u_r and u_θ for each $t\geqq 0$, in view of (7), is

(16)
$$u_r(r,\ \theta,\ t)=\frac{U\pi a}{2r}\left[AJ_1\left(\frac{r}{a}\right)+BN_1\left(\frac{r}{a}\right)\right]\cos\theta.\exp\left(-\frac{\nu}{a^2}t\right)$$
$$u_\theta(r,\ \theta,\ t)=-\frac{U\pi}{2}\left[AJ_1'\left(\frac{r}{a}\right)+BN_1'\left(\frac{r}{a}\right)\right]\sin\theta.\exp\left(-\frac{\nu}{a^2}t\right).$$

It is obvious that the solution (16) is subjected to the following boundary conditions for each $t\geqq 0$:

(17)
$$u_r|_{r=a}=U\cos\theta.\exp\left(-\frac{\nu}{a^2}t\right),\quad u_\theta|_{r=a}=-U\sin\theta.\exp\left(-\frac{\nu}{a^2}t\right)$$
$$u_r|_{r=\infty}=0,\quad u_\theta|_{r=\infty}=0.$$

The first two conditions (17) can be written in the following vector form

(18)
$$\bar{u}_s=\bar{U}\exp\left(-\frac{\nu}{a^2}t\right).$$

Due to $\nu>0$, it is evident from (18), that the cylinder makes delaying motion subjected to exponential law. In case $\nu=0$ the cylinder motion becomes uniformly one with constant speed $\bar{U}$.

Integrating system (3) and determining the constant of integration from the condition $p_\infty=p_0$ follows the pressure distribution $p(r,\ \theta,\ t)$. We give the final result as

$$p(r,\ \theta,\ t)=p_0-\frac{\rho}{2}\left(u_r^2+u_\theta^2-\frac{\Phi^2}{a^2}\right),$$

where u_r and u_θ are given by (16) while

$$\Phi(r,\ \theta,\ t)=Ua\left[AJ_1\left(\frac{r}{a}\right)+BN_1\left(\frac{r}{a}\right)\right]\sin\theta\ .\ \exp\left(-\frac{\nu}{a^2}t\right).$$

In such a way the stated initial-boundary value problem has completely been solved.

The stated problem allows a generalization by extension of the initial boundary conditions (6) to the following

$$u_r|_{r=a}=U\alpha\cos\theta,\ u_\theta|_{r=a}=-U\beta\sin\theta,\ u_r|_{r=\infty}=0,\ u_\theta|_{r=\infty}=0,$$

where α and β are dimensionless parameters. Then the system determining the constants A and B, which corresponds to the system (14) is

$$AJ_1(1)+BN_1(1)=\alpha$$
$$AJ_1'(1)+BN'(1)=\beta.$$

The above system always has unique solution. In case $\alpha=1$, $\beta=1$ follows boundary conditions (6), but when $\alpha\neq1$, $\beta\neq1$ they become disturbed. In specific case $\alpha=1$, and β could be arbitrarily chosen following boundary condition of inviscid fluid. This being so it is significant that the Navier-Stokes differential equations could have vortex solution sub ject to boundary conditions of inviscid fluid.

It is worth also noticing that the solution derived here has the essential property when $\nu\to0$ then the solution tends to the solution of the Euler differential equations subject to boundary condition of viscous fluid, with no-slip condition at the cylinder surface. In the light of this fact the Oseen's concept, concerning the impossibility Euler equation solution to be derived, in the particular case $\nu=0$, from Navier-Stokes equation solution [3], needs a revision.

References:

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