БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София, 1985, год. XVI, № 4

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia, 1985, Year XVI, No 4

The Three-Dimensional Problem for Anisotropic Media and Solids

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Many problems in mechanics and theoretical physics demand solving second order systems of linear partial differential equations. Using numerical methods in these cases is not always convenient, because of the difficulties for the results to be used for an extensive analysis. On the other hand, these problems are extremely difficult to be solved analytically, because of the lack of general enough methods. Usually the Fourier method for separating variables is used because of it's simplicity and convenience, but there is an important disadvantage that it can be applied only for a restricted class of problems. There are other methods developed, too, which deal only with very special tasks.

In this paper a general method is proposed for obtaining very broad classes of partial solutions for second order systems of partial differential equations of three arguments. Obtaining a complete solution of the problem becomes possible by using the method developed by the author for reducing the number of arguments in systems of partial differential equations [1, 2, 3, 4]. This leads to a very general and effective transformation of the linear system of partial differential equations into systems of ordinary differential equations, where many convenient methods are available for their solution.

One very important partial case for the theory and practice is the three-dimensional problem of elasticity, which we will discuss later in the paper.

Let us have a system

$$D^{kl}_{2pq}\frac{\partial^2 u^p}{\partial x_k \partial x_l}+D^l_{1pq}\frac{\partial u^p}{\partial x_l}+D_{0pq}u^p=0, \tag{1}$$

where

$$k,\ l=1,\ 2,\ 3,$$

$$p,\ q=1,\ 2, \dots\ Q.$$

We have to mention that here and further on everywhere if same letters are used in expressions — no matter upper or lower indices, powers, orders or derivatives, marked as variable — it means that this is a sum of such expressions with respect to those letters, the sign Σ designating a sum being omitted. The index q means that this is a system of equations. We will use the index p for functions and indices k and l for the independent variables.

In the case of anisotropic theory of elasticity (1) becomes (2)

(2) $$D^{kl}_{2pq}\frac{\partial^2 u^p}{\partial x_k \partial x_l}=0,$$

where

$$k,\ l=1,\ 2,\ 3$$
$$p,\ q=1,\ 2,\ 3$$

or in an expanded form

$$A_{11}\frac{\partial^2u^1}{\partial x_1^2}+A_{66}\frac{\partial^2u^1}{\partial x_2^2}+A_{55}\frac{\partial^2u^1}{\partial x_3^2}+(A_{16}+A_{61})\frac{\partial^2u^1}{\partial x_1\partial x_2}+(A_{56}+A_{65})\frac{\partial^2u^1}{\partial x_2\partial x_3}$$
$$+(A_{15}+A_{51})\frac{\partial^2u^1}{\partial x_1\partial x_3}+A_{16}\frac{\partial^2u^2}{\partial x_1^2}+A_{62}\frac{\partial^2u^2}{\partial x_2^2}+A_{54}\frac{\partial^2u^2}{\partial x_3^2}+(A_{12}+A_{66})\frac{\partial^2u^2}{\partial x_1\partial x_2}$$
$$+(A_{64}+A_{52})\frac{\partial^2u^2}{\partial x_2\partial x_3}+(A_{14}+A_{56})\frac{\partial^2u^2}{\partial x_1\partial x_3}+A_{15}\frac{\partial^2u^3}{\partial x_1^2}+A_{64}\frac{\partial^2u^3}{\partial x_2^2}+A_{53}\frac{\partial^2u^3}{\partial x_3^2}$$
$$+(A_{14}+A_{65})\frac{\partial^2u^3}{\partial x_1\partial x_2}+(A_{54}+A_{63})\frac{\partial^2u^3}{\partial x_2\partial x_3}+(A_{13}+A_{55})\frac{\partial^2u^3}{\partial x_1\partial x_3}=0,$$

(3) $$A_{61}\frac{\partial^2u^1}{\partial x_1^2}+A_{26}\frac{\partial^2u^1}{\partial x_2^2}+A_{45}\frac{\partial^2u^1}{\partial x_3^2}+(A_{21}+A_{66})\frac{\partial^2u^1}{\partial x_1\partial x_2}+(A_{25}+A_{46})\frac{\partial^2u^1}{\partial x_2\partial x_3}$$
$$+(A_{41}+A_{65})\frac{\partial^2u^1}{\partial x_1\partial x_3}+A_{66}\frac{\partial^2u^2}{\partial x_1^2}+A_{22}\frac{\partial^2u^2}{\partial x_2^2}+A_{44}\frac{\partial^2u^2}{\partial x_3^2}+(A_{26}+A_{62})\frac{\partial^2u^2}{\partial x_1\partial x_2}$$
$$+(A_{24}+A_{42})\frac{\partial^2u^2}{\partial x_2\partial x_3}+(A_{46}+A_{64})\frac{\partial^2u^2}{\partial x_1\partial x_3}+A_{65}\frac{\partial^2u^3}{\partial x_1^2}+A_{24}\frac{\partial^2u^3}{\partial x_2^2}+A_{43}\frac{\partial^2u^3}{\partial x_3^2}$$
$$+(A_{25}+A_{64})\frac{\partial^2u^3}{\partial x_1\partial x_2}+(A_{23}+A_{44})\frac{\partial^2u^3}{\partial x_2\partial x_3}+(A_{45}+A_{63})\frac{\partial^2u^3}{\partial x_1\partial x_3}=0,$$

$$A_{51}\frac{\partial^2u^1}{\partial x_1^2}+A_{46}\frac{\partial^2u^1}{\partial x_2^2}+A_{35}\frac{\partial^2u^1}{\partial x_3^2}+(A_{41}+A_{56})\frac{\partial^2u^1}{\partial x_1\partial x_2}+(A_{36}+A_{45})\frac{\partial^2u^1}{\partial x_2\partial x_3}$$
$$+(A_{31}+A_{55})\frac{\partial^2u^1}{\partial x_1\partial x_3}+A_{56}\frac{\partial^2u^2}{\partial x_1^2}+A_{42}\frac{\partial^2u^2}{\partial x_2^2}+A_{34}\frac{\partial^2u^2}{\partial x_3^2}+(A_{46}+A_{52})\frac{\partial^2u^2}{\partial x_1\partial x_2}$$
$$+(A_{32}+A_{44})\frac{\partial^2u^2}{\partial x_2\partial x_3}+(A_{36}+A_{54})\frac{\partial^2u^2}{\partial x_1\partial x_3}+A_{55}\frac{\partial^2u^3}{\partial x_1^2}+A_{44}\frac{\partial^2u^3}{\partial x_2^2}+A_{33}\frac{\partial^2u^3}{\partial x_3^2}$$
$$+(A_{54}+A_{45})\frac{\partial^2u^3}{\partial x_1\partial x_2}+(A_{34}+A_{43})\frac{\partial^2u^3}{\partial x_2\partial x_3}+(A_{35}+A_{53})\frac{\partial^2u^3}{\partial x_1\partial x_3}=0.$$

Here in the full anisotropic media the number of constants of elasticity is 21. In the special cases of orthotropic, axially symmetric, symmetric with respect to a plane etc., the number of the mentioned constants is reduced and in the most simple case of an isotropic body it is two.

Let us return now to the general case (1). According to the mentioned method, we put

(4) $$u^p(x_1,\ x_2,\ x_3)=X_i(x_1)\,.\,Y_j(x_2)\,.\,Z^p_{ij}(x_3),$$

where

$$i=0,\ 1,\ 2,\dots\ m_1-1$$
$$j=0,\ 1,\ 2,\dots\ m_2-1,$$

i. e. the solution u^p is searched using classes of linear independent functions $X_i(x_1)$ and $Y_j(x_2)$. It mustn't be mistaken with the Fourier method, where there are no sums, i. e. there is only one product of functions with separated variables. The solution of (4) is much more general and it is not in the class of separated variables. Individual products in (4) are not solutions of (1), only their sum is a solution.

In accordance with (1) the derivatives $\frac{\partial u^p}{\partial x^l}$, $\frac{\partial^2 u^p}{\partial x^k \partial x^l}$ etc. must be expressed with the help of same independent functions, i. e.

$$X_i' = a_i^{i'} X_{i'} \qquad (5)$$
$$Y_j' = b^{j'} Y_{j'},$$

where

$$i' = 0, 1, 2, \dots m_1 - 1$$
$$j' = 0, 1, 2, \dots m_2 - 1.$$

It's not difficult to be seen that for the discussed problem, without restricting the generality, condition (4) could be transformed in (6) and (5) in (7)

$$u^p(x_1, x_2, x_3) = X^{(i)}(x_1) \,.\, Y^{(j)}(x_2) \,.\, Z_{ij}^{(0)p}(x_3), \qquad (6)$$

$$X^{(m_1)} = c_i X^{(i)} \quad (i = 0, 1, 2, \dots m_1 - 1), \qquad (7)$$
$$Y^{(m_2)} = d_j Y^{(j)} \quad (i = 0, 1, 2, \dots m_2 - 1).$$

If we denote

$$\mathbf{M}_1 = \begin{bmatrix} 0 & 1 & 0 \dots 0 \\ 0 & 0 & 1 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 \\ c_0 & c_1 & c_2 \dots c_{m_1-1} \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 1 & 0 \dots 0 \\ 0 & 0 & 1 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 \\ d_0 & d_1 & d_2 \dots d_{m_2-1} \end{bmatrix}, \qquad (8)$$

we can receive

$$X^{(i+\nu)} = [\mathbf{M}_1^\nu]_{i'}^i X^{(i')}, \quad (i' = 0, 1, 2, \dots m_1 - 1), \qquad (9)$$
$$Y^{(j+\nu)} = [\mathbf{M}_2^\nu]_{j'}^j Y^{(j')}, \quad (j' = 0, 1, 2, \dots m_2 - 1).$$

The solution of (7) in the practically most interesting case, when

$$X^{(m_1)} = -\sum_{i=1}^{m_1/2}{}' C_{m_1/2}^i X^{(m_1 - 2i)} \alpha^{2i}, \quad Y^{(m_2)} = -\sum_{j=1}^{m_2/2}{}' C_{m_2/2}^j \mathbf{Y}^{(m_2 - 2j)} \beta^{2j}, \qquad (10)$$

is

$$X(x_1) = A_\xi(\alpha) x_1^\xi \sin \alpha x_1 + \bar{A}_\xi(\alpha) x_1^\xi \cos \alpha x_1, \qquad (11)$$
$$Y(x_2) = B_\eta(\beta) x_2^\eta \sin \beta x_2 + \bar{B}_\eta(\beta) x_2^\eta \cos \beta x_2,$$

where α and β are free parameters, and $A_\xi(\alpha)$ and $B_\eta(\beta)$ are constants of integration, depending on them. Now, after differentiating (6) it follows

$$\frac{\partial u^p}{\partial x_l} = \delta_l^2 X^{(i)} Y^{(j+1)} Z_{ij}^{(0)p} + \delta_l^1 X^{(i+1)} Y^{(j)} Z_{ij}^{(0)p} + \delta_i^3 X^{(i)} Y^{(j)} Z_{ij}^{(1)p},$$

(12)
$$\frac{\partial^2 u^p}{\partial x_k \partial x_l} = \delta_k^2 \delta_l^2 X^{(i)} Y^{(j+2)} Z_{ij}^{(0)p} + \delta_k^1 \delta_l^1 X^{(i+2)} Y^{(j)} Z_{ij}^{(0)p}$$
$$+ \delta_k^3 \delta_l^3 X^{(i)} Y^{(j)} Z_{ij}^{(2)p} + \delta_k^2 \delta_l^1 X^{(i+1)} Y^{(j+1)} Z_{ij}^{(0)p} + \delta_k^1 \delta_l^2 X^{(i+1)} Y^{(j+1)} Z_{ij}^{(0)p} + \delta_k^2 \delta_l^3 X^{(i)} Y^{(j+1)} Z_{ij}^{(1)p}$$
$$+ \delta_k^1 \delta_l^3 X^{(i+1)} Y^{(j)} Z_{ij}^{(1)p} + \delta_k^3 \delta_l^1 X^{(i+1)} Y^{(j)} Z_{ij}^{(0)p} + \delta_k^3 \delta_l^2 X^{(i)} Y^{(j+1)} Z_{ij}^{(1)p}.$$

Replacing (6) and (12) in (1) we obtain

(13)
$$D_{2pq}^{kl} \{\delta_k^1 \delta_l^1 [M_1^2]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(0)p} + \delta_k^1 \delta_l^2 [M_1^1]_{i'}^i [M_2^1]_{j'}^j Z_{ij}^{(0)p} + \delta_k^1 \delta_l^3 [M_1^1]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(1)p}$$
$$+ \delta_k^2 \delta_l^1 [M_1^1]_{i'}^i [M_2^1]_{j'}^j Z_{ij}^{(0)p} + \delta_k^2 \delta_l^2 [M_1^0]_{i'}^i [M_2^2]_{j'}^j Z_{ij}^{(0)p} + \delta_k^2 \delta_l^3 [M_1^0]_{i'}^i [M_2^1]_{j'}^j Z_{ij}^{(1)p}$$
$$+ \delta_k^3 \delta_l^1 [M_1^1]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(1)p} + \delta_k^3 \delta_l^2 [M_1^0]_{i'}^i [M_2^1]_{j'}^j Z_{ij}^{(1)p} + \delta_k^3 \delta_l^3 [M_1^0]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(2)p}\} X^{(i')} . Y^{(j')}$$
$$+ D_{1pq}^l \{\delta_l^1 [M_1^1]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(0)p} + \delta_l^2 [M_1^0]_{i'}^i [M_2^1]_{j'}^j Z_{ij}^{(0)p} + \delta_l^3 [M_1^0]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(1)p}\} X^{(i')} . Y^{(j')}$$
$$+ D_{0pq} \{[M_1^0]_{i'}^i [M_2^0]_{j'}^j Z_{ij}^{(0)p}\} X^{(i')} . Y^{(j')} = 0$$

or after processing

(14)
$$\{D_{2pq}^{33} [M_1^0]_{i'}^i [M_2^0]_{j'}^j\} Z_{ij}^{(2)p} + \{2(D_{2pq}^{13} [M_1^1]_{i'}^i [M_2^0]_{j'}^j + D_{2pq}^{23} [M_1^0]_{i'}^i [M_2^1]_{j'}^j)$$
$$+ D_{1pq}^3 [M_1^0]_{i'}^i [M_2^0]_{j'}^j\} Z_{ij}^{(1)p} + \{D_{2pq}^{11} [M_1^2]_{i'}^i [M_2^0]_{j'}^j + 2D_{2pq}^2 [M_1^1] [M_2^1]_{j'}^j$$
$$+ D_{2pq}^{22} [M_1^0]_{i'}^i [M_2^2]_{j'}^j + D_{1pq}^1 [M_1^1]_{i'}^i [M_2^0]_{j'}^j + D_{1pq}^2 [M_1^0]_{i'}^i [M_2^1]_{j'}^j$$
$$+ D_{0pq} [M_1^0]_{i'}^i [M_2^0]_{j'}^j\} Z_{ij}^{(0)p} = 0.$$

This is a system of $Q \times m_1 \times m_2$ ordinary differential equations for the functions $Z_{ij}^p(x_3)$. So as a result of applying the method, the given system of partial differential equations (1) is reduced to three groups of ordinary differential equtions (7) and (14), the solutions of everyone of which give free parameters α and β and constants of integration to be found from the boundary conditions.

If we denote

(15)
$$L_{2pqi'j'}^{ij} = D_{2pq}^{33} [M_1^0]_{i'}^i [M_2^0]_{j'}^j,$$
$$L_{1pqi'j'}^{ij} = 2(D_{2pq}^{13} [M_1^1]_{i'}^i [M_2^0]_{j'}^j + D_{2pq}^{23} [M_1^0]_{i'}^i [M_2^1]_{j'}^j) + D_{1pq}^3 [M_1^0]_{i'}^i [M_2^0]_{j'}^j,$$
$$L_{0pqi'j'}^{ij} = D_{2pq}^{11} [M_1^2]_{i'}^i [M_2^0]_{j'}^j + 2D_{2pq}^{12} [M_1^1]_{i'}^i [M_2^1]_{j'}^j + D_{2pq}^{22} [M_1^0]_{i'}^i [M_2^2]_{j'}^j$$
$$+ D_{1pq}^1 [M_1^1]_{i'}^i [M_2^0]_{j'}^j + D_{1pq}^2 [M_1^0]_{i'}^i [M_2^1]_{j'}^j + D_{0pq} [M_1^0]_{i'}^i [M_2^0]_{j'}^j,$$

then (14) turns to (16)

(16)
$$L_{2pqi'j'}^{ij} Z_{ij}^{(2)p} + L_{1pqi'j'}^{ij} Z_{ij}^{(1)p} + L_{0pqi'j'}^{ij} Z_{ij}^{(0)p} = 0$$

or

$$L_{\nu pqi'j'}^{ij} Z_{ij}^{(\nu)p} = 0, \quad (\nu = 0, 1, 2).$$

It is evident from the discussions till now that D_{2pq}^{kl}, D_{1pq}^l and D_{0pq} could be functions of x_3 and a suitable known method (including numerical) may be applied for solving (16). In our case, as it is well known, the solutions of the system of ordinary differential equations with constant coefficients are

$$(17) \qquad Z_{ij}^{(0)p\theta}=\{C_\rho(\alpha,\ \beta)\}^{p+(i+m_1j)Q}\,.\,z^\rho\,.\,e^{\lambda_\theta z},$$

where

$$(\rho=0,\ 1,\ 2,\ldots\ m_1m_2-1).$$

Here the components of vectors $\{C_\rho(\alpha,\ \beta)\}$ are constants of integration, depending on the parameters — roots λ_θ.

Let us denote with K the matrix with elements

$$(18) \qquad [\mathrm{K}(\lambda_\theta)]_{p+(i+m_1j)Q}^{q+(i'+m_1j')Q}=(\lambda_\theta^2 L_{2pqi'j'}^{ij}+\lambda_\theta \mathrm{L}_{1pqi'j'}^{ij}+\mathrm{L}_{0pqi'j'}^{ij}),$$

where $\lambda_\theta(\alpha,\ \beta)$ are obtained from

$$(19) \qquad \det|\mathrm{K}(\lambda_\theta)|=0.$$

It is easy to prove that only $2\times Q$ roots $\lambda_\theta(\alpha,\ \beta)$ are independent, everyone of which is $m_1\times m_2$ multiple.

As it can be seen in [2], the determination of λ_θ could be easily performed by solving the less difficult "basic" determinant of order Q

$$(20) \qquad \det|D(\alpha,\ \beta,\ \gamma)|=0$$

that can be received directly from the initial system (1) formally replacing

$$(21) \qquad \begin{array}{lll} \dfrac{\partial u^p}{\partial x_1}\to i\alpha, & \dfrac{\partial^2 u^p}{\partial x_1^2}\to-\alpha^2, & \dfrac{\partial^2 u}{\partial x_1\partial x_2}\to-\alpha\beta, \\[2ex] \dfrac{\partial u^p}{\partial x_2}\to i\beta, & \dfrac{\partial^2 u^p}{\partial x_2^2}\to-\beta^2, & \dfrac{\partial^2 u}{\partial x_2\partial x_3}\to i\beta\gamma, \\[2ex] \dfrac{\partial u^p}{\partial x_3}\to\gamma, & \dfrac{\partial^2 u^p}{\partial x_3^2}\to\gamma^2, & \dfrac{\partial^2 u}{\partial x_3\partial x_1}\to i\alpha\gamma. \end{array}$$

In the case of the anisotropic theory of elasticity from (3) using (21) we have

$$(22) \qquad \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix}=0\to\gamma_{1,2,3,4,5,6}(\alpha,\ \beta)=\ \ldots,$$

where

$$\begin{array}{l} B_{12}=B_{21} \\ B_{13}=B_{31} \quad \text{if } A_{ij}=A_{ji}, \\ B_{23}=B_{32} \end{array}$$

$$(23) \qquad B_{11}=A_{55}\gamma^2+i[\alpha(A_{15}+A_{51})+\beta(A_{56}+A_{65})]\gamma-[\alpha^2A_{11}+\beta^2A_{66}+\alpha\beta(A_{16}+A_{61})],$$

$$B_{22}=A_{44}\gamma^2+i[\alpha(A_{46}+A_{64})+\beta(A_{24}+A_{42})]\gamma-[\alpha^2A_{66}+\beta^2A_{22}+\alpha\beta(A_{26}+A_{62})],$$

$$B_{33}=A_{33}\gamma^2+i[\alpha(A_{35}+A_{53})+\beta(A_{34}+A_{43})]\gamma-[\alpha^2A_{55}+\beta^2A_{44}+\alpha\beta(A_{45}+A_{54})],$$

$$B_{12}=B_{21}=A_{45}\gamma^2+i[\alpha(A_{14}+A_{56})+\beta(A_{46}+A_{25})]\gamma-[\alpha^2A_{16}+\beta^2A_{26}+\alpha\beta(A_{12}+A_{66})],$$

$$B_{23}=B_{32}=A_{34}\gamma^2+i[\alpha(A_{36}+A_{45})+\beta(A_{23}+A_{44})]\gamma-[\alpha^2A_{56}+\beta^2A_{24}+\alpha\beta(A_{46}+A_{25})],$$

$$B_{31}=B_{13}=A_{35}\gamma^2+i[\alpha(A_{13}+A_{55})+\beta(A_{36}+A_{45})]\gamma-[\alpha^2A_{15}+\beta^2A_{46}+\alpha\beta(A_{14}+A_{56})],$$

i. e. in the case of complete anisotropy there are 6 independent roots λ_0 which are $m_1 \times m_2$ multiple.

Now with the help of (17) and (18), (16) becomes

$$(24)\qquad \binom{s}{0}[\mathbf{K}^{(0)}]\{C_\rho\}+\binom{s+1}{1}[\mathbf{K}^{(1)}]\{C_{\rho+1}\}+\binom{s+2}{2}[\mathbf{K}^{(2)}]\{C_{\rho+2}\}=0,$$

or

$$\binom{s+\nu}{\nu}[\mathbf{K}^{(\nu)}]\{C_{\rho+\nu}\}=0, \quad (\nu=0,\ 1,\ 2).$$

From (24) a following recurrent dependence could be derived

$$(25)\qquad \{C_\rho(\alpha,\ \beta)\}=-[\mathbf{L}_\rho]^{-1}.[\mathbf{K}^{(0)}].\{C_{\rho-1}(\alpha,\ \beta)\},$$

where

$$[\mathbf{L}_s]=s.\ [\mathbf{K}^{(1)}],$$

$$(26)\qquad [\mathbf{L}_\rho]=\binom{\rho}{1}[\mathbf{K}^{(1)}]-\binom{9+1}{2}[\mathbf{K}^{(2)}][\mathbf{L}_{\rho+1}]^{-1}.[\mathbf{K}^{(0)}],$$

$$\det|\mathbf{L}_\rho|\neq 0,$$

where

$$(\rho=s,\ s-1,\ldots,\ 1).$$

The system (24) consists of linear dependent algebraic equations. As a parameter can be choosen, the vector $\{C_0(\alpha,\ \beta\}$ and every other vector $\{C_\rho(\alpha,\ \beta)\}$ will be its function

$$(27)\qquad \{C_\rho(\alpha,\ \beta)\}=(-1)^\rho\prod_{i=\rho}^{1}([\mathbf{L}_\rho]^{-1}.[\mathbf{K}^{(0)}]).\{C_0(\alpha,\ \beta)\},$$

or

$$(28)\qquad \{C_\rho(\alpha,\ \beta)\}=[\mathbf{N}_\rho].\{C_0(\alpha,\ \beta)\},$$

and

$$(29)\qquad \{C_\rho(\alpha,\ \beta)\}^{\rho+(i+m_1j)Q}=[\mathbf{N}_\rho]_\varkappa^{\rho+(i+m_1j)Q}.\{C_0(\alpha,\ \beta)\}^\varkappa,$$

where

$$(30)\qquad [\mathbf{N}_\rho]=(-1)^\rho\prod_{i=\rho}^{1}([\mathbf{L}_\rho]^{-1}.[\mathbf{K}^{(0)}],$$

for

$$\rho=1,\ 2,\ldots\ s.$$

Putting (29) in (17) we receive

$$(31)\qquad Z_{ij}^{\rho}=\sum_{\theta}[\mathbf{N}_\rho]_\varkappa^{\rho+(i+m_1j)Q}.\{C_0(\alpha,\ \beta)\}^\varkappa.z^\rho\, e^{\lambda_0 z}.$$

The expressions (11) and (31) give us at the end the class of partial solutions (6) of the initial system (1).

As it was noticed, the determination of the three groups constants of integration $A_\xi(\alpha)$, $B_\eta(\beta)$ and $\{C_0(\alpha,\ \beta)\}$ for every value of the parameters α and β, must be done so that the boundary conditions are satisfied. These conditions could be given analytically or discretely on surfaces, lines or at points.

In the case of the three- dimensional problem for anisotropic media and solids, the boundary conditions are given directly in displacement as external loading or they are mixed. We must append to the formulae used so long, the expressions for the stresses. In the case of complete anisotropy we have

$$\sigma_{kl} = A_{kqlp} \cdot \varepsilon_{lp}, \tag{32}$$

or

$$\sigma_{kq} = \frac{1}{2} A_{kqlp}\left(\frac{\partial u^l}{\partial x_p} + \frac{\partial u^p}{\partial x_l}\right), \tag{33}$$

or

$$\sigma_{kq} = \frac{1}{2}(A_{kqlp} + A_{kqpl})\frac{\partial u^p}{\partial x_l}. \tag{34}$$

Using (12) we can receive

$$\sigma_{kq} = \frac{1}{2}\{[(A_{kq1p} + A_{kqp1})[M_1^1]^i_{i'}[M_2^0]^j_{j'} + (A_{kq2p} + A_{kqp2})[M_1^0]^i_{i'}[M_2^1]^j_{j'} \cdot Z^{(0)p}_{ij} + [(A_{kq3p} + A_{kqp3})[M_1^0]^i_{i'}[M_2^0]^j_{j'}] \cdot Z^{(1)p}_{ij}\} \cdot X^{(i')} \cdot Y^{(j')}. \tag{35}$$

Without entering into details, we shall only mention that the method proposed may be extended on non-homogeneous systems of linear partial differential equations without essential complications and therefore the method may be applied as an approximate one in the case of non-linear differential equations after a proper linearization. It may be also applied in cases where the coefficients of the basic system of partial differential equtions are not constant and are given as functions of x_3.

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Received 10. 03. 1985