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Kinematics and Kinetostatics of Spatial Mechanisms
Part I: Kinematics of Spatial Mechanisms

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Simulating the kinematics of spatial mechanisms lies in the basis of the works concerning kinematic analysis [1, 2, 3, 4, 5]. Theoretical formulations, based on coordinates transformations and matrix equations [1, 2, 6, 7, 8], as well as on the screw theory [3, 4, 8], are directly related to the application of computers used for automating the analysis of kinematic chains of the most intricate type, which results in creating powerful algorithms and computer programs for analysis and design [9, 10, 11, 12, 13].

The most general problem of kinematic analysis is in defining all characteristics of link motion, covering both the relative position of mutually connected links and their absolute position, velocities, and accelerations, depending on the constant (kinematic) parameters. The complete study of the spatial kinematic chain, which is an operating mechanical system, consists in defining the loading of the links as a function of the mass parameters of the mechanism, and the external loading; this functional dependence involves the kinematic characteristics. Therefore, kinematic analysis should be considered from the point of view of the possibilities to use it as a stage in the complete analysis of the mechanical structure, hence for the purposes of designing the latter.

The paper suggests a generalized algorithm and a computer program for simulating the kinematics of open and closed spatial mechanisms on the basis of coordinates transformations and matric equations. The analysis of closed kinematic chains is reduced to the simultaneous analysis of a set of one-contour chains that make up the poly-contour mechanism. For this purpose, generalized matrices are used concerning the relative position of links; this makes it possible that the computer program created for solving the direct and the inverse problem of kinematics for open spatial chains [14] could be iteratively used in analyzing poly-contour chains. Numerical results are directed towards their further use in the next stage: the power analysis.

1. Kinematic Model of Open Spatial Chains

The kinematic model of an open spatial chain shows the functional dependence between the constant parameters of the kinematic chain and the characteristics of motion which include: the relative position of links, as well as the derivatives of the relative motions; velocity and acceleration of motion of links and some of their specific points with respect to absolute space.

The application of the kinematic model in open kinematic chains is most often connected with the solution of the direct and the inverse problems of kinematics.

The functional dependence between the constant parameters of the chain and the kinematic characteristics is determined by a successive transformation of the output link coordinates and the driven links connected with it, with respect to the absolute coordinate system.

The mutual position of two bodies is determined by the three Euler angles formed between the coordinate axes of the bodies and the coordinates of the origin of one of the coordinate systems with respect to the other. In mechanisms consisting of pairs of Class 5 only (rotational and translatory), the mutual position of two interconnected bodies (links), moving with a spatial motion and having coordinate systems oriented along the axes of the pairs pertaining to them, can be determined by means of the parameters of Denavit and Hartenberg [1].

The matrix

$$\tau_{i-1,i}=\begin{bmatrix} \cos X_{i-1}\widehat{\ }X_i & \cos X_{i-1}\widehat{\ }Y_i & \cos X_{i-1}\widehat{\ }Z_i \\ \cos Y_{i-1}\widehat{\ }X_i & \cos Y_{i-1}\widehat{\ }Y_i & \cos Y_{i-1}\widehat{\ }Z_i \\ \cos Z_{i-1}\widehat{\ }X_i & \cos Z_{i-1}\widehat{\ }Y_i & \cos Z_{i-1}\widehat{\ }Z_i \end{bmatrix}$$

illustrates the rotation of the i-th coordinate system with respect to the $i-$ 1st coordinate system. The coefficients of this matrix are invariant with respect to the parameters of location of the two coordinate systems. Three of them give a single-valued determination of the mutual rotation of the coordinate systems. Point M pertaining to the i-th coordinate system with origin O_i, and having a radius-vector $\rho_{i,M} =[X_{i,M} Y_{i,M} Z_{i,M}]^T$ is transformed with respect to the $i-$ 1st coordinate system by means of a matrix equation:

(1) $$\rho_{i-1,M}=\tau_{i-1,i}\cdot\rho_{i,M}+\rho_{i-1,O_i}.$$

Vector $P_{i,M}$ with a point ot application M and coordinates $P_{i,M}=[Px_{i,M}\ Py_{i,M}\ Pz_{i,M}]^T$ is transformed with respect to the $i-$ 1st coordinate system by a matrix equation:

(2) $$P_{i-1,M}=\tau_{i-1,i}\cdot P_{i,M}.$$

After a succession of matrix transformations, equation (1) becomes more complex. The introduction of homogeneous coordinates [1] of the radius-vector $r_{i,M}=|\ \rho_{i,M}\ 1\ |^T$, and the generalized matrix

$$T_{i-1,i}=\left[\begin{array}{c|c} \tau_{i-1,i} & \\ \cdots\cdots & r_{i-1,O_i} \\ 0 & \end{array}\right]$$

enables the presentation of equation (1) in the following form:

(3) $$r_{i-1,M}=T_{i-1,i}\cdot r_{i,M}$$

and upon effecting successive coordinate transformations of the mutually connected coordinate systems, successively indexed by subscripts 0 through n, we get

(4) $$r_{0,M}=T_{0n}\cdot r_{n,M},$$

where

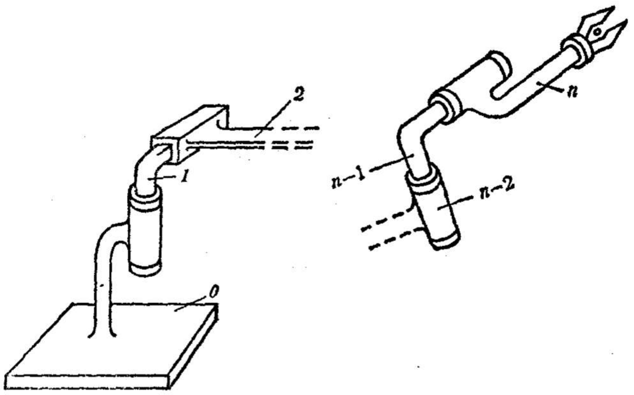
(5) $$T_{0n}=T_{01}\cdot T_{12}\dots T_{n-1,n}.$$

Such a succession of mutually connected bodies represents a robot having n links, successively numbered from 0 — frame to n — output link (Fig 1).

Equations (4) and (5) reflect the implicit functional dependence between the constant parameters of the open chain and the kinematic characteristics. In a robot having pairs of Class 5, these characteristics are the relative displacements Q in the joints, where $Q=[q_1 q_2 \dots q_n]^T$, while the matrix $T_{i-1,i}$ is a function of q_i, $i=1, 2, \dots, n$.

The functional dependence between the velocities $\dot{Q}=[\dot{q}_1 \dot{q}_2 \dots \dot{q}_n]^T$ and the constant parameters of the chain is given by equations

$$\dot{r}_{0,M}=\dot{T}_{0n} \cdot r_{n,M}, \tag{6}$$



Фиг. 1

$$\dot{T}_{0n}=\frac{\partial T_{0n}}{\partial q_1} \cdot \dot{q}_1+\frac{\partial T_{0,n}}{\partial q_2} \cdot \dot{q}_2+\dots+\frac{\partial T_{0,n}}{\partial q_n} \cdot \dot{q}_n, \tag{7}$$

where

$$\frac{\partial T_{0n}}{\partial q_i}=T_{01} \cdot T_{12} \dots \frac{\partial T_{i-1,i}}{\partial q_i} \dots T_{n-1,n}. \tag{8}$$

The functional dependence between the accelerations $\ddot{Q}=[\ddot{q}_1 \ddot{q}_2 \dots \ddot{q}_n]^T$ and the constant parameters of the chain is given by equations

$$\ddot{r}_{0,M}=\ddot{T}_{0n} \cdot r_{n,M}, \tag{9}$$

$$\ddot{T}_{0n}=\sum_{i=1}^{n} \frac{\partial T_{0n}}{\partial q_i} \cdot \ddot{q}_i+\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 T_{0n}}{\partial q_i \partial q_j} \cdot \dot{q}_i \dot{q}_j, \tag{10}$$

where

$$\frac{\partial^2 T_{0n}}{\partial q_i \partial q_j}=T_{01} \cdot T_{12} \dots \frac{\partial T_{i-1,i}}{\partial q_i} \dots \frac{\partial T_{j-1,j}}{\partial q_j} \dots T_{n-1,n}, \tag{11}$$

$$\frac{\partial^2 T_{0n}}{\partial q_j^2}=T_{01} \cdot T_{12} \dots \frac{\partial^2 T_{i-1,i}}{\partial q_i^2} \dots T_{n-1,n}. \tag{12}$$

In equations (4) and (5) the linearly independent elements are six: three elements from equation (4), giving the location of point M, and three diagonal elements of matrix T_{0n} from equation (5). Similarly, when considering systems (6) and (7), and (9) and (10), respectively, the linearly independent elements are six, and they reflect the velocity and acceleration of point M, as well as the angular velocity and acceleration of the n -th coordinate system [15]. Taking into consideration the degrees of freedom of the chain, the linearly independent elements of matrix T_{0n}, and the respective elements of matrices $\dot{T}_{0n}$, $\ddot{T}_{0n}$, for a chain with $n \geq 6$, equations (4, 5), (6, 7), and (9, 10) are presented in the following form:

(13) $$H=\begin{bmatrix} \rho_{0,M} \\ tr(\tau_{0n}) \end{bmatrix},$$

(14) $$H1=\begin{bmatrix} \dot{\rho}_{0,M} \\ tr(\dot{\tau}_{0n}) \end{bmatrix}=DH.\dot{Q},$$

(15) $$H2=\begin{bmatrix} \ddot{\rho}_{0,M} \\ tr(\ddot{\tau}_{0n}) \end{bmatrix}=DH.\ddot{Q}+\sum_{i=1}^{n} DDH_i.\dot{Q}.\dot{q}_i,$$

where $DH=\frac{\partial H}{\partial Q}$, $DDH_i=\frac{\partial DH}{\partial g_i}$ consist of linearly independent elements of matrices $\frac{\partial T_{0n}}{\partial q_i}$, $\frac{\partial^2 T_{0n}}{\partial q_i \partial q_j}$, $\frac{\partial^2 T_{0n}}{\partial q_j^2}$. This is given in detail in [14].

The relation between the vectors of angular velocity $\omega_{0,n}$ and angular acceleration $\varepsilon_{0,n}$ of the n-th link, and matrices $\dot{\tau}_{0n}$ and $\ddot{\tau}_{0n}$ is

(16) $$\omega_{0,n}=\dot{\tau}_{0n}.\tau_{0n}^{T},$$

(17) $$\varepsilon_{0,n}=\ddot{\tau}_{0n}.\tau_{0n}^{T}+\dot{\tau}_{0n}.\dot{\tau}_{0n}^{T}.$$

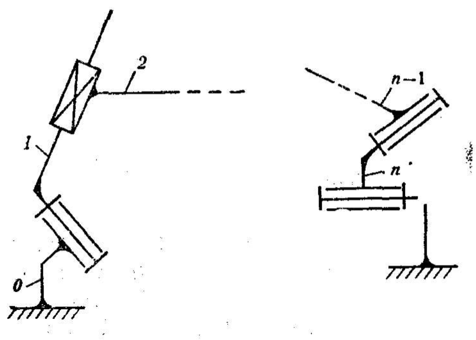
2. Kinematic Model of Poly-contour Spatial Chains

In closed one-contour spatial kinematic chains (Fig. 2) made up of binary links only, it is possible to orientate the coordinate systems, pertaining to the links, along the axes of the pairs of Class 5. The successive coordinate transformations of the mutually connected coordinate systems along the closed chain is given by equation

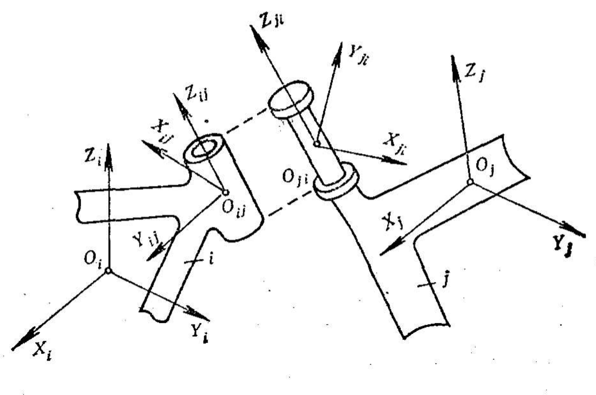
(18) $$T_{01}.T_{12}...T_{n-1,n}.T_{n0}=E,$$

where E is a 4-by-4 identity matrix.

In closed poly-contour spatial chains, the presence of links having more pairs does not allow for a coordinate system to be oriented for each pair, so that this coordi-



Фиг. 2



Фиг. 3

nate system should be one and only one for each link [16]. Therefore, each link of the chain is identified with a freely oriented coordinate system. The kinematic parameters of the link are determined by the coordinate systems oriented in each pair These coordinate systems do not change their position with respect to the link coordinates in the process of motion. Fig. 3 shows a way of orienting the coordinate systems of two links, i and j, of a poly-contour chain, the links being connected with a common pair $ij\,(ji)$. Coordinate systems, with subscripts corresponding to the subscripts of the links, are connected to the links. Pair ij is connected with a coordinate system which also pertains to the i-th link and is indexed by ij; pair ji is connected with a coordinate system indexed by ji. With the coordinate systems thus selected, it is possible to apply the well-known methods by using Euler coordinates or the parameters of Denavit and Hartenberg [1], in order to account for the location of the ij-th coordinate system with respect to the i-th, as well as the location of the ji-th coordinate system with respect to the j-th coordinate system.

When axis Z_{ij} coincides with the axis of pair ij, and when axis Z_{ji} coincides with the axis of pair ji (Fig. 3), then the relative position of the mutually connected links depends on the angle between axes X_{ij} and X_{ji} for rotational pairs (angle φ_{ij}), or it depends on the distance $\overline{O_{ij}O_{ji}}$ between the coordinate systems for translatory pairs (parameter S_{ij}).

According to the above-mentioned, the matrices of the relative position of the j-th link with respect to the i-th link are the following:

$$\tau_{ij}=\tau_{i,ij}\cdot\begin{bmatrix}\cos\varphi_{ij} & -\sin\varphi_{ij} & 0\\ \sin\varphi_{ij} & \cos\varphi_{ij} & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\tau^T_{j,ji}, \tag{19}$$

$$T_{ij}=T_{i,ij}.\begin{bmatrix}\cos\varphi_{ij} & -\sin\varphi_{ij} & 0 & 0\\ \sin\varphi_{ij} & \cos\varphi_{ij} & 0 & 0\\ 0 & 0 & 1 & S_{ij}\\ 0 & 0 & 0 & 1\end{bmatrix}.T^{-1}_{j,ji}, \tag{20}$$

where

$$T^{-1}_{j,ji}=\left[\begin{array}{c:c}\tau^T_{j,ji} & \tau^T_{j,ji}\cdot(-\rho_{j,oji})\\ \hdashline 0 & 1\end{array}\right]. \tag{21}$$

In equations (19, 20, 21), $\tau_{i,ij}$, $\tau_{j,ji}$, $T_{i,ij}$, and $T_{j,ji}$, are constant matrices, φ_{ij} is a variable parameter in a rotational pair $ij\,(ji)$, and S_{ij} is a variable parameter in a translatory pair $ij\,(ji)$.

By reducing the mathematical description of the mutually connected links to matrices of the relative position, the successive coordinate transformations in tracing the path of a closed spatial mechanism are presented in a way that is similar to the way open one-contour chains are presented, i. e., in the following form:

$$T^m=T^m_{ij}.T^m_{jk}\ldots T^m_{li}=E,\quad m=1,\,2,\ldots,\,q, \tag{22}$$

where q is the number of contours of the mechanism, required for the complete mathematical description, while subscripts ij, jk ... denote the pairs pertaining to the m-th contour. In equations (22) T_{ij} depends only on the parameter q_{ij} which is equal to φ_{ij} for rotational pairs, and it is equal to S_{ij} for translatory pairs. Relative velocities and accelerations of motion of the links are determined similarly to '" 10) from equations

$$\dot{T}^m=\frac{\partial T^m}{\partial Q}\cdot\dot{Q}=0,\quad m=1,\,2,\ldots,\,q, \tag{23}$$

$$(24)\qquad \ddot{T}^m=\frac{\partial \dot{T}^m}{\partial Q}\cdot\ddot{Q}+\sum_i\frac{\partial^2\dot{T}^m}{\partial Q\,.\,\partial q_i}\cdot\dot{Q}\cdot\dot{q}_i=0,\quad m=1,2,\ldots,q,$$

where 0 is a 4-by-4 zero matrix, while vector $Q=[q_1q_2\ldots q_p]^T=[q_{ij}q_{jk}\ldots q_{li}]^T$ is a vector of the ordered relative displacements in the joints with a dimension p — number of pairs (of Class 5). q_{ij} or q_{ji} participate in vector Q, on the condition that $q_{ij}=-q_{ji}$. In forced-motion mechanisms (the degrees of freedom are controlled by power output mechanisms), vector Q is arranged in the following ordered fashion:

$$(25)\qquad Q=\begin{bmatrix}Q_f\\ Q_s\end{bmatrix},$$

where Q_s is the vector of the driven displacements.

According to (20), the partial derivatives of matrix T^m:

$$\frac{\partial T^m}{\partial q_{ij}},\ \frac{\partial^2 T^m}{\partial q_{ij}^2},\ \frac{\partial^2 T^m}{\partial q_{ij}\,\partial q_{jk}},$$

are determined from

$$\frac{\partial T_{ij}}{\partial q_{ij}}=T_{i,ij}\cdot\begin{bmatrix}-\sin\varphi_{ij} & -\cos\varphi_{ij} & 0 & 0\\ \cos\varphi_{ij} & -\sin\varphi_{ij} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\cdot T_{j,ji}^{-1}=-\frac{\partial T_{ij}}{\partial q_{ji}}$$

for a rotational pair; and from

$$\frac{\partial T_{ij}}{\partial q_{ij}}=T_{i,ij}\cdot\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\cdot T_{j,ji}^{-1}=-\frac{\partial T_{ij}}{\partial q_{ij}}$$

for a translatory pair. The order of differentiation depends on the arrangement of the elements in vector Q, as well as on the direction of tracing around the path of the links (from the i-th link to the j-th link, or vice versa) when describing the contours. We have $\frac{\partial^2 T_{ij}}{\partial q_{ij}^2}$ for a rotational pair, and $\frac{\partial^2 T_{ij}}{\partial q_{ij}^2}=0$ for a translatory pair. Similarly to equations (13, 14, 15), the linearly independent terms of matrices T^m, $\dot{T}^m$, and $\ddot{T}^m$, yield matrices $H=[H^1H^2\ldots H^q]^T$, $H1=[H1^1H1^2\ldots H1^q]^T$, and $H2=[H2^1H2^2\ldots H2^q]^T$; while from the linearly independent elements of matrices $\frac{\partial T^m}{\partial Q}$, and $\frac{\partial^2 T^m}{\partial Q\,\partial q_i}$, we get matrices $DH=[DH^1DH^2\ldots DH^q]^T$, $DDH_i=[DDH_iDDH_i\ldots DDH_i]^T$, where matrices DH and DDH_i, similarly to (25), are arranged in the following order: $DH=[DH_fDH_s]$, $DDH_i=[DDHi_fDDHi_s]$.

3. Kinematic Analysis of Spatial Mechanisms

The kinematic model of spatial mechanisms suggested here can be used both for the purposes of analysis and for synthesis. The model is used here for solving the direct and the inverse problem of kinematics, both for open and for closed spatial kinematic chains.

The solution of the direct problem of kinematics for open kinematic chains is comparatively easy, and it is reduced to finding the location, the velocities and accelerations of the output link (matrices H, $H1$, $H2$) with preset relative displacements, velocities and accelerations in the joints (vectors Q, $\dot{Q}$, $\ddot{Q}$). Practically, the explicit equations (4), (5), (6), (7), (9, 10) should be solved with respect to r_M, T_{0n}, $\dot{r}_M$, $\dot{T}_{0n}$, $\ddot{r}_M$, $\ddot{T}_{0n}$.

For solving the inverse problem — which consists in finding the vector of the generalized coordinates with a preset location of the output link (matrix $\bar{H}$) — the method of Newton for solving non-linear equations is used, namely: concerning a configuration Q^i in which equations (13) of the type

$$\bar{H}=H(Q^i) \tag{26}$$

are not satisfied, such corrections ΔQ^i are sought, for which

$$[H(Q^i)-\bar{H}]+DH(Q^i)\,.\,\Delta Q^i=0. \tag{27}$$

If $H(Q^{i+1})-\bar{H}\leq\varepsilon$, where $Q^{i+1}=Q^i+\Delta Q^i$, and ε is a sufficiently small, preset number, then $Q=Q^{i+1}$ is the solution sought; otherwise, equation (27) is solved again. With vector Q being defined, and with preset velocities and accelerations of motion of the output link (matrices $\overline{H1}$ and $\overline{H2}$), the systems of equations (14) and (15) are successively solved

$$\overline{H1}=DH\,.\,\dot{Q}, \tag{28}$$

$$\overline{H2}-\sum_{i=1}^{n}DDH_i\,.\,\dot{Q}\,.\,\dot{q}_i=Dh\,.\,\ddot{Q}, \tag{29}$$

linear with respect to the unknown $\dot{Q}$ and $\ddot{Q}$.

The kinematic analysis of closed poly-contour spatial chains is reduced to finding the motions of the mutually connected links, with preset values of the driven displacements Q_s, $\dot{Q}_s$, $\ddot{Q}_s$. This problem is similar to the inverse problem concerning open chains, having in mind that

$$\bar{H}=[\bar{H}^1\,\bar{H}^2\,\ldots\,\bar{H}^q]^T, \tag{30}$$

where $\bar{H}^i=[0\ 0\ 0\ 1\ 1\ 1]^T$, $i=1, 2, \ldots, q$.

$$\overline{H1}=-DH_s\dot{Q}_s=DH_f\,.\,\dot{Q}_f, \tag{31}$$

$$\overline{H2}=-DH_s\,.\,\ddot{Q}_s-\sum_{i=1}^{p}DDH_i\,\dot{Q}\,.\,\dot{q}_i=DH_f\,.\,\ddot{Q}_f \tag{32}$$

whereupon, the system of non-linear equations (30) is solved according to the previously-mentioned iteration method of Newton with respect to the unknown $\Delta Q_f^i(\Delta Q_s^i=0)$. Equations (31) and (32) are systems of linear equations with respect to the unknown $\dot{Q}_f$ and $\ddot{Q}_f$.

References

1. Denavit, J., R. Hartenberg. A Kinematic Notation for Lower Pair Mechanisms Based on Matrices, Trans. ASME, ser. E, 1955, Vol. 22, No 2, 201—206
2. Uicker, J., J. Denavit, R. Hartenberg. An Iterative Method for the Displacement Analysis of Spatial Mechanisms, Trans. ASME, ser. E, 1964, Vol. 31, No 3, 309—314
3. Диментберг, Ф. М. Винтовое исчисление и его приложения в механике. М., Наука, 1965 г.
4. Константинов, М., М. Марков. Първи предавателни функции на пространствени кинематични вериги, Теоретична и приложна механика, БАН, 1977, Т. 8, № 2, 18—23

5. Литвин, Ф. Л. Определение функции положения пространственого механизма способом условного размыкания контура, М., „Машиноведение", 1970, №3
6. Hartenberg, R., J. Denavit. A Kinematic Notation for Lower Pair Mechanisms Based on Matrices, Journal of appl. mechanics, 1954, paper No 54—A—34
7. Минков, К. Роботика. Механика на манипулиращи системи. Софийски университет „Кл. Охридски", 1985
8. Bottema, O., B. Roth. Theoretical Kinematics, Amsterdam etc.: Publ. Co., North-Holland, 1979
9. Шет, Уикер мл. Программа синтеза механизмов: Система машинного анализа и проектирования механизмов различного назначения. Динамические системы и управления, 1972, № 2, 193—202
10. Паул, Крайчинович. Применение вычислительных машин для анализа плоских механизмов. Прикладная механика, 1970, № 3, 128—137
11. Пейсах, Э. Е. Алгоритм синтеза пространственных передаточных рычажных механизмов. Труды ЛИТЛП. 1971, № 11, 294—297
12. Göcke, M., J. Rößner. KOGEAN — ein Algoritmus zur vollautomatischen Analyse ebener Mechanismen von Berechnung bis zur lochbandge-steuerten Zeichnung — Ausgabe, WZ der TH Karl—Marx—Stadt, 1973, H. 1
13. Лилов, Л. Механо-математично моделиране на системи от взаимно свързани тела, Докторска дисертация, ИМБ, БАН, С., 1984
14. Konstantinov, M., P. Genova, E. Zakhariev. Direct Kinematic Control of Industrial Manipulators and Robots, Proceedings of the 11th Int. Symp. on Industrial Robots and Robot Exhibit, Tokyo, 1981
15. Константинов, М., П. Генова, Е. Захариев. Някои проблеми на численото решаване на кинематичната обратна задача на роботиката, Год. на ВМЕИ „Ленин", 37, № 8 1982—1983
16. Воскреенский, В. В., Б. И. Павлов. Методы решения задач кинематики и динамики пространственных механизмов на ЭБМ, сб. „Кинематика, динамика и точность механизмов", М., „Машиностроение". 1984

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