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Similarity analysis of a dynamic problem of coupled thermoelasticity

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1. Introduction

An effective method of analysing one-dimensional problems is to construct similarity solutions which, as a general rule, have a simple space-time structure. The essential property of similarity is that the profiles of the examined quantities remain constant in time. Thus the similarity solution keeps the main features of the solution. Therefore, on the one hand, qualitative analysis of the solution could be carried out, and, on the other hand, similarity solutions could be used as a convenient test for verifying the accuracy of the numerical procedures. This is of great value in relation with nonlinear problems.

The application of the method of similarity solutions to one-dimensional problems transforms the partial differential equations (PDE) into ordinary differential equations (ODE). From mathematical point of view it simplifies the problems and in a number of cases exact special solutions can be found.

In nonlinear problems, since the principle of superposition does not hold, exact special solutions sometimes appear not to be useful. However, these special solutions represent the asymptotics of a wide class of other more general solutions that correspond to various initial conditions [1, 2]. Under these circumstances the value of exact special solutions increases considerably.

Similarity solutions for thermoelastic problems are studied in [3—6]. In [3, 4] solutions of the type of travelling wave for the linear dynamic problem are investigated. The dynamic problem with finite velocity of heat transfer is considered in [5]. In [6] two classes of similarity solutions are obtained for the dynamic problem with temperature dependent coefficient of thermal conductivity.

The purpose of the present paper is to establish some similarity solutions of a one-dimensional dynamic problem of coupled thermoelasticity. The material is assumed to be homogeneous and isotropic, the coefficient of thermal conductivity is considered to be strain dependent. A class of similarity solutions of the type of travelling wave is obtained. Another class of solutions is constructed by the method of generalized separation of variables.

2. The governing equations

For the one-dimensional case the governing equations of coupled thermoelasticity are [7, 8]:

(1) $$\rho\frac{\partial^2 u}{\partial t^2}-(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(3\lambda+2\mu)\alpha\frac{\partial\theta}{\partial x}=F(x,t), \quad \text{(motion)}$$

(2) $$c\rho\frac{\partial\theta}{\partial t}-\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right)+(3\lambda+2\mu)\alpha T_0\frac{\partial^2 u}{\partial x\partial t}=W(x,\ t),\quad \text{(heat transfer)},$$

where x is the space variable, t — time, u — displacement, $\theta=T-T_0$ — change of the body temperature from a reference one T_0, k — coefficient of heat conductivity, λ and μ — Lamé's elastic constants, c — specific heat, α — coefficient of thermal expansion, ρ — density, F — body forces, and W — heat sources.

It is convenient to write the equations (1), (2) in dimensionless form [9]:

(3) $$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+a_1\frac{\partial\theta}{\partial x}=F(x,\ t),$$

(4) $$\frac{\partial\theta}{\partial t}-\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right)+a^2\frac{\partial^2 u}{\partial x\partial t}=W(x,\ t),$$

where

$$a_1=(3\lambda+2\mu)\alpha T_0/(\lambda+2\mu),\quad a_2=(3\lambda+2\mu)\alpha/c\rho,$$

and the same latter is retained to denote the corresponding dimensionless quantity.

The parameter $\eta=a_1a_2$ is called a coupling parameter. For some materials $\eta\ll 1$. For example, the data for four common metals at 20°C are [7]:

Aluminium	3.56×10^{-2}
Iron (Steel)	1.14×10^{-2}
Copper	1.68×10^{-2}
Lead	7.33×10^{-2}

In general, the coefficient of thermal conductivity k depends on the temperature, its gradient and the strain [3]. In the present analysis k will be assumed to be strain dependent as follows:

(5) $$k=(\partial u/\partial x)^n,\quad n>0,\quad n=\text{const}.$$

3. Solution of the type travelling wave

The system (3), (4) is of a mixed hyperbolic — parabolic type. Such systems admit solution representing waves travelling without change of shape or magnitude, that is, the waves have constant profile. Mathematically such solutions depend on x and t through the variable $s=x-Dt$, where D is the constant velocity of the wave. Thus a solution in the following form is considered:

(6) $$u(x,\ t)=u(s),\quad \theta(x,\ t)=\theta(s),\quad s=x-Dt.$$

Substituting (6) into the system (3), (4) an equivalent system of ODE is obtained

(7) $$(D^2-1)\frac{d^2u}{ds^2}+a_1\frac{d\theta}{ds}=F(s),$$
$$D\frac{d\theta}{ds}+\frac{d}{ds}\left[\left(\frac{du}{ds}\right)^n\frac{d\theta}{ds}\right]+a_2D\frac{d^2u}{ds^2}=W(s).$$

Consider a wave propagation in an infinite elastic medium. For simplicity, let us take the special case in which the medium is unstrained, and its initial temperature is zero. Thus the above system will be integrated by considering the conditions:

(8) $$\theta(x_1)=0,\quad \{(du/ds)^n(d\theta/ds)\}_{s=x_1}=0,\quad u(x_1)=0,\quad (du/ds)_{s=x_1}=0.$$

It is assumed that the wave front is at the point $s=x_1$, and this point moves with constant velocity. Presuming that the body is not subject to body forces and heat sources (i. e., $F=0$, $W=0$) the following particular temperature and displacement solution is found:

(9)
$$\theta(s)=[B(s-x_1)]^{1/n},$$
$$u(s)=\left(\frac{a_1}{1-D^2}\right)^{n+1}[(n+1)A]^{-1}[B(s-x_1)]^{1+1/n}.$$

Here the following notations are introduced:

$$A=D(a_1a_2+1-D^2)/(D^2-1), \quad B=nA[(1-D^2)/a_1].$$

Taking into account a consequence of the Clausius-Duhem inequality, the velocity D should satisfy the following condition:

(10)
$$D\in\{(0,\ 1)\cup(\sqrt{1+a_1a_2},\ +\infty)\}$$

(the proof is given in the Appendix).

For $1\leq D\leq\sqrt{1+a_1a_2}$ the system (7) has no suitable solution.

Once the temperature and displacement are determined, the stress function can be expressed as:

(11)
$$\sigma(s)=\frac{du}{ds}-a_1\theta=\frac{a_1D}{1-D^2}\theta(s)=D[nA(s-x_1)]^{1/n}.$$

Note that the trivial solution

(12)
$$\theta(x,\ t)=0, \quad u(x,\ t)=0$$

is also a particular solution of the homogeneous system (3), (4).

The particular solutions (9) and (12) can be combined to construct the functions:

(13)
$$\theta(x,\ t)=\begin{cases}[B(x-Dt-x_1)]^{1/n}, & 0\leq x\leq x_1+Dt\\ 0, & x>x_1+Dt\end{cases}$$

(14)
$$u(x,\ t)=\begin{cases}\left(\frac{a_1}{1-D^2}\right)^{n+1}[(n+1)A]^{-1}[B(x-Dt-x_1)]^{1+1/n}, & 0\leq x\leq x_1+Dt\\ 0, & x>x_1+Dt,\end{cases}$$

which represent the solution of the original systems (3), (4) with the following specia boundary and initial conditions:

(15)
$$\theta(0,\ t)=[B(-Dt-x_1)]^{1/n}, \quad t>0$$
$$u(0,\ t)=\left(\frac{a_1}{1-D^2}\right)^{n+1}[(n+1)A]^{-1}[B(-Dt-x_1)]^{1+1/n}, \quad t>0$$

(16)
$$\theta(x,\ 0)=\begin{cases}[B(x-x_1)]^{1/n}, & 0<x\leq x_1\\ 0, & x>x_1\end{cases}$$
$$u(x,\ 0)=\begin{cases}\left(\frac{a_1}{1-D^2}\right)^{n+1}[(n+1)A]^{-1}[B(x-x_1)]^{1+1/n}, & 0<x\leq x_1\\ 0, & x>x_1.\end{cases}$$

4. Method of generalized separation of variables

Another class of similarity solutions can be obtained using the method of generalized separation of variables [10]. The solution will be sought in the similarity representation:

(17) $$\theta(x, t)=l_1(t)\tilde{\theta}(\xi), \quad u(x, t)=l_2(t)\tilde{u}(\xi),$$

where ξ is the similarity variable, defined as

(18) $$\xi=x/\varphi(t).$$

The main objective is to reduce the PDE (3), (4) to a system of ODE in ξ by determing appropriate functions $l_1(t)$, $l_2(t)$, $\varphi(t)$. Such functions can be derived as follows.

Inserting the above relations into the equations (3), (4) renders the system:

(19)
$$\begin{aligned}
&\dot{l}_2(t)\tilde{u}(\xi)+\varphi^{-1}(t)l_2(t)[2\varphi^{-1}(t)\dot{\varphi}^2(t)-2\dot{\varphi}(t)\,l_2^{-1}(t)\,\dot{l}_2(t)-\ddot{\varphi}(t)]\,\xi\,\frac{d\tilde{u}}{d\xi}\\
&+\varphi^{-2}(t)l_2(t)\,[\xi^2\dot{\varphi}^2(t)-1]\frac{d^2\tilde{u}}{d\xi^2}+a_1\varphi^{-1}(t)l_1(t)\frac{d\tilde{\theta}}{d\xi}=0,\\
&\dot{l}_1(t)\tilde{\theta}(\xi)-\varphi^{-1}(t)\dot{\varphi}(t)l_1(t)\,\xi\frac{d\tilde{\theta}}{d\xi}-\varphi^{-n-2}(t)l_1(t)\,l_2^n(t)\frac{d}{d\xi}\left[\left(\frac{d\tilde{u}}{d\xi}\right)^n\frac{d\tilde{\theta}}{d\xi}\right]\\
&+a_2\varphi^{-1}(t)\,[\dot{l}_2(t)-l_2(t)\varphi^{-1}(t)\dot{\varphi}(t)]\frac{d\tilde{u}}{d\xi}-a_2l_2(t)\varphi^{-2}(t)\dot{\varphi}(t)\xi\frac{d^2\tilde{u}}{d\xi^2}=0,
\end{aligned}$$

where a dot designates a time derivative.

To get differential equations for the unknown functions $\tilde{\theta}(\xi)$, $\tilde{u}(\xi)$, it is necessary to separate the variables. For this purpose we should have:

$$\xi^2\dot{\varphi}^2(t)-1=p(\xi),$$

i. e.,

(20) $$\dot{\varphi}(t)=A_1^2, \quad A_1=\text{const}.$$

Hence

(21) $$\varphi(t)=A_1t+A_0, \quad A_1\neq 0, \quad A_0\neq 0.$$

The functions $l_1(t)$, $l_2(t)$ must satisfy the corresponding conditions of variable separation. Let us divide the first equation (19) by $\varphi^{-2}(t)l_2(t)$ and the second equation by $\varphi^{-n-2}(t)l_1(t)l_2^n(t)$, and take (20) into account. Now let us set

(22) $$\frac{\varphi^{n+1}(t)}{l_2^n(t)}=B_1, \quad \frac{\varphi(t)l_1(t)}{l_2(t)}=B_2,$$

where B_1 and B_2 are arbitrary separation constants. It can be easily shown that all other terms in (19), depending on t, become constants. Making use of (22) we arrive at the following results

(23) $$l_1(t)=B_2\,[B_1^{-1}\varphi(t)]^{1/n}, \quad l_2(t)=B_1\,[B_1^{-1}\varphi(t)]^{1+1/n},$$

i. e.,

(24) $$l_1(t)=B_2\,[B_1^{-1}(A_1t+A_0)]^{1/n}, \quad l_2(t)=B_1\,[B_1^{-1}(A_1t+A_0)]^{1+1/n},$$

(25) $$\xi=x/(A_1t+A_0).$$

In the simple case when $A_1=A_0=B_1=B_2=1$, we obtain:

(26) $$l_1(t)=(t+1)^{1/n}, \quad l_2(t)=(t+1)^{1+1/n}, \quad \xi=x/(t+1).$$

The corresponding formulae (26) for some concrete values of n are:

1) $n=1$, i. e., $k=\partial u/\partial x$,

$l_1(t)=t+1$, $l_2(t)=(t+1)^2$;

2) $n=2$, i. e., $k=(\partial u/\partial x)^2$,

$l_1(t)=\sqrt{t+1}$, $l_2(t)=(t+1)\sqrt{t+1}$.

On the account of the foregoing results, the functions $\tilde{u}(\xi)$, $\tilde{\theta}(\xi)$ can be determined by integrating the following system of ODE:

$$n^2(A_1^2\xi^2-1)\frac{d^2\tilde{u}}{d\xi^2}-2nA_1^2\xi\frac{d\tilde{u}}{d\xi}+A_1(n+1)\tilde{u}(\xi)+a_1n^2B_2\frac{d\tilde{\theta}}{d\xi}=0 \tag{27}$$

$$nB_2\frac{d}{d\xi}\left[\left(\frac{d\tilde{u}}{d\xi}\right)^n\frac{d\tilde{\theta}}{d\xi}\right]+nA_1B_1B_2\xi\frac{d\tilde{\theta}}{d\xi}-n^2A_1B_1B_2\tilde{\theta}(\xi)-a_2A_1B_1\frac{d\tilde{u}}{d\xi}+na_2A_1B_1\xi\frac{d^2\tilde{u}}{d\xi^2}=0. \tag{28}$$

The only problem left is to associate appropriate initial conditions with the system (27), (28). Let us consider the following initial-boundary-value problem in the half-space $x\geqq 0$:

$$\begin{aligned} &u(0,t)=l_2(t), \quad \theta(0,t)=l_1(t), \\ &u(x,0)=0, \quad \frac{\partial u}{\partial x}(x,0)=0, \quad \theta(x,0)=0, \end{aligned} \tag{29}$$

which renders to

$$\tilde{u}(0)=1, \quad \tilde{\theta}(0)=1, \tag{30}$$

$$\tilde{u}(\xi_0)=0, \quad \frac{d\tilde{u}}{d\xi}(\xi_0)=0, \tilde{\theta}(\xi_0)=0. \tag{31}$$

The value of ξ_0 is not known beforehand. The wavefront is determined by the formula (25):

$$x_f=\xi_0(A_1t+A_0),$$

and the velocity of the front can be calculated using the formula

$$v=dx_f/dt=A_1\xi_0. \tag{32}$$

As the system (27), (28) is of second order, therefore it is necessary to know the first derivatives of the functions $\tilde{u}(\xi)$, $\tilde{\theta}(\xi)$ at the wavefront $\xi=\xi_0$. It can be seen from (32) that the wave velocity does not depend on the time t. With this in mind we recall the solution (9), (10) for the constant wave velocity and seek series expansions of the functions $\tilde{\theta}(\xi)$, $\tilde{u}(\xi)$ in the vicinity of the point $\xi=\xi_0$ in the form

$$\begin{aligned} &\tilde{\theta}(\xi)=[E_1(\xi-\xi_0)]^{1/n}+O[(\xi-\xi_0)^{1+1/n}], \\ &\tilde{u}(\xi)=E_2[E_1(\xi-\xi_0)]^{1+1/n}+O[(\xi-\xi_0)^{2+1/n}]. \end{aligned} \tag{33}$$

For the constants E_1 and E_2, the following expressions are found:

$$E_1=n\xi_0A_1B_1(A_1^2\xi_0^2-1-a_1a_2)(1-A_1^2\xi_0^2)^{n-1}/(a_1B_2)^n,$$

$$E_2=na_1B_2/E_1(n+1)(1-A_1^2\xi_0^2).$$

Note that for $A_1=B_1=B_2=1$, the values of E_1 and E_2 are equal to the corresponding constants in the formula (9), which have been expected.

Now we can calculate the values of the functions $\tilde{\theta}(\xi)$, $\tilde{u}(\xi)$, $d\tilde{\theta}/d\xi$, $d\tilde{u}/d\xi$ with an acceptable accuracy at the point $\xi_1=\xi_0-0$ (see the formulae (33)):

$$\begin{aligned} &\tilde{\theta}(\xi_1)=[E_1(\xi_1-\xi_0)]^{1/n}+\ldots, \quad \frac{d\tilde{\theta}}{d\xi}(\xi_1)=\frac{E_1}{n}[E_1(\xi_1-\xi_0)]^{1/n-1}+\ldots \\ &\tilde{u}(\xi_1)=E_2[E_1(\xi_1-\xi_0)]^{1+1/n}+\ldots, \quad \frac{d\tilde{u}}{d\xi}(\xi_1)=\left(1+\frac{1}{n}\right)E_1E_2[E_1(\xi_1-\xi_0)]^{1/n}+\ldots, \end{aligned} \tag{34}$$

Having the conditions (30), (34) the system (27), (28) can be integrated numerically for $0<\xi<\xi_1=\xi_0-0$ by using the traditional shooting method. The parameter of shooting is the state of the wavefront ξ_0.

The numerical treatment of the corresponding two-point boundary problems for the system of ODE (27), (28) is a matter of future investigation.

Conclusions

Similarity solutions are rarely applied to thermoelasticity problems. The analytical techniques presented here are both simple and systematic and the solutions found should be of theoretical interest. They could also be used as a test for the numerical procedures. These solutions could be considered as intermediate asymptotics. For arbitrary initial conditions, however, the asymptotic behaviour should be proved.

Appendix

Integrating the system (7), taking into account the conditions (8), and presuming that $F=0$, $W=0$, we obtain

$$\frac{du}{ds}=-\frac{a_1}{D^2-1}\,\theta, \tag{A1}$$

$$-\left(\frac{du}{ds}\right)^n\frac{d\theta}{ds}=D\theta+A_2D\frac{du}{ds}\,. \tag{A2}$$

Let us replace du/ds from (A1) into the right-hand side of (A2):

$$-\left(\frac{du}{ds}\right)^n\frac{d\theta}{ds}=\frac{D(D^2-1-a_1a_2)}{D^2-1}\,\theta.$$

Let us multiply this equality by $d\theta/ds$. Then we arrive at the following result:

$$\left[-\left(\frac{du}{ds}\right)^n\frac{d\theta}{ds}\right]\frac{d\theta}{ds}=\frac{D(D^2-1-a_1a_2)}{D^2-1}\left(\theta\frac{d\theta}{ds}\right). \tag{A3}$$

According to an well-known consequence of the Clausius-Duhem inequality (namely the direction of the heat vector is opposite to the spatial rate of change of the temperature), it follows that the left-hand side of (A3) is always negative. This fact yield the inequality:

$$D(D^2-1-a_1a_2)/(D^2-1)>0,$$

which easily leads to the condition (10).

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