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Restricted Length Configurations

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Abstract. The paper deals with the problem of finding a restricted length plane curve which minimizes a convex functional invariant under isometric transformations. Necessary and sufficient conditions characterizing the solution are given. The particular problem of determining the restricted length minimal energy configuration of an elastic string is considered in details.

1. Problem Statement

Consider the problem of finding a function $x(.)\in C^2[-1, 1]$ satisfying the conditions

(1) $$x(-1)=x(1)=0;\quad \dot{x}(-1)=u\leq 0,\quad \dot{x}(1)=v>0,$$

(2) $$\int_{-1}^{1}(1+\dot{x}^2(t))^{1/2}dt=\int_{(-1,0)}^{(1,0)}ds\leq 2L\ (L>1)$$

and minimizing the functional $K(.)$: $C^2[-1, 1]\to R_+$,

(3) $$K(x)=\int_{-1}^{1}\frac{\ddot{x}^{2m}(t)\,dt}{D^2(\dot{x}(t))},$$

where $m\geq 1$ is an integer and $D\in C^1(R)$ is an even and positive valued function.

If we suppose that K is invariant under rotations in the (t, x)-plane then by necessity

4) $$D(p)=(1+p^2)^{(6m-1)/4}.$$

Note that (4) yields

(5) $$K(x)=\int_{-1}^{1}\ddot{x}^{2m}(t)(1+\dot{x}^2(t))^{1/2-3m}dt=\int_{(-1,0)}^{(1,0)}k^{2m}(s)ds\to\min,$$

where $k(.)$ is the natural parametrization of the curvature. If $m=1$ and $x(.)$ describes the configuration of an elastic string passing through the points $(-1,0)$, $(1,0)$ with slopes u, v, then $K(x)$ as defined from (5) is proportional to the full strain energy. Hence the solution of (1), (2), (5) with $m=1$ gives the restricted length minimal energy configuration of an elastic string with fixed ends and slopes.

Obviously (1)—(3) can be viewed as a convex two-point Hermite interpolation problem. If $m=1$ and $w=(u^2+v^2)^{1/2}$ is small then the solution of the unrestricted problem (1), (5) is approximately described by a 3rd degree polynomial.

Setting $\dot{x}(.)=p(.)$ the problem (1)—(3) can be rewritten as

$$J(p)=\int_{-1}^{1}\frac{\dot{p}^{2m}(t)\,dt}{D^2(p(t))}\to\min \tag{6}$$

subject to the constraints

$$p(-1)=u,\ p(1)=v, \tag{7}$$

$$\int_{-1}^{1}p(t)dt=0, \tag{8}$$

$$\int_{-1}^{1}(1+p^2(t))^{1/2}dt\le 2L\quad (L>1). \tag{9}$$

Denote the problem (6)—(9) by $\mathbf{P}(u, v, L)$ and let $\tilde{p}(.)$ be the solution of the unconstrained problem $\mathbf{P}(u, v, \infty)$. Then the constraint (9) is active if and only if

$$L<\tilde{L}=\frac{1}{2}\int_{-1}^{1}(1+\tilde{p}^2(t))^{1/2}dt.$$

2. Characterization of the Solution

Introduce the auxiliary functional $\bar{J}(.):\mathbf{C}^1[-1,1]\to\mathbf{R}_+$,

$$\bar{J}(p)=J(p)+\bar{B}p+\bar{C}(1+p^2)^{1/2},$$

where $\bar{B}$, $\bar{C}$ are Lagrange multipliers. The standard Ouler-Lagrange conditions for minimization of $\bar{J}$ give the following characterization of the solution:

The function $\hat{p}(.)\in\mathbf{C}^1[-1,1]$ solves (6)—(9) if and only if

$$\dot{\hat{p}}(t)=(A+B\hat{p}(t)+C(1+\hat{p}^2(t))^{1/2})^{1/2m}D^{1/m}(\hat{p}(t))=R(\hat{p}(t);A,B,C),\ t\in[-1,1], \tag{10}$$

where the constants A, B, C are determined from

$$\int_{u}^{v}\frac{dp}{R(p;A,B,C)}=2, \tag{11}$$

$$\int_{u}^{v}\frac{p\,dp}{R(p;A,B,C)}=0, \tag{12}$$

$$(13) \qquad \int_u^v \frac{(1+p^2)^{1/2}\,\mathrm{d}p}{R(p\,;A,B,C)} = 2L\,(L<\tilde{L}).$$

Let $\tilde{A}$, $\tilde{B}$ be the solution of the system

$$(14) \qquad \int_u^v \frac{\mathrm{d}p}{R(p\,;A,B,0)} = 2,$$

$$(15) \qquad \int_u^v \frac{p\mathrm{d}p}{R(p\,;A,B,0)} = 0.$$

Then

$$L = 0.5 \int_{-1}^{1} (1+\tilde{p}^2(t))^{1/2}\,\mathrm{d}t = 0.5 \int_u^v \frac{(1+p^2)^{1/2}\mathrm{d}p}{R(p\,;\tilde{A},\tilde{B},0)},$$

where $\tilde{p}(.)$ is the solution of the unrestricted problem $\mathbf{P}(u, v, \infty)$ defined from (10)—(12).

The analysis of the system (11)—(13) for determining A, B, C shows that for each $u\leq 0$, $v>0$ and $1<L<\tilde{L}$ there exists a solution A, B, C such that $C>0$. Moreover, $B=0$ if and only if $u+v=0$.

Setting $A/C=a$, $B/C=b$ one has

$$(16) \qquad \int_u^v \frac{p\mathrm{d}p}{R(p\,;a,b,1)} = 0,$$

$$(17) \qquad \left[\int_u^v \frac{(1+p^2)^{1/2}\mathrm{d}p}{R(p\,;a,b,1)}\right] \Big/ \left[\int_u^v \frac{\mathrm{d}p}{R(p\,;a,b,1)}\right] = L$$

and

$$A=aC,\ B=bC,\ C=\left[0.5\int_u^v \frac{\mathrm{d}p}{R(p\,;a,b,1)}\right]^{2m}.$$

Moreover, it follows from (10) and (6)—(9) that

$$J(\hat{p}) = A + LC = (a+L)C.$$

Hence the problem is reduced to the solution of the system (16), (17) for the unknown constants a, b. In the next section we consider in more details the symmetric case $u+v=0$ for $m=1$.

3. Minimal Energy Symmetric Configurations

In what follows we assume that $u+v=0$, $m=1$ and that $D(.)$ is defined from (4), i. e. $D(p)=(1+p^2)^{-5/4}$. This corresponds to the minimal energy configuration of an elastic string or a bar with restricted length, fixed ends and symmetric slopes

$$J(p)=\int_{-1}^{1} \dot{p}^2(t)(1+p^2(t))^{-5/2}\,dt \to \min, \tag{18}$$

$$p(-1)=-v,\ p(1)=v,\ (v>0), \tag{19}$$

$$\int_{-1}^{1} (1+p^2(t))^{1/2}\,dt \leq 2L,\ (L>1), \tag{20}$$

$$\int_{-1}^{1} p(t)\,dt=0. \tag{21}$$

It is easy to see that here $B=b=0$ (the condition (21) is always fulfilled for this choice of B) and (10) becomes

$$\dot{p}(t)=(A+C(1+p^2(t))^{1/2}(1+p^2(t))^{5/4},\ t\in[-1,1]. \tag{22}$$

Consider first the unconstrained problem (18), (19) for which $C=0$ and $A=\tilde{A}$ is determined from

$$A=\tilde{A}(v)=\int_0^v (1+p^2)^{-5/4}\,dp=2G^2(v),$$

where $G(v)=2E(g(v), 1/\sqrt{2})-F(g(v), 1/\sqrt{2})$, $g(v)=\arccos(1+v^2)^{-1/4}$ and $F(g, r)$ $E(g, r)$ are the first and second kind elliptic integrals

$$F(g, r)=\int_0^g (1-r^2\sin^2 p)^{-1/2}\,dp,\ E(g, r)=\int_0^g (1-r^2\sin^2 p)^{1/2}\,dp.$$

Having in mind (19) the length of the optimal unconstrained curve

$$\{(t, \tilde{x}(t)) : t\in[-1, 1]\},\ \dot{\tilde{x}}(t)=\tilde{p}(t),\ \text{is}\ 2\tilde{L}(v),$$

$$\tilde{L}(v)=\frac{1}{\tilde{A}^{1/2}(v)}\int_0^v (1+p^2)^{-3/4}\,dp=F(g(v), 1/\sqrt{2})/G(v),$$

while $\tilde{p}(.)$ is given by

$$\tilde{p}(t)=G^{-1}(tG(v))$$

($G^{-1}(.)$ is the inverse of $G(.)$). In particular the constraint (20) is active if and only if $1<L<\tilde{L}(v)$.

Consider now the general case $L<\tilde{L}(v)$.

Define the function $f_v(.)\colon \mathbf{I}\to \mathbf{R}_+$, $\mathbf{I}=(-1, \infty)$, from

$$f_v(a)=\frac{\int_0^v (1+p^2)^{-3/4}(a+(1+p^2)^{1/2})^{-1/2}\,dp}{\int_0^v (1+p^2)^{-5/4}(a+(1+p^2)^{1/2})^{-1/2}\,dp}. \tag{23}$$

Straightforward computations show that $f_v(.)$ is increasing, and

$$\lim_{a\downarrow -1} f_v(a)=1, \quad \lim_{a\to\infty} f_v(a)=\tilde{L}(v).$$

Hence $f_v(\mathrm{I})=(1, L(v))$.

As it follows from (17) the constants A, C in (22) are uniquely determined from

(24) $$A=a(v, L)\,C(v, L),\ C=C(v, L),$$

where

(25) $$C(v, L)=\left[\int_0^v (1+p^2)^{-5/4}(a(v, L)+(1+p^2)^{1/2})^{-1/2}dp\right]^2$$

and $a=a(v, L)$ is the unique solution of the equation

(26) $$f_v(a)=L.$$

Setting $V=(1+v^2)^{-1/2}$ the function $f_v(.)$ admits the following representation:

(i) $-1<a<0$

$$f_v(a)=aF(g_{-1}, r_{-1})/((a+1)\,\Pi(g_{-1}, r_{-1}^2, r_{-1})-F(g_{-1}, r_{-1})),$$

where

$$g_{-1}=\arccos((a+1+aV)/(2+2aV)),\ r_{-1}=(2a/(a-1))^{1/2}$$

and $\Pi(g, n, r)$ is the third kind elliptic integral

$$\Pi(g, n, r)=\int_0^g (1+n\sin^2 p)^{-1}(1-r^2\sin^2 p)^{-1/2}\,dp.$$

(ii) $a=0$

$$f_v(0)=(1+v^{-2})^{1/2}\operatorname{arctg} v.$$

(iii) $0<a<1$

$$f_v(a)=aF(g_0, r_0)/((a+1)E(g_0, r_0)-F(g_0, r_0)),$$

where

$$g_0=\arccos((1+V)/2),\ r_0=(2a/(a+1))^{1/2}.$$

(iv) $a=1$

$$f_v(1)=2\ln W/(2(2-2V)^{1/2}+\ln W),$$

where $W=(3-V+2(2-2V)^{1/2})/(V+1)$.

(v) $a>1$

$$f_v(a)=F(g_1, r_1)/(2E(g_1, r_1)-F(g_1, r_1)),$$

where

$$g_1=\arccos((1+aV)/(a+1)),\ r_1=((1+a)/2a)^{1/2}.$$

It is worth mentioning that only for

$$L=L_c(v)=(1+v^{-2})^{1/2}\operatorname{arctg} v$$

(see (ii) above) the optimal curve is a part ot a circle.

An important characteristic of the optimal curve $\{(t, \widehat{x}(t)) : t \in [-.1,1]\}$, $\dot{\widehat{x}}(t)=\widehat{p}(t)$, is its deflection

$$h=-x(0)=\int_0^1 \widehat{p}(t)\mathrm{d}t=\int_0^v p(A+C(1+p^2)^{1/2})^{-1/2}(1+p^2)^{-5/4}\mathrm{d}p,$$

where A, C are defined from (24)—(26). The explicit expression for h is

$$h=((1+a)^{1/2}-(1+aV)^{1/2})/(aC^{1/2}),\ a\neq 0,$$
$$h=(1-V)/(vV),\ a=0,$$
$$h=(1-V^{1/2})/\tilde{A}^{1/2}(v),\ a=\infty,$$

where $a=a(v, L)$, $C=C(v, L)$, $V=(1+v^2)^{-1/2}$.

Note finally that the limit case $v\to\infty$ is simply obtained from the above formulae setting $V=0$. In particular

$\tilde{A}(\infty)=\Gamma^2(.75)/(\sqrt{2}\,\Gamma(1.5))\approx 1.1983$,

$\tilde{L}(\infty)=2.25(\Gamma(1.25)/\Gamma(1.75))^2\approx 2.1882$, etc.

4. The Lagrange and Hermite Multipoint Restricted Length Problems

Let $N+1$ points (t_i, x_i) $(N\geq 2)$ be given

$$-1=t_0<\cdots<t_N=1;\ x_0=x_N=0;\ x_1^2+\cdots+x_{N-1}^2>0.$$

Then the Lagrange restricted length problem is

$$K(x)=\int_{-1}^{1}\ddot{x}^2(t)(1+\dot{x}^2(t))^{-5/2}\,\mathrm{d}t\to\min \tag{27}$$

subject to the constraints

$$x(t_i)=x_i;\ i=0,\ldots,N \tag{28}$$

$$\int_{-1}^{1}(1+\dot{x}^2(t))^{1/2}\,\mathrm{d}t\leq 2L, \tag{29}$$

where

$$L>0.5\sum_{i=0}^{N-1}((t_{i+1}-t_i)^2+(x_{i+1}-x_i)^2)^{1/2}.$$

The Hermite problem includes the additional restrictions

$$\dot{x}(t_i)=y_i;\ i=0,\ldots,N, \tag{30}$$

where $y_0,\ldots,y_N$ are given numbers.

These statements correspond to the configuration of a restricted length elastic string passing through the points (t_i, x_i) with free (the Lagrange problem) or fixed (the Hermite problem) slopes.

The solutions of (27)–(29) and (27)–(30) are given in terms of second order nonlinear multipoint boundary value problems arising after imbedding the problems in optimal control frameworks and applying the Pontryagin maximum principle [1] (see e. g. [2], where the Lagrange problem is considered for $K(x)$ replaced with $\bar{K}(x)=\|x\|^2_{L_2}$).

It must be pointed out that the solution of the unconstrained Lagrange problem (27), (28) is approximately described by a cubic spline provided the number $d=(d_1^2+\cdots+d_N^2)^{1/2}$, $d_i=(x_{i+1}-x_i)/(t_{i+1}-t_i)$, is small (i. e. if $K(x)$ is replaced with $\bar{K}(x)$). The exact solution, however, is a generalized spline: at each interval (t_i, t_{i+1}) the derivative $p_i(.)$ of $x(.)$ satisfies an equation of type (10), and the fitting conditions $\dot{p}_i(t_{i+1})=\dot{p}_{i+1}(t_{i+1})$ are fulfilled.

References

1. Alekseev, V. M., V. M. Tihomirov, S. V. Fomin. Optimal Control. Nauka, M. 1982 (in Russian).
2. Dontchev, A. L., G. I. Iliev, M. M. Konstantinov. Constrained interpolation via optimal control. Proc. 14th Spring Conf. of UBM, Bourgas, April 1985, 385-392.

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