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Antisymmetric distribution of thermal stresses in the vicinity of a flat external crack in an infinite solid

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ABSTRACT: — An attempt has been made to solve antisymmetric distribution of thermal stress in the vicinity of a flat external crack in an infinite solid. It is assumed that the temperature is prescribed inside the crack while the flux at the outside of the crack. The problem is reduced into a pair of dual integral equations and the closed form solution has been obtained.

1. Introduction

Uflyand [1] was the first to solve equations of elastic equilibrium appropriate to an infinite solid containing a flat crack covering the outside of a circle. In this paper, he employed toroidal coordinates and Boussinesq-Popkovich solutions of the equations of elastic equilibrium. Subsequently, Lowengrub and Sneddon [2] studied the asymmetric distribution of stress in the vicinity of an external crack in an infinite elastic body, employing William's representation of the displacement vector in terms of harmonic functions, while Shail [3] has studied the distribution of thermoelastic stresses in the vicinity of an external crack in an infinite solid. The even and odd distributions of temperature respectively have been considered by [3] with regard to the plane of the crack. Das [4] has extended his studies to some axially symmetric thermal stress distributions in an elastic solid containing an external crack. Recently Lal [5] has taken up axially symmetric thermal stress distributions in a transversely isotropic solid containing an external crack. By using integral transform techniques, Das [4] and Lal [5] reduced the problems into a pair of dual integral equations and a closed form solution was obtained.

In the present paper we use cylindrical coordinates (r, θ, z) and choose our unit of length to be radius of the circle. It is assumed that the surfaces of the crack are free from stress and are subjected to a prescribed temperature within the crack while outside the crack, flux is prescribed. The problem is reduced into a pair of dual integral equations and closed form solution is derived by using Hankel transform technique. As an example, the solution for the two point loading problem has also been attempted.

2. Formulation of the problem and boundary conditions

Consider an infinite, isotropic, elastic medium which contains a flat external crack whose faces are stress free. Using cylindrical polar coordinates (r, θ, z), the faces of the crack are described by the relations $z=0$, $r \geq 1$, $0 \leq \theta \leq 2\pi$, with a suitable choice of unit of length. There is established in the solid a steady temperature field $T(r, \theta, z)$ where T is the deviation of the absolute temperature from the temperature of the solid in a state of zero stress and strain. In the absence of body forces or heat sources within the medium the steady state equations of classical thermoelasticity, the temperature field satisfies the partial differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=0. \tag{2.1}$$

The equations of equilibrium in cylindrical coordinates in terms of the displacements u_r, u_θ and u_z in the r, θ and z directions are

$$\begin{aligned}
&\nabla^2 u_r+\frac{1}{(1-2\eta)}\frac{\partial \Delta}{\partial r}-\frac{1}{r}\left(2\frac{\partial u_\theta}{r\partial \theta}+\frac{u_r}{r}\right)=0,\\
&\nabla^2 u_\theta+\frac{1}{(1-2\eta)}\frac{\partial \Delta}{r\partial \theta}-\frac{1}{r}\left(\frac{u_\theta}{r}-2\frac{\partial u_r}{\partial \theta}\right)=0,\\
&\nabla^2 u_z+\frac{1}{(1-2\eta)}\frac{\partial \Delta}{\partial z}=0,
\end{aligned} \tag{2.2}$$

where ∇^2 and η detnote the Laplacian operator and Possion's respectively and Δ denotes the dilatation which is given by

$$\Delta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_\theta}{r\partial \theta}+\frac{\partial u_z}{\partial z}\ . \tag{2.3}$$

The boundary conditions for the temperature field on the plane $z=0$ are assumed to be in the form

$$\begin{aligned}
&T(r, \theta, 0)=\sum_{m=0}^{\infty} T^1_m(r)\cos(m\theta); \quad 0\leq\theta\leq 2\pi; \quad 1\leq r<\infty,\\
&\frac{\partial T(r, \theta, 0)}{\partial z}=-\sum_{m=0}^{\infty} R_m(r)\cos(m\theta); \quad 0\leq\theta\leq 2\pi; \quad 0\leq r<1,
\end{aligned} \tag{2.4}$$

where $T^1_m(r)$ and $R_m(r)$ are known functions of r.

As a result of the symmetry we can postulate that the shearing stress vanishes on the boundary plane of the half-space. In this way, we obtain the boundary conditions.

$$u_z(r, \theta, 0)=0; \quad 0\leq\theta\leq 2\pi; \quad 0\leq r\leq 1, \tag{2.5}$$

$$\begin{aligned}
\sigma_{zz}(r, \theta, 0)&=-p(r, \theta); \quad 0\leq\theta\leq 2\pi; \quad r>1,\\
&=-\sum_{m=0}^{\infty} P_m(r)\cos(m\theta),
\end{aligned} \tag{2.6}$$

$$\left.\begin{array}{l}\sigma_{rz}(r,\ \theta,\ 0)=0\\ \sigma_{\theta z}(r,\ \theta,\ 0)=0\end{array}\right\};\quad 0\leqq\theta\leqq 2\pi\,;\quad r\geqq 0 \tag{2.7}$$

and we assume that $p(r,\ \theta)$ is an even function of θ and $P_m(r)$ is a known function of r.

In section 3 the mixed boundary value problem for the half-space $z\geqq 0$ posed by these equations is reduced, by the use of Hankel transform technique, to the solution of a pair of dual integral equations whose solution is known by Sneddon [6]. The use of results is illustrated in section 4 by the consideration of a special type loading.

3. Solution of the problem

(a) The Heat Conduction Problem:

A suitable integral representation of the temperature function satisfying equation (2.1) in the form

$$T(r,\ \theta,\ z)={\sum_{m=0}^{\infty}}'\int_0^{\infty} A(\xi)\cos(m\theta)e^{-\xi z}J_m(\xi r)\,\mathrm{d}\xi, \tag{3.1}$$

where $A(\xi)$ is a function of ξ only to be determined from the boundary condition (2.4). We may write

$$T_m(r,\ z)=\int_0^{\infty} A(\xi)\,e^{-\xi z}J_m(\xi r)\,\mathrm{d}\xi, \tag{3.2}$$

where $T_m(r,\ z)$ is the solution of differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\,\frac{\partial T}{\partial r}-\frac{m^2}{r^2}T+\frac{\partial^2 T}{\partial z^2}=0. \tag{3.3}$$

Using the equation (3.1), the boundary conditions (2.4) are satisfied, if the function $A(\xi)$ is the solution of the pair of dual integral equations

$$\int_0^{\infty}\xi A(\xi)J_m(\xi r)\mathrm{d}\xi=R_m(r)\,;\quad 0\leqq r<1, \tag{3.4}$$

$$\int_0^{\infty}A(\xi)J_m(\xi r)\mathrm{d}\xi=T_m^1(r),\quad 1\leq r<\infty. \tag{3.5}$$

The dual integral equations (3.4) and (3.5) have already been considered by Gordon [7], Copson [8]. Putting $\beta=0$, $\alpha=-\frac{1}{2}$ and $\nu=m$ in their solutions, we may take

$$A(\xi)=\sqrt{\frac{2}{\xi}}\left\{\int_0^1\sqrt{r}\ \varphi_1(r)J_m(\xi r)\mathrm{d}r+\int_1^r\sqrt{r}\ \varphi_2(r)J_m(\xi r)\,\mathrm{d}r,\right. \tag{3.6}$$

where

$$\varphi_1(r)=\frac{1}{r^m\sqrt{\pi}}\int_0^r \frac{r^{m+1}R_m(r)}{\sqrt{(r^2-t^2)}}\,\mathrm{d}r,\quad 0\leq r<1,$$

(3.7)

$$\varphi_2(r)=-\frac{r^m}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{t^{1-m}T_m^1(t)}{\sqrt{(t^2-r^2)}}\,\mathrm{d}t,\quad 1\leq r<\infty.$$

With these values of $A(\xi)$, the temperature and flux fields are thus completely determined.

(b) Determination of Displacements and Stresses:

Now, we proceed to solve the mechanical part of the problem to evaluate displacements and stresses.

For the determination of displacements and stresses, a particular solution of the equations of equilibrium (2.2) can be taken according to Muki [9] as

$$u_{r_1}=K\frac{\partial B}{\partial r};\quad u_{z_1}=K\frac{\partial B}{\partial z}, \tag{3.8}$$

with corresponding stress components given by

(3.9)

$$(\sigma_{zz_1}/2\mu)=K\left(\frac{\partial^2 B}{\partial z^2}-T\right),$$

$$(\sigma_{\theta z_1}/2\mu)=K\frac{\partial^2 B}{r\partial\theta\partial z};\quad (\sigma_{rz_1}/2\mu)=K\frac{\partial^2 B}{\partial r\partial z},$$

where $K=(1+\eta)\varepsilon/(1-\eta)$, ε is the coefficient of linear expansion and B is a particular solution of the equation

$$\nabla^2 B(r,\,\theta,\,z)=T(r,\,\theta,\,z). \tag{3.10}$$

Let us assume B as

$$B(r,\,\theta,\,z)=\sum_{m=0}^{\infty}B_m(r,\,z)\cos(m\theta), \tag{3.11}$$

and T is defined by equations (3.1) and (3.2) as

$$T(r,\,\theta,\,z)=\sum_{m=0}T_m(r,\,z)\cos(m\theta). \tag{3.12}$$

Now using the equations (3.10) to (3.12), the following equation has been obtained

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)B_m(r,\,z)=T_m(r,\,z). \tag{3.13}$$

If we multiply both sides by $rJ_m(\xi r)$ in equation (3.13) and integrate it with respect to r over the range $(0,\,\infty)$, it is found

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\xi^2\right)L_m(\xi,\,z)=M_m(\xi,\,z), \tag{3.14}$$

where

$$L_m(\xi,\ z)=\int\limits_0^\infty rB_m(r,\ z)J_m(\xi r)\,\mathrm{d}r,$$

$$M_m(\xi,\ z)=\int\limits_0^\infty rT_m(r,\ z)\,J_m(\xi r)\,\mathrm{d}r.$$

Now we may take

$$L_m(\xi,\ z)=-\{zA(\xi)e^{-\xi z}\}/2\xi^2, \tag{3.15}$$

using the equations (3.14) and (3.15), we obtain

$$M_m(\xi,\ z)=\{A(\xi)e^{-\xi z}\}/\xi. \tag{3.16}$$

Now the transformation of the expressions (3.8) and (3.9) for the displacements and stresses into relations involving $L_m(\xi,\ z)$, $M_m(\xi,\ z)$ and their derivatives are obtained as follows

$$\begin{aligned}
u_{r_1}&=-(K/2)\sum_{m=0}^{\infty}\left[\int\limits_0^\infty \xi^2 L_m J_{m+1}(\xi r)\mathrm{d}\xi-\int\limits_0^\infty \xi^2 L_m J_{m-1}(\xi r)\mathrm{d}\xi\right]\cos(m\theta),\\
u_{z_1}&=K\sum_{m=0}^{\infty}\left[\int\limits_0^\infty \xi\,\frac{\mathrm{d}L_m}{\mathrm{d}z}\,J_m(\xi r)\,\mathrm{d}\xi\right]\cos(m\theta),\\
(\sigma_{zz_1}/2\mu)&=K\sum_{m=0}^{\infty}\left[\int\limits_0^\infty \xi^3 L_m J_m(\xi r)\mathrm{d}\xi\right]\cos(m\theta),\\
(\sigma_{\theta z_1}/\mu)&=K\sum_{m=0}^{\infty}\left[\int\limits_0^\infty \xi^2\,\frac{\mathrm{d}L_m}{\mathrm{d}z}J_{m+1}(\xi r)\mathrm{d}\xi\right.\\
&\qquad\left.+\int\limits_0^\infty \xi^2\,\frac{\mathrm{d}L_m}{\mathrm{d}z}J_{m-1}(\xi r)\mathrm{d}\xi\right]\sin(m\theta),\\
(\sigma_{rz_1}/\mu)&=K\sum_{m=0}^{\infty}\left[\int\limits_0^\infty \xi^2\,\frac{\mathrm{d}L_m}{\mathrm{d}z}\,J_{m+1}(\xi r)\mathrm{d}\xi\right.\\
&\qquad\left.-\int\limits_0^\infty \xi^2\,\frac{\mathrm{d}L_m}{\mathrm{d}z}\,J_{m-1}(\xi r)\mathrm{d}\xi\right]\cos(m\theta),
\end{aligned} \tag{3.17}$$

where $L_m(\xi,\ z)$ is defined by equation (3.15).

To satisfy the boundary conditions (2.5) to (2.7), we superpose on (3.17), the complementary displacements and stresses according to Muki [9] are given by

$$u_{r_2}=-\frac{\partial^2\varphi}{\partial r\partial z}+\frac{2}{r}\ \frac{\partial\psi}{\partial\theta};\quad u_{z_2}=2(1+\eta)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2},$$

$$(3.18)\qquad \begin{aligned}(\sigma_{zz_2}/2\mu)&=\frac{\partial}{\partial z}\left[(2-\eta)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\right],\\ (\sigma_{\theta z_2}/2\mu)&=\frac{1}{r}\ \frac{\partial}{\partial\theta}\left[(1-\eta)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\right]-\frac{\partial^2\psi}{\partial r\partial z},\\ (\sigma_{rz_2}/2\mu)&=\frac{\partial}{\partial r}\left[\left(1-\eta\right)\nabla^2\varphi-\frac{\partial^2\varphi}{\partial z^2}\right]+\frac{\partial^2\psi}{r\partial\theta\partial z},\end{aligned}$$

where φ and ψ satisfy the equations

$$(3.19)\qquad \nabla^4\varphi=0;\quad \nabla^2\psi=0.$$

Expanding the biharmonic function φ and the harmonic function ψ in the following forms

$$(3.20)\qquad \begin{aligned}\varphi(r,\theta,z)&=\sum_{m=0}^{\infty}\varphi_m(r,z)\cos(m\theta),\\ \psi(r,\theta,z)&=\sum_{m=0}^{\infty}\psi_m(r,z)\cos(m\theta),\end{aligned}$$

it is easily estimated that φ_m and ψ_m are the solutions of the following partial differential equations

$$(3.21)\qquad \nabla^4\varphi_m=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\ \frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)^2\varphi_m=0,$$

$$(3.22)\qquad \nabla^2\psi_m=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\ \frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z^2}\right)\psi_m=0,$$

and writing

$$(3.23)\qquad \begin{aligned}G_m(\xi,z)&=\int_0^{\infty}r\,\varphi_m(r,z)J_m(\xi r)\,\mathrm{d}r,\\ H_m(\xi,z)&=\int_0^{\infty}r\psi_m(r,z)J_m(\xi r)\mathrm{d}r,\end{aligned}$$

we obtain that $G_m(\xi,z)$, $H_m(\xi,z)$ are the Hankel transforms of $\varphi_m(r,z)$ and $\psi_m(r,z)$ and must be the solutions of the ordinary differential equations

$$(3.24)\qquad \begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\xi^2\right)^2G_m(\xi,z)=0,\\ &\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\xi^2\right)H_m(\xi,z)=0.\end{aligned}$$

The valid solutions of (3.24) can be written as

$$(3.25)\qquad \begin{aligned}G_m(\xi,z)&=(A_1+B_1z)e^{-\xi z},\\ H_m(\xi,z)&=C_1e^{-\xi z},\end{aligned}$$

where A_1, B_1 and C_1 are the functions of ξ only and are to be determined from the boundary conditions (2.7).

Now the Henkel inversion of (3.23) gives

$$\varphi_m(r, z)=\int_0^\infty \xi G_m(\xi, z)J_m(\xi r)\mathrm{d}\xi \tag{3.26}$$

$$\psi_m(r, z)=\int_0^\infty \xi H_m(\xi, z)J_m(\xi r)\mathrm{d}\xi.$$

The complementary displacements and stresses are respectively obtained from (3.18) by using (3.20) and (3.26) which are given by

$$u_{r_2}=\frac{1}{2}\sum_{m=0}^{\infty}\int_0^\infty\left\{\frac{\mathrm{d}G_m}{\mathrm{d}z}+2H_m\right\}\xi^2J_{m+1}(\xi r)\mathrm{d}\xi$$

$$-\int_0^\infty\left\{\frac{\mathrm{d}G_m}{\mathrm{d}z}+2H_m\right\}\xi^2J_{m-1}(\xi r)\mathrm{d}\xi]\cos(m\theta),$$

$$u_{z_2}=\sum_{m=0}^{\infty}\left[\int_0^\infty\left\{(1-2\eta)\frac{\mathrm{d}^2G_m}{\mathrm{d}z^2}-2(1-\eta)\xi^2G_m\right\}\xi J_m(\xi r)\mathrm{d}\xi\right]\cos(m\theta),$$

$$\frac{\sigma_{zz_2}}{2\mu}=\sum_{m=0}^{\infty}\left[\int_0^\infty\left\{(1-\eta)\frac{\mathrm{d}^3G_m}{\mathrm{d}z^3}-2(1-\eta)\xi^2\frac{\mathrm{d}G_m}{\mathrm{d}z}\right\}\xi J_m(\xi r)\mathrm{d}\xi\right]\cos(m\theta),$$

$$\frac{\sigma_{\theta z_2}}{\mu}=\sum_{m=0}^{\infty}\left[\int_0^\infty\left\{\eta\frac{\mathrm{d}^2G_m}{\mathrm{d}z^2}+(1-\eta)\xi^2G_m+\frac{\mathrm{d}H_m}{\mathrm{d}z}\right\}\xi^2J_{m+1}(\xi r)\mathrm{d}\xi\right. \tag{3.27}$$

$$\left.+\int_0^\infty\left\{\eta\frac{\mathrm{d}^2G_m}{\mathrm{d}z^2}+(1-\eta)\xi^2G_m-\frac{\mathrm{d}H_m}{\mathrm{d}z}\right\}\xi^2J_{m-1}(\xi r)\mathrm{d}\xi\right]\sin(m\theta),$$

$$\frac{\sigma_{rz_2}}{\mu}=\sum_{m=0}^{\infty}{}'\left[\int_0^\infty\left\{\eta\frac{\mathrm{d}^2G_m}{\mathrm{d}z^2}+(1-\eta)\xi^2G_m+\frac{\mathrm{d}H_m}{\mathrm{d}z}\right\}\xi^2J_{m+1}(\xi r)\mathrm{d}\xi\right.$$

$$\left.-\int_0^\infty\left\{\eta\frac{\mathrm{d}^2Gm}{\mathrm{d}z^2}+(1-\eta)\xi^2G_m-\frac{\mathrm{d}H_m}{\mathrm{d}z}\right\}\xi^2J_{m-1}(\xi r)\mathrm{d}\xi\right]\sin(m\theta),$$

where $G_m(\xi, r)$ and $H_m(\xi, z)$ are given equation (3.25). Thus the resultant displacements and stresses are obtained by summing up the corresponding relations of (3.17) and (3.27) and are given as

$$u_r=u_{r_1}+u_{r_2},\quad u_z=u_{z_1}+u_{z_2}, \tag{3.28}$$

$$\sigma_{\theta z}=\sigma_{\theta z_1}+\sigma_{\theta z_2},\quad \sigma_{rz}=\sigma_{rz_1}+\sigma_{rz_2},$$

$$\sigma_{zz}=\sigma_{zz_1}+\sigma_{zz_2}.$$

The boundary condition (2.7) with the aid of (3.28) yield

$$\begin{aligned}[\xi^2A_1-2\eta\xi B_1-(KA\xi^{-2})/2]\,[J_{m+1}(\xi r)\\ -J_{m-1}(\xi r)]-\xi C_1[J_{m+1}(\xi r)+J_{m-1}(\xi r)]=0,\end{aligned} \tag{3.29}$$

$$\begin{aligned}[\xi^2A_1-2\eta\xi B_1-(KA\xi^{-2})/2]\,[J_{m+1}(\xi r)\\ +J_{m+1}(\xi r)]-\xi C_1[J_{m+1}(\xi r)-J_{m-1}(\xi r)]=0.\end{aligned} \tag{3.30}$$

To satisfy the boundary conditions we assume $C_1=0$ in equations (3.29) and (3.30) and these will give

$$B_1=(KA\xi^{-3}/4\eta)-\xi A_1, \tag{3.31}$$

where A_1 is a known function and defined in equation (3.6). Now the boundary conditions (2.5) and (2.6) with the help of equations (3.28) and (3.31) are reduced to dual integral equations

$$\int_0^\infty \xi^{-1}g(\xi)J_m(\xi r)\mathrm{d}\xi=0, \quad 0\leq r<1, \tag{3.32}$$

$$\int_0^\infty g(\xi)J_m(\xi r)\mathrm{d}\xi=F_m(r), \quad r>1, \tag{3.33}$$

where

$$\begin{aligned}\xi^{-1}g(\xi)&=(4\eta-1)A_1(\xi)+\{(1-\eta)K/2\eta\}A(\xi),\\ F_m(r)&=\frac{(1-4\eta)}{4\mu\eta}P_m(r)+\frac{(4\eta^2-2\eta-1)K}{8\eta^2}\int_0^\infty A(\xi)J_m(\xi r)\mathrm{d}\xi.\end{aligned} \tag{3.34}$$

The dual integral equation (3.32) anp (3.33) have been considered by Noble [10], Williams [11] and Lowengrub and Sneddon [12]. Putting $\beta=1/2$ $\nu=m$ in the general solution of Lowengrub and Sneddon, we may take

$$g(\xi)=(2\xi^3/\pi)^{1/2}\int_1^\infty t^{m+1/2}H_m(t)J_{m-1/2}(\xi t)\mathrm{d}t, \tag{3.35a}$$

where

$$H_m(t)=\int_t^\infty \frac{r^{1-m}F_m(r)\mathrm{d}r}{\sqrt{(r^2-t^2)}}\,. \tag{3.35b}$$

The general solution of the problem is obtained by substituting the values from (3.35) into displacements expressions.

5. Solution for the two-point loading problem

As an example, we consider the case in which point forces of magnitude P act at the points $(a,\ 0,\ 0+)$, $(a,\ 0,\ 0-)$, $(a>1)$ of the crack surface. Expanding the Dirac function $\delta(\varphi)$ in a Fourier series we find that $F_0(r)$ and $F_m(r)$ can be written as follows

$$F_0(r)=\frac{P}{2r}\delta(r-a), \quad F_m(r)=\frac{P}{r}\,\delta(r-a), \tag{4.1}$$

and hence that

$$H_0(t)=\frac{P}{2}(a^2-t^2)^{-1/2}H(a-t)\,; \quad H_m(t)=P\mathrm{a}^{-m}(a^2-t^2)^{-1/2}\,H(a-t), \tag{4.2}$$

where H is Heavy side step function. Thus we see that

$$\int\limits_1^r \frac{t^{2m}H_m(t)\mathrm{d}t}{\sqrt{(r^2-t^2)}}=Pa^{-m}\int\limits_1^r \frac{t^{2m}H(a-t)\mathrm{d}t}{\sqrt{[(r^2-t^2)\,(a^2-t^2)}}\,. \tag{4.3}$$

Now, the two cases $1<r<a$ and $r>a$ must be treated separately. For $1<t<r<a$, we have the equation

$$\int\limits_1^\infty \frac{t^{2m}\mathrm{d}t}{\sqrt{[(r^2-t^2)\,(a^2-t^2)}}=\frac{r^{2m}}{a}\,I_m(k_1,\ \beta_1), \tag{4.4}$$

where

$$I_m(k,\ \beta)=\int\limits_\beta^{\pi/2}\frac{\sin^{2m}\theta\mathrm{d}\theta}{\sqrt{\ [1-k^2\sin^2\theta]}}, \tag{4.5}$$

and

$$k_1=r/a, \quad \beta_1=\sin^{-1}(1/r), \tag{4.6}$$

so that, in the usual notation for elliptic integrals [13], we have the formula

$$u_z(r,\ \theta,\ 0)=\frac{2(1-\eta)P}{\pi a^2}[K_1(k_1)-F(k_1,\ \beta_1)$$

$$+2a\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^m I_m(k_1,\ \beta_1)\cos{(m\theta)}], \quad 1<r<a. \tag{4.7}$$

Similarly, if $r>a$ we find that

$$\int\limits_1^r \frac{t^{2m}H_m(t)\mathrm{d}t}{\sqrt{(r^2-t^2)}}=\frac{Pa^m}{r}\,I_m(k_2,\ \beta_2),$$

where

$$k_2=a/r, \quad \beta_2=\sin^{-1}(1/a), \tag{4.8}$$

so that

$$u_z(r,\ \theta,\ 0)=\frac{2(1-\eta)P}{\pi r^2}[K_1(k_2)-F(k_2,\ \beta_2)$$

$$+2r\sum_{m=1}^{\infty}\left(\frac{a}{r}\right)^m I_m(k_2,\ \beta_2)\cos{(m\theta)}], \quad r>a. \tag{4.9}$$

The integrals (4.5) can be evaluated numerically for any particular values of m, k, β.

The solution of this problem when temperature, flux and loading function $P(r, \theta)$ do not depend on angle θ assumes a much simpler form. By putting $m=0$, the problem is reduced to a particular case of symmetrical loading.

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