БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ BULGARIAN ACADEMY OF SCIENCES
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА THEORETICAL AND APPLIED MECHANICS
София, 1988, год. XIX, № 4 Sofia, 1988, Year XIX, No 4

Plastic localization bands at nonhomogeneous plane strain and coupled thermo-plastic processes

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1. Introduction

Plastic metal forming processes are considered. The mechanical and the thermal processes taking place are highly coupled. The heat source due to the dissipated plastic strain energy is taken into account and the material is plastically hardening, rate sensitive and thermosoftening. The initiation of bands, inside which the plastic strain, the strain-rate and the temperature are considerably higher than in the surrounding is often observed at such processes [1,2]. Determining the condition under which these bands initiate, as well as their position during the thermomechanical process is important for technological applications. Inside those bands cracking may take place, which leads to fracture. The initiaion of plastic localization bands is explained by means of local instability during plastic deformation [3, 4]. They are determined as regions, inside which bifurcation of the solution of the boundary value problem takes place. It is proved that they are shear bands if microfracture and void nucleation are neglected [3, 5, 10]. Two following methods are used for determining the localization lines in the case of homogeneous strain field.

1. Direct analysis of the differential equations system describing the process and application of a bifurcation condition or of a condition for the change of the type of the differential equations in the case if elastic-plastic rate-nonsensitive materials are considered [3, 6, 7].

2. Application of the perturbation method and of a condition for infinite growth of perturbation amplitudes, if visco-plastic materials are considered [8, 9. 10].

The plastic localization in the case of a nonhomogeneos strain field was numerically investigated in [11, 12].

Initiation of localization bands in a nonhomogeneous stress and strain field will be considered. The bands may be curvilinear. The material under consideration is isotropic, plastically and dynamically hardening and thermally softening. Plastic strains are large and thermoelastic parts are neglected. The necessary conditions for initiation of plastic localization bands at plain strain will be established. A possibility to consider the plastic localization bands as lines of tangential velocity discontinuity will be shown.

2. Basic equations of the thermo-plastic process

The model of a rigid-plastic body is applied. Its deformation depends on the parameters of thermomechanical process: the plastic strain, the strain rate and the temperature. Plastical incompressibility is assumed. A Lagrangian description of the process will be applied. A moving orthogonal convective coordinate system ($0\xi^k$, $k=1, 2, 3$) is introduced. The yield limit changes isotropically with the parameters of the process, according to the Mises yield condition

$$F=\frac{1}{2}s^i_j s^j_i-\tau_p^2=0,\quad (i,\ j=1,\ 2,\ 3), \tag{2.1}$$

where s^i_j is the deviator of the stress tensor σ^i_j, τ_p is the shear yield limit. The latter depends on the strain intensity γ, on the strain rate intensity β and on the temperature θ, $\tau_p=\tau_p(\gamma,\ \beta,\ \theta)$. The strain rate tensor λ^i_j coincides with the plastic strain rate tensor. It is assumed that $\gamma=\int\limits_0^t \beta\, dt$, where $\beta=\sqrt{\frac{1}{2}\lambda^i_j\lambda^j_i}$ or $\beta=\frac{\partial\gamma}{\partial t}=\dot\gamma$. The dot above denotes partial time differentiation. The strain tensor $\varepsilon_{ij}=\int\limits_0^t\lambda_{ij}\, dt$, $\lambda_{ij}=\dot\varepsilon_{ij}$. The compatibillity conditions are:

$$\lambda_{ij}=\frac{1}{2}(v_{i|j}+v_{j|i}), \tag{2.2}$$

where v_i is the velocity vector. The vertical bar denotes contravariant differentiation.

The flow rule, associated with the yield condition is

$$\lambda^i_j=\dot\lambda s^i_j,\ \lambda^m_m=0 \tag{2.3}$$

where according to (2.1)

$$\dot\lambda=\frac{\beta}{\tau_p}\geqq 0. \tag{2.4}$$

$\dot\lambda$ may be expressed in terms of the process parameters time derivaties, using the condition

$$\dot F=s^i_k\dot\sigma^k_i-2\tau_p(\tau_\gamma\dot\gamma+\tau_\theta\dot\theta+\tau_\beta\dot\beta)=0, \tag{2.5}$$

where

$$\tau_\gamma=\frac{\partial\tau_p}{\partial\gamma},\quad \tau_\beta=\frac{\partial\tau_p}{\partial\beta},\quad \tau_\theta=\frac{\partial\tau_p}{\partial\theta}. \tag{2.6}$$

Equations (2.4) and (2.5) yield

$$\tau_\gamma\dot\lambda=\frac{1}{2\tau_p^2}(s^k_l\dot\sigma^l_k-2\tau_p\tau_\theta\dot\theta-2\tau_p\tau_\beta\dot\beta). \tag{2.7}$$

Introducing (2.7) into (2.3), we obtain the following form of the flow rule:

$$\lambda^i_j=\frac{L}{2\tau_p^2}s^i_j, \tag{2.8}$$

where

$$L=\frac{1}{\tau_\gamma}(s^k_l\dot\sigma^l_k-2\tau_p\tau_\theta\dot\theta-2\tau_p\tau_\beta\dot\beta) \tag{2.9}$$

and $\lambda^i_j\neq 0$ if $F=0$ and $L>0$.

The temperature equation has the form

$$\dot{\theta} = \frac{k_T}{\rho c_T} T^i{}_{|i} + Q, \tag{2.10}$$

where

$$Q = \frac{k}{\rho c_T} \sigma^i_j \lambda^j_i, \quad T_i = \theta_{\lambda i}. \tag{2.11}$$

Q is the rate of dissipation during plastic deformation, converted into heat; k_T is the conductivity coefficient; ρ is the material density; c_T is the specific heat supply and k is assumed to take values between 0.8 and 0.9 and determines the fraction of the strain rate energy which turns into heat.

The expression (2.11) may be written in the following form:

$$Q = \frac{2k}{\rho c_T} \tau_p \beta \tag{2.12}$$

as $\sigma^i_j \lambda^j_i = 2\tau_p . \beta \geqq 0$, according to the requirement that the rate of dissipation should be positive during plastic deformation [17].

The equations of equilibrium are

$$\sigma^i_{j/i}{}^i = 0 \tag{2.13}$$

if the body forces are neglected.

The basic equations describing the process are (2.1), (2.2), (2.3), (2.10) and (2.13).

3. Plastic localization bands in plane strain

Consider a body under plane strain. A coupled thermoplastic process takes place in it. Assume that at time t_0 bifurcation of the rates of the process parameters starts to take place inside a band with a width $2d$. At a time $t > t_0$ localization of the plastic deformation inside the band is available. The condition for initiation of such a band will be determined as conditions of bifurcation of the solution. Up to the bifuraction time the process parameters are nonhomogeneously distributed inside the body. This leads to nonlinear bifurcation bands.

We introduce a convective curvilinear coordinate system (ξ, η) in the plane strain planes $x_3 = \text{const}$, so that $\xi = \text{const}$ are lines parallel to the middle line $\mathscr{L}$ of the band and $\eta = \text{const}$ are straight lines, normal to $\mathscr{L}$. Let $\Lambda_t(\xi \in [-d, d], \eta \in [\eta_1, \eta_2]) \subset \Omega_t$ be the region of the band in the $x_3 = \text{const}$ plane. Ω_t is the space occupied by the body at time t. The band is thin and material. The straight normals hypothesis known from the shell theory is assumed. The material time derivatives if the process parameters will coincide with their partial time derivatives. The equations describing the process take the form

$$\begin{aligned}
&\lambda_{\alpha\beta} = \frac{1}{2}(v_{\alpha/\beta} + v_{\beta/\alpha}),\\
&\sigma_{\alpha\beta|\beta} = 0,\\
&\lambda_{\alpha\beta} = \dot{\lambda} s_{\alpha\beta}, \quad \dot{\lambda} = \frac{\dot{\gamma}}{\tau_p}, \quad (\alpha,\ \beta = \xi,\ \eta),\\
&\lambda_{\alpha\alpha} = 0,\\
&\dot{\theta} = \frac{k_T}{\rho c_T} \theta_{|\alpha\alpha} + Q,\\
&Q = \frac{2k}{\rho c_T} \tau_p \dot{\gamma}, \quad \dot{\gamma} = \beta,
\end{aligned} \tag{3.1}$$

where dot above denotes partial time derivative $\frac{\partial}{\partial t}$. All the variables in (3.1) are functions of ξ, η and t. Bifurcation of the solution is understood as follows: Let the process parameters be unique in the whole region Ω_{t_0}, at time t_0. The process parameters are $\Phi=\{u_\alpha, \sigma_{\alpha\beta}, \varepsilon_{\alpha\beta}, \theta\}(\xi, \eta, t)$ where u_α is the displacement vector in the $x_3=\text{const}$ plane. Assume that at time $t=t_0+\Delta t$ at least two solutions are admissible in $\Lambda_t : \Phi^{(1)}(\xi, \eta, t_0+\Delta t)$ and $\Phi^{(2)}(\xi, \eta, t_0+\Delta t)$. This means that the rates of the process parameters bifurcate

$$\text{(3.2)} \qquad \begin{gathered} \dot{\Phi}^{(i)}(\xi, \eta, t)=\lim_{\Delta t\to 0}\frac{\Phi^{(i)}(\xi, \eta, t_0+\Delta t)-\Phi^{(i)}(\xi, \eta, t_0)}{t}, \quad (i=1, 2), \\ \Delta\dot{\Phi}=\dot{\Phi}^{(1)}-\dot{\Phi}^{(2)}, \quad (\xi, \eta)\in\Lambda_{t_0}. \end{gathered}$$

As it was already mentioned, the localization band in an incompressible rigid-plastic material is a shear band [7]. This leads to certain type of bifurcation of the solution for the strain rates

$$\text{(3.3)} \qquad \begin{gathered} \Delta\lambda_{\xi\xi}=0 \quad \Delta\lambda_{\eta\eta}=0 \\ \Delta\lambda_{\eta\xi}\neq 0 \qquad\qquad (\xi, \eta)\in\Lambda_t. \end{gathered}$$

It follows from (3.1) and (3.3) that $\Delta v_\xi=\text{const}$, $\Delta v_\eta=\Delta v_\eta(\xi)$ $\xi\in[-d, d]$. Taking into account that the solution is unique outside the band and that no discontinuities are available along the band boundaries, it follows:

$$\text{(3.4)} \qquad \Delta v_\xi=0, \quad \Delta v_\eta(-d)=\Delta v_\eta(d)=0.$$

The expression (3.3) may be written in the following form [7]:

$$\text{(3.5)} \qquad \begin{gathered} \Delta v_{\alpha|\beta}=g_\alpha(\xi)\, n_\beta(\eta), \\ \Delta\lambda_{\alpha\beta}=\frac{1}{2}(g_\alpha n_\beta+g_\beta n^\alpha), \\ \Delta\lambda_{\alpha\alpha}=g_\alpha n_\alpha=0, \\ g_\alpha(-d)=g_\alpha(d)=0, \end{gathered}$$

n_α is the unit normal vector to the line $\xi=\text{const}$ at the point under consideration defined by the coordinate η. The vector g_α at that point is perpendicular to n_α and tangential to the $\xi=\text{const}$ line.

The equilibrium conditions involve the relation

$$\text{(3.6)} \qquad \Delta\dot{\sigma}_{\alpha\beta}\,.\,n_\beta=0.$$

It follows from the flow rule (3.1) that

$$\text{(3.7)} \qquad \Delta\dot{\varepsilon}_{\alpha\beta}=\Delta\dot{\lambda}s_{\alpha\beta}.$$

Taking into account (3.5) and (3.7), we obtain

$$\text{(3.8)} \qquad s_{\alpha\beta}=\frac{1}{2}(\mu_\alpha n_\beta+\mu_\beta n_\alpha), \quad \mu_\alpha=\frac{1}{\Delta\dot{\lambda}}\,g_\alpha.$$

The condtion $\Delta\dot{\lambda}\neq 0$ should be fulfilled, in order to make plastic localization in the band possible. Equations (2.7) and (3.1) yield

$$\text{(3.9)} \qquad \begin{gathered} \tau_\gamma\,\Delta\dot{\lambda}\frac{1}{2\tau_p^2}s_{\alpha\beta}\;\Delta\dot{\sigma}_{\alpha\beta}-\frac{\tau_\theta}{\tau_p}\Delta\dot{\theta}-\frac{\tau_\beta}{\tau_p}\;\Delta\dot{\beta}, \\ \Delta\dot{\lambda}=\frac{\Delta\dot{\gamma}}{\tau_p}. \end{gathered}$$

It is taken into account in (3.9) that τ_p, τ_γ, τ_θ and τ_β do not bifurcate at time t. This follows directly from the condition $\Delta\dot{F}=0$.

Relations (3.5) and (3.6) yield

$$s_{\alpha\beta}\Delta\dot{\sigma}_{\alpha\beta}=\frac{1}{2}(\mu_\alpha n_\beta+\mu_\beta n_\alpha)\quad\Delta\dot{\sigma}_{\alpha\beta}=0. \tag{3.10}$$

The relation $(3.1)_4$ together with the fact that θ, as well as its space time derivatives are unique at time t, yields

$$\Delta\dot{\theta}=\frac{2k\tau_p^2}{\rho c_T}\,\Delta\dot{\lambda}. \tag{3.11}$$

The last relation explicitly shows that the mechanical and the thermal processes are coupled.

Substituing (3.10) and (3.11) into (3.9), we obtain

$$\Delta\dot{\gamma}\,\left(\tau_\gamma+\frac{2k}{\rho c_T}\tau_p\tau_\theta\right)+\tau_\beta\,\Delta\dot{\beta}=0. \tag{3.12}$$

This is the condition required, which should be fulfilled in Λ_{t_0}, so that bifurcation of the solution at time t could be available. We shall now consider different cases at which equation (3.12) is fulfilled:

1. $\Delta\dot{\gamma}=0$, $\Delta\dot{\beta}=0$. No bifurcation is available.
2. $\Delta\dot{\gamma}\neq0$, $\Delta\dot{\beta}\neq0$. Strain rates and strain accelerations bifurcate. The following subcases are possible:

a) Arbitrary bifurcations $\Delta\dot{\gamma}$ and $\Delta\dot{\beta}$. The relation (3.12) is fulfilled if

$$\tau_\gamma+\frac{2k}{\rho c_T}\tau_p\tau_\theta=0,\quad \tau_\beta=0. \tag{3.13}$$

Equation (3.13) should be interpreted as a differential relation between τ_p and its derivatives according to the process parameters γ, β, θ. It must be fulfilled in the (τ, γ, β, θ) space on the yield surface $\tau=\tau_p(\gamma,\beta,\theta)$, $\tau=\sqrt{\frac{1}{2}\,s_{\alpha\beta}\,s_{\alpha\beta}}$. The condition (3.13) coincides with the equality in the condition given by Bai [8], obtained by means of the perturbation method for shear bands and homogeneous strain field.

b) Arbitrary bifurcations $\Delta\dot{\gamma}$ and $\Delta\dot{\beta}$, but

$$\tau_\gamma=0,\quad \tau_\theta=0,\quad \tau_\beta=0. \tag{3.14}$$

This is the condition for absolute maximum of $\tau=\tau_p(\gamma,\beta,\theta)$ in the parameter space (τ, γ, β, θ).

c) A relationship between both bifurcations exists of the form $\Delta\dot{\beta}=k_\gamma\Delta\dot{\gamma}$, where k_γ may depend on the process parameters at time t_0, but not on the bifurcated values. A single condition is then obtained

$$\tau_\gamma+\frac{2k}{\rho c_T}\,\tau_p\,\tau_\theta+k_\gamma\,\tau_\beta=0. \tag{3.15}$$

In the particular case if $k_\gamma=\text{const}$, the relation proposed by Pomey [14] is obtained. This linear relationship between $\Delta\dot{\gamma}$ and $\Delta\dot{\beta}$ is also assumed in the perturbation method applied in [8, 9]. In [13] the relationship $\frac{\mathrm{d}}{dt}(\Delta\dot{\gamma})=\Delta\dot{\beta}$ is assumed, which is typical for the postbifurcation state inside the band.

3. $\Delta\dot{\gamma}\neq 0$, $\Delta\beta=0$. Only strain rates bifurcate. The conditon $(3.13)_1$ should be then fulfilled at arbitrary values of τ_β. This is possible also if $\tau_\theta=\tau_\gamma=\tau_\beta=0$.
4. $\Delta\dot{\gamma}=0$, $\Delta\dot{\beta}\neq 0$. Only strain accelerations bifurcate. In that case $\tau_\beta=0$ at arbitrary values of τ_θ and τ_γ.

The localization development in the band during the postbifurcation process is governed by additional conditions [13].

The conditions derived above are the necessary conditions for bifurcation band initiation. Inside the band plastic localization and sharp changes of the velocities, tangential to the middle line $\mathcal{L}$ takes place. The stress-, strain-, velocity- and temperature field at two-dimensional boundary-value problems are usually determined numerically, using the finite element method, the finite difference method, the boundary element method, etc. As soon as a bifurcation band initiates, the sharp change in the velocity field would lead to numerical difficulties. This could be avoided if the band of sharp changes is replaced by a discontinuity line, on which the jump conditions are prescribed. This possibillity will be discussed in the following part.

4. The plastic localization band as a tangential velocity discontinuity line

4.1. *Definition of the discontinuity line*

We consider the plastic localization band at plane strain as a line at which the tangential velocities are discontinuous. The line is obtained from the band if its width tends to zero ($2d\rightarrow 0$). The line coincides then with the middle line $\mathcal{L}$ of the band [16]. Let $v_\eta^+(\eta,t)=\lim\limits_{d\to 0} v_\eta(d,\eta,t)$ and $v_\eta^-(\eta,t)=\lim\limits_{d\to 0} v_\eta(-d,\eta,t)$. The discontinuity $[v_\eta]=v_\eta^+-v_\rho^-$ is then defined on $\mathcal{L}$. As the localization band is a shear band, $\mathcal{L}$ will be a shear line and $[v_\xi]=0$. The line $\mathcal{L}$ is a material one, described by the relation $x_i=\hat{x}_i(\eta,t)$, $(i=1,2)$.

The geometrical first order compatibility conditions on $\mathcal{L}$ are [15]

$$[v_{i,j}]=G_i n_j+[v_{i,\eta}]\cdot t_j \quad (i,\ j=1,2), \tag{4.1}$$

where $G_i=[v_{i,k}]\,n_k$ and t_i and n_i are the tangential and the normal unit vectors to the line $\mathcal{L}$ at an arbitrary point M, belonging to $\mathcal{L}$. The position of M on $\mathcal{L}$ is defined by its curvilinear coordinate η. Introduce a local rectangular cartesian coordinate system Mtn. Its origin coincides with the point M and the axes are tangential and normal to $\mathcal{L}$. The velocities in the coordinate system Mtn are $v_i=v_t t_i+v_n n_i$. The following relations are then fulfilled on $\mathcal{L}$:

$$\begin{gathered}[v_t]\neq 0, \qquad [v_n]=0\\ [v_i]=[v_t]\,t_i.\end{gathered} \tag{4.2}$$

Introducing (4.2) into (4.1) yields

$$[v_{i,j}]=G_i n_j+[v_t]_{,\eta}t_i t_j+[v_t]t_{i,\eta}\,t_j. \tag{4.3}$$

Let α be the angle closed by the tangent Mt and the Ox_1 axis. α is a function of η and $t_{i,\eta}=\alpha_{,\eta}n_i$. The condition (4.3) takes then the form:

$$[v_{i,j}]=G_i n_j+[v_t]_{,\eta}t_i t_j+[v_t]\alpha_{,\eta}\,n_i t_j. \tag{4.4}$$

In the Mtn coordinate system (4.4) has the form

$$\begin{aligned}&[v_{n,n}]=G_n \quad [v_{n,t}]=[v_t]\alpha_{,\eta}\\ &[v_{t,n}]=G_t \quad [v_{t,t}]=[v_t]_{,\eta}.\end{aligned} \tag{4.5}$$

It is worth mentioning that $[v_{n,t}] \neq [v_{\xi,\eta}]$; both are velocity gradient discontinuities in two different coordinate systems — the cartesian *Mtn* system and the curvilinear (ξ, η) system.

4.2. *Strain rate discontinuities*

The shear strain rate λ_{nt} tends to infinity on $\mathscr{L}$ as the band width tends to zero

$$\lambda_{nt}\Big|_{\mathscr{L}} = \frac{1}{2}\left(\frac{\partial v_t}{\partial n} + \frac{\partial v_n}{\partial t}\right)\Big|_{\mathscr{L}} \to \infty \quad \text{if} \quad d \to 0. \tag{4.6}$$

The rest of the strain rate components are finite an $\mathscr{L}$. This is a well-known fact for discontinuity lines, which are shear lines [17].

The geometrical relations (2.2) lead to the following relation between strain rate- and velocity discontinuites:

$$[\lambda_{ij}] = \frac{1}{2}\{(G_i + [v_t]\alpha_{,\eta} t_i)n_j + (G_j + [v_t]\,\alpha_{,\eta} t_j)n_i\} \tag{4.7}$$

or in the *Mtn* coordinate system:

$$\begin{aligned} [\lambda_{nn}] &= [v_{n,n}] = G_n \\ [\lambda_{tt}] &= [v_{t,t}] = [v_t]_{,\eta} \\ [\lambda_{nt}] &= \frac{1}{2}([v_{n,t}] + [v_{t,n}]) = \frac{1}{2}([v_t]\alpha_{,\eta} + G_t). \end{aligned} \tag{4.8}$$

4.3. *Stress discontinuities*

The equations of equilibrium (2.13) yield

$$[\sigma_{ij}]n_j = 0 \tag{4.9}$$

or

$$[\sigma_{nn}] = 0, \quad [\sigma_{nt}] = 0. \tag{4.10}$$

Applying the flow rule (2.3) to points belonging to $\mathscr{L}$, we obtain

$$\frac{\lambda_{nn}}{\lambda_{nt}}\Big|_{\mathscr{L}} = \frac{s_{nn}}{s_{nt}}\Big|_{\mathscr{L}}, \qquad \frac{\lambda_{tt}}{\lambda_{nt}}\Big|_{\mathscr{L}} = \frac{s_{tt}}{s_{nt}}\Big|_{\mathscr{L}}. \tag{4.11}$$

Since $\lambda_{nt} \to \infty$ but λ_{nn} and λ_{tt} are finite on $\mathscr{L}$ we obtain

$$\begin{gathered} s_{nn}|_{\mathscr{L}} = s_{tt}|_{\mathscr{L}} = 0 \\ \sigma_{nn}|_{\mathscr{L}} = \sigma_{tt}|_{\mathscr{L}} = \sigma, \end{gathered} \tag{4.12}$$

where $\sigma = \frac{1}{3}\sigma_{kk}$ is the hydrostatic stress.

Relations (4. 10) and (4.12) yield that $[\sigma_{tt}] = 0$, or

$$[\sigma_{ij}] = 0, \quad (i, j = 1, 2). \tag{4.13}$$

The Mises' yield condition (2.1), applied to points belonging to $\mathscr{L}$ yields that $s_{nt}|_{\mathscr{L}} = \sigma_{nt}|_{\mathscr{L}} = \pm\tau_p|_{\mathscr{L}}$. In the case of a rigid-perfectly plastic material $\tau_p = \text{const}$ and the condition $[\sigma_{nt}] = 0$ leads then to $[\tau_p] = 0$. This is admissible and hence the application of a model, containing a discontinuity line brings no problems. If the material is plastically and dynamically hardening then $\tau_p = \tau_p(\gamma, \beta)$. The yield limit continuity required by the Mises' yield condition and the relation (4.13) will be then violated as σ_{nt} is continuous on $\mathscr{L}$ but τ_p, depending on β, would be discontinuous either. This leads to the well-known fact that tangential velocity discontinuity lines do not exist in hardening materials [17]. The discontinunity line which we introduced here represents the experimentally observed in hardening materials localization band. But turning the band into a line if $d \to 0$ needs additional assumptions about the material properties of the line. These additional assumptions defining the properties of the line will make possible the application of discontinuity lines in hardening materials.

We assume that the material points belonging to the line $\mathscr{L}$ have another yield limit $\bar{\tau}_p$, so that

$$\sigma_{nt}\big|_{\mathscr{L}} = \pm\bar{\tau}_p. \tag{4.14}$$

This assumption may also be connected with the fact that the properties of the material inside the band are different compared with those outside the band, due to the structural changes during the localized plastic deformation [18].

4.4. *Discontinuities in the temperature field*

The localized plastic deformation on $\mathscr{L}$ leads to a linear heat source there. The following discontinuities exist in the temperature field:

$$\begin{gathered} [\theta_{,n}] \neq 0, \quad [\theta_{,t}] = 0 \\ [\theta] = 0, \quad [\dot{\theta}] = 0. \end{gathered} \tag{4.15}$$

Integration of the temperature equation (2.10) in the interval $[-d, d]$ and letting $d \to 0$ leads to [11]

$$[\theta_{,n}] = -\rho \frac{c_T}{k_T} \bar{Q}\Big|_{\mathscr{L}}, \tag{4.16}$$

where

$$\bar{Q}\Big|_{\mathscr{L}} = \frac{k}{\rho c_T}\sigma_{nt}\,[v_t] = \frac{k}{\rho c_T}\bar{\tau}_p\,|[v_t]|. \tag{4.17}$$

It follows from $(4.15)_4$ that

$$\left[\frac{d\theta}{dt}\right] = [\theta_{,n}]\,v_n + \theta_{,t}\,[v_t] \tag{4.18}$$

and taking into account (4.16) and (4.17), we obtain

$$\frac{d\theta}{dt} = -\frac{k}{k_T} v_n \bar{\tau}_p\,|[v_t]| + \theta_{,t}[v_t] = \left(\theta_{,t} - v_n \frac{k}{k_T}\bar{\tau}_p \operatorname{sign}[v_t]\right)[v_t]. \tag{4.19}$$

This relationship contains the temperature and the velocity discontinuities and is due to the coupling of the mechanical and the thermal processes.

4.5. *Relationship between the discontinuities*

It follows from the flow rule (2.3) and from the stress continuity on $\mathscr{L}$ (4.13) that

$$[\lambda_{ij}] = \left[\frac{d\lambda}{dt}\right] s_{ij} \quad (i, j = 1, 2) \tag{4.20}$$

or

$$[\lambda_{nn}]=\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]s_{nn}=0,$$

$$(4.21)\qquad [\lambda_{tt}]=\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]s_{tt}=0,$$

$$[\lambda_{nt}]=\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]s_{nt}=\pm\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]\bar{\tau}_p,$$

as $\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]=\frac{1}{\tau_p}\left[\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right]\neq 0$ and $s_{nn}=s_{tt}=0$, $s_{nt}=\pm\bar{\tau}_p$ on $\mathscr{L}$. Equations (4.21) and (4.8) yield

$$(4.22)\qquad \begin{aligned} &G_n=0, \quad [v_t]_{,\eta}=0\\ &\frac{1}{2}([v_t]\alpha_{,\eta}+G_t)=\pm\left[\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right]\bar{\tau}_p. \end{aligned}$$

The result $[v_t]_{,\eta}=0$ shows that the tangential velocity has a constant jump on $\mathscr{L}$, no matter that the field outside $\mathscr{L}$ is nonhomogenous. This condition is well known for the case of perfectly plastic materials [17]. After assuming (4.14) the same condition holds for hardening materials and nonhomogeneous strain field.

Using (2.4), (2.7) and (4.14), we obtain

$$(4.23)\qquad [\beta]=\left[\frac{\mathrm{d}\sigma_{nt}}{\mathrm{d}t}\cdot\frac{1}{\tau_\gamma}\right]+\left[\frac{\tau_\theta}{\tau_\gamma}\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\right]+\left[\frac{\tau_\beta}{\tau_\gamma}\,\frac{\mathrm{d}\beta}{\mathrm{d}t}\right].$$

The jump conditions derived above may be used in a numerical solution of a boundary-value problem for thermoplastic metal forming with an existing plastic localization band.

Aknowledgement

The results of this paper are obtained in the course of an official cooperation between the Lehrstuhl A für Mechanik mit Staatlichem Materialprüfamt für den Maschinenbau of the Technical University Munic and the Institute of Mechanics and Biomechanics of the Bulgarian Academy of Sciences. The investigations were financially supported by the Bulgarian Commitee of Sciences. The authors are highly indepted to Prof. Dr. rer. nat. Dr. mont. h. c. H. Lippmann for many interesting and useful discussions.

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Received 27. 04. 1988