БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ
ТЕОРЕТИЧНА И ПРИЛОЖНА МЕХАНИКА
София, 1985, год. XVI, № 1

BULGARIAN ACADEMY OF SCIENCES
THEORETICAL AND APPLIED MECHANICS
Sofia, 1985, Year XVI, No 1

A rule of nonsymmetric anisotropic hardening

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1. Introduction

A lot of experiments exist which show that the yield surface changes its shape if a preliminary plastic deformation had taken place. If the preliminary loading process is followed by total unloading, the subsequent yield surface of the reloading process changes, but keeps, more or less, its symmetric form [7, 16, 10]. In the case of partial unloading, the subsequent yield surface changes nonsymmetrically, moving in the preloading direction, and obtaining an extruded part in the same direction and flatness on the opposite side [1, 2, 3, 4, 5]. Such a phenomenon may take place in a structure under complex loading. During the prescribed loading programme, some regions of the structure could deform plastically, undergo later a partial elastic unloading and be later deformed plastically again. A theoretical model, describing the case of a symmetric subsequent yield surface is given in [9]. A lot of theoretical models exist, describing the nonsymmetrical character of the yield surface [2, 5, 6, 7]. The aim of this paper is to propose a nonsymmetric subsequent yield surface, for the case of partial unloading, based on the conception of a macrocomposite material structure. Metals, which are considered homogeneous on macrolevel, consist of two components on microlevel — the component of grains and that of grain boundaries. Both components have different mechanical properties. This microcomposite structure was discussed in [8, 9]. According to the mechanical model proposed there, microstresses appear in the body, due to plastic deformation.

2. Thermodynamical analysis

Assume that the specific free enthalpy function z exists [14]. It depends on the state variables: the stress tensor σ_{ij} and the absolute temperature θ_t as well as on the internal state variables. We assume the plastic strain tensor ε^p_{ij} and the microstress tensor σ^μ_{ij} to be internal state variables. The microstress tensor σ^μ_{ij} is thermodynamically conjugated with the inelastic strain tensor ε^μ_{ij}, which is part of the inelastic strain ε^p_{ij}. We assume that a one-to-one valued relation between σ^μ_{ij} and ε^μ_{ij} exists. We involve the function $Z=-\rho_0 z$, where ρ_0 is the initial material density. The material is plastically incompressible ($\varepsilon^p_{kk}=0$). In that case only the microstress deviator $s^\mu_{ij}=f_{ij}(\varepsilon^\mu_{kl})$ appears in Z. The considered model describes linear elastic and thermal properties, hence Z takes the form:

$$(1)\qquad Z=\frac{1}{2}H_{ijkl}\sigma_{ij}\sigma_{kl}+\bar{s}_{ij}\varepsilon^p_{ij}+\alpha_T\delta_{ij}\Delta\theta_t+\rho_0c_T\theta_t\ln\frac{\theta_t}{\theta_{t0}}-\rho_0c_T\Delta\theta_t+Z_0$$

where H_{ijkl} is the elastic resistance tensor, the components of which are the elastic material constants; $\alpha_T=\text{const}$ is the thermal expansion coefficient; $c_T=\text{const}$ is the thermal capacity coefficient; Z_0 is the initial value of Z; $\Delta\theta_t=\theta_t-\theta_{t0}$, θ_{t0} is the initial temperature distribution; $\bar{s}_{ij}=s_{ij}-s^\mu_{ij}$ is the active stress deviator [11]; s_{ij} is the stress deviator; δ_{ij} is Kronecker's delta.

Isothermal processes are considered, i. e. $\Delta\theta_t=0$. The function Z takes then the form:

$$(2)\qquad Z=\frac{1}{2}H_{ijkl}\sigma_{ij}\sigma_{kl}+[s_{ij}-f_{ij}(\varepsilon^\mu_{kl})]\varepsilon^p_{ij}+Z_0.$$

The active stress deviator $\bar{s}_{ij}$ appears in Z, as the real stresses, which are acting in the infinitessimal neighbourhood of a body point correspond to the active stresses $\bar{\sigma}_{ij}=\sigma_{ij}-\sigma^\mu_{ij}$.

The condition that the thermodynamical process is admissible, leads to the following three relations:

$$(3)\qquad \varepsilon_{ij}=\frac{\partial Z}{\partial\sigma_{ij}}=H_{ijkl}\sigma_{kl}+\varepsilon^p_{ij}=\varepsilon^e_{ij}+\varepsilon^p_{ij}.$$

Hence the assumption made in the model that the strain consists of an elastic and a plastic part is obtained.

$$(4)\qquad P^a_{ij}=\frac{\partial Z}{\partial\varepsilon^p_{ij}}=\bar{s}_{ij},\quad P^\mu_{ij}=\frac{\partial Z}{\partial\varepsilon^\mu_{ij}}=-M_{ijkl}\varepsilon^\mu_{kl},\quad M_{ijkl}=\frac{\partial f_{ij}}{\partial\varepsilon^\mu_{kl}}.$$

It is seen that the active stress deviator is thermodynamically conjugated with the plastic strain tensor, which corresponds to the microstructural model of the process.

$$(5)\qquad P^a_{ij}\dot{\varepsilon}^p_{ij}+P^\mu_{ij}\dot{\varepsilon}^\mu_{ij}=\bar{s}_{ij}\dot{\varepsilon}^p_{ij}-M_{ijkl}\varepsilon^\mu_{kl}\dot{\varepsilon}^\mu_{ij}=\bar{s}_{ij}\dot{\varepsilon}^p_{ij}-\dot{s}^\mu_{ij}\varepsilon^\mu_{ij}\geq 0.$$

$$\dot{s}^\mu_{ij}=M_{ijkl}\dot{\varepsilon}^\mu_{kl}$$

The analysis of that relation in the case of partial unloading shows that the assumption that $\dot{\varepsilon}^p_{ij}$ is directed along the normal to the yield surface $F=0$ in the stress space, is in accordance with the second thermodynamic law. We assume that the sign of the components of the inelastic strain tensor ε^μ_{ij} coincides with the sign of the components of the plastic strain tensor ε^p_{ij}.

3. Yield condition

The microstructural analysis of the model leads to the assumption that the yield condition on macrolevel depends on the stress tensor σ_{ij} as well as on the microstress tensor σ^μ_{ij}:

$$(6)\qquad F(\sigma_{ij},\ \sigma^\mu_{ij})=0.$$

Taking into account that the yield condition must be an isotropic function of those tensors and assuming plastic incompressibility of the material ($\varepsilon^p_{kk}=0$) and neglecting the influence of the invariants of an order higher than two, F takes the form:

(7) $$F(s_{ij}s_{ij},\quad s^{\mu}_{ij}s^{\mu}_{ij},\quad s_{ij}s^{\mu}_{ij})=0$$

where

$$s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\,\delta_{ij},\quad s^{\mu}_{ij}=\sigma^{\mu}_{ij}-\frac{1}{3}\sigma^{\mu}_{kk}\,\delta_{ij}.$$

The experiments mentioned above [1, 4, 5] show that the change of the form of the yield surface depends on the directions of the preliminary and of the subsequent loading. In order to describe this phenomenon we introduce the factor χ:

(8) $$\chi=\frac{(s_{mn}-s^{\mu}_{mn})s^{\mu}_{mn}}{\sqrt{\frac{1}{4}(s_{ij}-s^{\mu}_{ij})(s_{ij}-s^{\mu}_{ij})s^{\mu}_{kl}\,s^{\mu}_{kl}}}.$$

We assume a form of the yield condition (7) similar to the one, given in [9, 10], but with the addition of a new function $\varphi(\chi)$, taking into account the nonsymmetry of the yield surface during the subsequent loading:

(9) $$F=\frac{1}{2}(I_{ijkl}+As^{\mu}_{ij}s^{\mu}_{kl})(s_{ij}-s^{\mu}_{ij})(s_{kl}-s^{\mu}_{kl})-[\varphi(\chi)+\tau^*_p]^2=0$$

or $F=II^2_{\bar{s}}+\frac{1}{2}A\Sigma^2_{\text{III}}-[\varphi(\chi)+\tau^*_p]^2=0$

where $II^2_{\bar{s}}=\frac{1}{2}\bar{s}_{ij}\,\bar{s}_{ij},\quad \Sigma_{\text{III}}=\bar{s}_{ij}s^{\mu}_{ij},\quad II^2_{s^{\mu}}=\frac{1}{2}s^{\mu}_{ij}\,s^{\mu}_{ij},$

$$\bar{s}_{ij}=s_{ij}-s^{\mu}_{ij},\quad \chi=\frac{\Sigma_{\text{III}}}{II_{\bar{s}}II_{s^{\mu}}}.$$

There are a lot of published experimental data, which show that the yield surface depends on the preliminary plastic deformation [14], on the preliminary strain rate [15], on the actual plastic deformation [16] and on the actual strain rate [9, 10]. Therefore it seems convenient to introduce the following invariants as parameters of the process:

(10) $$\gamma_h=\int_{t_0}^{t_f}\sqrt{\frac{1}{2}\dot{e}^p_{ij}\dot{e}^p_{ij}}\,dt,\quad e^p_{ij}=\varepsilon^p_{ij}$$

is a measure, taking into account the preliminary plastic deformation, which had taken place in the time interval $[t_0, t_f]$.

(11) $$\beta_h=\frac{1}{t_f-t_0}\int_{t_0}^{t_f}\sqrt{\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}}\,dt,\quad e_{ij}=\varepsilon_{ij}-\frac{1}{3}\varepsilon_{kk}\,\delta_{ij}$$

is a measure, taking into account the preliminary strain rate in the same time interval.

(12) $$\gamma=\sqrt{\frac{1}{2}e^p_{ij}e^p_{ij}}$$

is the actual plastic strain intensity, and

(13) $$\beta=\sqrt{\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}}$$

is the actual strain rate intensity.

The four parameters (10) to (13) build the arranged sequence $\pi=\{\gamma_h, \beta_h, \gamma, \beta\}$ and are parameters of the process. We shall consider only processes at which $\beta=\text{const}$. The material functions A, τ^*_p, as well as the coefficients of function $\varphi(\chi)$ are assumed to depend on π.

The rate of microstresses is given by the following relation [9]:

(14) $$\dot{s}^{\mu}_{ij}=\dot{\gamma}Re^p_{ij}+Q\dot{e}^p_{ij}$$

where R and Q are material functions, depending on the parameters of the process π. It is seen that microstresses develop as a result of plastic deformation. During an initial process, if there were no preliminary plastic deformations, and if the yield limit was not exceeded, microstresses do not exist.

Relations (9) and (11) yield the following expression for the yield hardening factor h:

(15) $$h^{-1}=|R\,\mathrm{II}_F\Sigma_{\mathrm{I}}+Q\Sigma_{\mathrm{II}}+F_\gamma\,\mathrm{II}_F|$$

where

$$\Sigma_{\mathrm{I}}=R_{ij}e^p_{ij}\quad \mathrm{II}^2_F=\frac{1}{2}F_{ij}F_{ij}\quad \Sigma_{\mathrm{II}}=S_{ij}F_{ij}$$

$$S_{ij}=\frac{\partial F}{\partial s^{\mu}_{ij}}=-\bar{s}_{ij}+A\Sigma_{\mathrm{III}}(\bar{s}_{ij}-s^{\mu}_{ij})-\frac{2(\varphi+\tau^*_p)\varphi'}{\mathrm{II}_{\bar{s}}\,\mathrm{II}_{s^{\mu}}}\left[\left(1+\frac{\Sigma_{\mathrm{III}}}{2\mathrm{II}^2_{\bar{s}}}\right)\bar{s}_{ij}-\left(1+\frac{\Sigma_{\mathrm{III}}}{2\mathrm{II}^2_{s^{\mu}}}\right)s^{\mu}_{ij}\right]$$

$$F_{ij}=\frac{\partial F}{\partial\bar{\sigma}_{ij}}=\bar{s}_{ij}+A\Sigma_{\mathrm{III}}\,s^{\mu}_{ij}-\frac{2(\varphi+\tau^*_p)\varphi'}{\mathrm{II}_{\bar{s}}\,\mathrm{II}_{s^{\mu}}}\left[s^{\mu}_{ij}-\frac{\Sigma_{\mathrm{III}}}{2\mathrm{II}^2_{\bar{s}}}\bar{s}_{ij}\right]$$

$$F_\gamma=\frac{\partial F}{\partial\gamma}=\frac{1}{2}\Sigma^2_{\mathrm{III}}\frac{\partial A}{\partial\gamma}-2(\varphi+\tau^*_p)\left(\frac{\partial\varphi}{\partial\gamma}+\frac{\partial\tau^*_p}{\partial\gamma}\right).$$

From a thermodynamical point of view, the plastic strain tensor ε^p_{ij}, and the microstress deviator s^{μ}_{ij} are internal state variables. The corresponding equations of evolution are the flow rule.

(16) $$\dot{\varepsilon}^p_{ij}=hF_{ij}F_{kl}\dot{\bar{\sigma}}_{kl}$$

where

$$h=\begin{cases}0 & \text{if } F<0 \text{ or } F=0 \text{ and } F_{ij}\dot{\bar{\sigma}}_{ij}\leq 0\\ >0\end{cases}$$

and equation (14).

The dependence of the yield condition on the invariant χ provides a possibility to describe the following phenomena, experimentally observed: extrusion of the subsequent yield surface in the preloading direction, flatness on the opposite side and existence or inexistence of a cross-effect.

To investigate the proposed yield condition (9) and to determine the form of the function $\varphi(\chi)$, we introduce a five dimensional deviatoric space, according to Iliushin [17]. A deviator d_{ij} transforms itself into a five-dimensional vector d_I (I=1, 2, ..., 5) according to the expressions:

(17) $$d_1=\sqrt{\frac{3}{2}}d_{11},\quad d_2=\sqrt{2}(d_{22}+d_{11}),\quad d_3=\sqrt{2}d_{12},\quad d_4=\sqrt{2}d_{23},$$
$$d_5=\sqrt{2}d_{31}.$$

The invariants which appear in the yield condition then take the form

(18) $$\mathrm{II}_{\bar{s}}^2=\frac{1}{2}\bar{s}_\mathrm{I}\bar{s}_\mathrm{I},\quad \Sigma_\mathrm{III}=\bar{s}_\mathrm{I}s_\mathrm{I}^\mu,\quad \mathrm{II}_{s^\mu}^2=\frac{1}{2}s_\mathrm{I}^\mu s_\mathrm{I}^\mu.$$

The yield condition in that space reads:

(19) $$F=\frac{1}{2}(s_\mathrm{I}-s_\mathrm{I}^\mu)(s_\mathrm{I}-s_\mathrm{I}^\mu)+\frac{1}{2}A[(s_\mathrm{I}-s_\mathrm{I}^\mu)s_\mathrm{I}^\mu]^2-[\varphi(\chi)+\tau_p^*]^2=0$$
$$\chi=\frac{\Sigma_\mathrm{III}}{\mathrm{II}_{\bar{s}}\mathrm{II}_{s^\mu}}=\frac{2\bar{s}_\mathrm{I}s_\mathrm{I}^\mu}{\sqrt{\bar{s}_J\bar{s}_J s_K^\mu s_K^\mu}}$$

or

(20) $$F=\mathcal{A}_{IJ}s_Is_J+2\mathcal{B}_Is_I+\mathcal{C}=0$$

where

(21) $$\mathcal{A}_{IJ}=\frac{1}{2}(\delta_{IJ}+As_I^\mu s_J^\mu)$$
$$\mathcal{B}_I=-\mathcal{A}_{IJ}s_J^\mu$$
$$\mathcal{C}=\mathcal{A}_{IJ}s_I^\mu s_J^\mu-[\varphi(\chi)+\tau_p^*]^2.$$

If $\varphi(\chi)=\mathrm{const}$, $\mathcal{C}$ does not depend on s_I and (20) is a second-order surface in the five-dimensional deviatoric space [18]. If $\det|\mathcal{A}_{IJ}|\neq 0$ this surface has a centre of symmetry 0^μ, given by x_K^0:

(22) $$x_K^0=-\mathcal{A}_{KJ}^{-1}\mathcal{B}_J=\mathcal{A}_{KJ}^{-1}\mathcal{A}_{JI}s_I^\mu=\delta_{KI}s_I^\mu=s_K^\mu.$$

Obviously the microstress deviator defines the centre of symmetry of the second order surface $F=0$, if $\varphi=\mathrm{const}$. This centre does not depend on $\mathcal{C}$, i. e. on the value of φ.

Translating the coordinate system in the centre of symmetry $\bar{s}_I=s_I-s_I^\mu$ we obtain the equation of the yield surface in the five dimensional active stress deviatoric space:

(23) $$F=\mathcal{A}_{IJ}\bar{s}_I\bar{s}_J+\mathcal{C}=0,\quad \mathcal{C}=-[\varphi(\chi)+\tau_p^*]^2.$$

If $\varphi=\mathrm{const}$, this is a central second-order surface. A necessary and sufficient condition providing that this surface does not degenerate, is (24):

(24) $$\det\begin{pmatrix}\mathcal{A}_{IJ} & 0\\ 0 & \mathcal{C}\end{pmatrix}=-\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\mathcal{C}\neq 0$$

where $\lambda_I(I=1, 2, \ldots, 5)$ are the eigenvalues of the matrix $\mathcal{A}_{IJ}$. The assumption made above ensures that $\lambda_I\neq 0$. Then $C\neq 0$ must be fulfilled, or $\varphi(\chi)\neq -\tau_p^*$.

According to Drucker's postulate [14], the yield surface must be closed and convex. This will be fulfilled if (23) is a hyperellipsoide at $\varphi=$ const. This requires $\lambda_I>0$, ($I=1, 2, \ldots, 5$). In the coordinate system, coinciding with the main directions of the matrix $\mathcal{A}_{IJ}$, the equation of the hyperellipsoide reads:

$$\sum_{I=1}^{5} \frac{\xi_I^2}{a_I^2}=1, \quad a_I^2=\frac{(\varphi+\tau^*_p)^2}{\lambda_I} \tag{25}$$

Introduce a spherical coordinate system (r, ϑ_1, ϑ_2, ϑ_3, ϑ_4) with an origin coinciding with 0_μ:

$$\begin{aligned}
\xi_1&=r\cos\vartheta_1\cos\vartheta_2\cos\vartheta_3\cos\vartheta_4=r\Phi_1\\
\xi_2&=r\sin\vartheta_1\cos\vartheta_2\cos\vartheta_3\cos\vartheta_4=r\Phi_2\\
\xi_3&=r\sin\vartheta_2\cos\vartheta_3\cos\vartheta_4=r\Phi_3\\
\xi_4&=r\sin\vartheta_3\cos\vartheta_4=r\Phi_4\\
\xi_5&=r\sin\vartheta_4=r\Phi_5.
\end{aligned} \tag{26}$$

Eqn. (25) then takes the form:

$$r=\frac{\varphi+\tau^*_p}{\sqrt{\sum_{I=1}^{5}\Phi_I^2\lambda_I}}=\tau_p. \tag{27}$$

Obviously $r<\tau_p$ corresponds to an elastic deformation and $r=\tau_p$ means that the yield limit is reached. At a fixed direction in the five-dimensional deviatoric space and at a given value of A, the yield limit depends on the term $\varphi^*=\varphi+\tau^*_p=\sqrt{-\mathscr{C}}$. The factor χ then takes the form:

$$\chi=\frac{\xi_I\xi_I^\mu}{\sqrt{\frac{1}{2}\xi_J\xi_J}\sqrt{\frac{1}{2}\xi_K^\mu\xi_K^\mu}}=\frac{\Phi_I\xi_I^\mu}{\sqrt{\frac{1}{2}\Phi_J\Phi_J}\sqrt{\frac{1}{2}\xi_K^\mu\xi_K^\mu}} \tag{28}$$

where ξ_I^μ represents the vector s_I^μ in the new coordinate system. It is seen that χ does not depend on r, i. e. eqn. (25) may be solved with respect to r and take the form (27) even if $\varphi(\chi)\neq$ const, in which case (20) is no more a second order surface. In the five-dimensional deviatoric space the factor χ represents the angle closed between the vectors ξ_I and ξ_I^μ.

At the beginning of the process, when no preliminary plastic deformation existed, $s_I^\mu=0$ and $\Sigma_{\text{III}}=0$, $\bar{s}_I=s_I$, $\text{II}_{\bar{s}}=\text{II}_s$. We assume that in this case $\varphi=0$ and $\tau^*_p=\tau^0_p$ is the initial shear yield limit. The yield condition (9) coincides then with the Mises yield condition. Plastic deformation originates microstresses $s_I^\mu\neq0$ and the initial Mises surface translates, rotates and deforms. On the basis of the assumed two-component material structure on microlevel and the experimental data for the shape of the subsequent yield surface, we assume that the latter consists of three regions. Two of them are parts of two concentric hyperellipsoids. The third region is a smooth convex transition hypersurface. The range of validity of the regions is determined by the factor χ, the extreme values of which are 2 and -2. We shall take function φ in the form:

$$\varphi=\begin{cases}\varphi_1=\text{const} & \text{if } \chi_1\leqq\chi\leqq 2 \text{ I region}\\ \varphi(\chi) & \text{if } \chi_2\leqq\chi\leqq\chi_1 \text{ III region}\\ \varphi_2=\text{const} & \text{if } -2\leqq\chi\leqq\chi_2 \text{ II region.}\end{cases} \tag{29}$$

Region I is part of a hyperelipsoid with diameters $a_I^2=\frac{(\overset{*}{\tau}_p+\varphi_1)^2}{\lambda_I}$. Region II is part of another hyperellipsoide, concentric to the first one, with diameters $b_I^2=\frac{(\overset{*}{\tau}_p+\varphi_2)^2}{\lambda_I}$. Region III is a transition hypersurface of higher order. The following conditions must be fulfilled, in order to ensure a smooth transition:

$$\begin{aligned}\varphi(\chi_1)&=\varphi_1 & \left.\frac{\partial\varphi}{\partial\chi}\right|_{\chi=\chi_1}&=0\\ \varphi(\chi_2)&=\varphi_2 & \left.\frac{\partial\varphi}{\partial\chi}\right|_{\chi=\chi_2}&=0.\end{aligned} \tag{30}$$

4. The plane stress state

Let us consider the two-dimensional case of plane stress state

$$\begin{aligned}&\sigma_{11}\neq0,\quad \sigma_{12}\neq0,\quad \sigma_{22}=\sigma_{33}=\sigma_{23}=\sigma_{13}=0\\ &\sigma^{\mu}_{11}\neq0,\quad \sigma^{\mu}_{12}\neq0,\quad \sigma^{\mu}_{22}=\sigma^{\mu}_{33}=\sigma^{\mu}_{23}=\sigma^{\mu}_{13}=0.\end{aligned} \tag{31}$$

The five-dimensional vector of the active stress deviator turns into a twodimensional one:

$$\bar{s}_1=\sqrt{\frac{2}{3}}\,\bar{\sigma}_{11},\qquad \bar{s}_2=\sqrt{2}\,\bar{\sigma}_{12}. \tag{32}$$

The corresponding invariants are:

$$\begin{aligned}\text{II}^2_{\bar{s}}&=\frac{1}{3}\bar{\sigma}^2_{11}+\bar{\sigma}^2_{12}=\frac{1}{2}(\bar{s}_1^2+\bar{s}_2^2)\\ \text{II}^2_{s^{\mu}}&=\frac{1}{3}\sigma^{\mu^2}_{11}+\sigma^{\mu^2}_{12}=\frac{1}{2}(s_1^{\mu^2}+s_2^{\mu^2})\\ \Sigma_{\text{III}}&=2\left(\frac{1}{3}\bar{\sigma}_{11}\sigma^{\mu}_{11}+\bar{\sigma}_{12}\sigma^{\mu}_{12}\right)=\bar{s}_1 s_1^{\mu}+\bar{s}_2 s_2^{\mu}\end{aligned} \tag{33}$$

and the yield condition (9) takes the form:

$$F=\frac{1}{3}\left(1+\frac{2}{3}A\sigma^{\mu^2}_{11}\right)\bar{\sigma}^2_{11}+(1+2A\sigma^{\mu^2}_{12})\bar{\sigma}^2_{12}+\frac{4}{3}A\sigma^{\mu}_{11}\sigma^{\mu}_{12}\bar{\sigma}_{11}\bar{\sigma}_{12}-[\varphi(\chi)+\overset{*}{\tau}_p]^2=0. \tag{34}$$

In the case when $\varphi=$const, this represents an ellipse in the $(\sigma_{11}, \sigma_{12})$ plane. The centre of the ellipse is $0'(\sigma^{\mu}_{11}, \sigma^{\mu}_{12})$. The diameters are:

$$a_{1,2}=\frac{\sqrt{2}(\overset{*}{\tau}_p+\varphi)}{\sqrt{S\pm\sqrt{p^2+\frac{16}{9}A^2\sigma^{\mu^2}_{12}\sigma^{\mu^2}_{11}}}} \tag{35}$$

where

$$S = \frac{4}{3} + \frac{2}{9} A\sigma_{11}^{\mu^2} + 2A\sigma_{12}^{\mu^2}$$

$$P = -\frac{2}{3} + \frac{2}{9} A\sigma_{11}^{\mu^2} - 2A\sigma_{12}^{\mu^2}.$$

They are rotated at an angle δ to the $0^{\mu}\sigma_{11}$ axis:

(36) $$\operatorname{tg}\delta = \frac{2A\sigma_{12}^{\mu}\sigma_{11}^{\mu}}{-1 + \frac{1}{3}A\sigma_{11}^{\mu^2} - 3A\sigma_{12}^{\mu^2}}.$$

Introducing polar coordinated

(37) $$\bar{s}_1 = r\cos\vartheta, \quad \operatorname{tg}\vartheta = \frac{\bar{s}_2}{\bar{s}_1} = \frac{\sqrt{3}\,\bar{\sigma}_{12}}{\bar{\sigma}_{11}}$$
$$\bar{s}_2 = r\sin\vartheta, \quad r^2 = \bar{s}_1^2 + \bar{s}_2^2 = 2\left(\frac{1}{3}\bar{\sigma}_{11}^2 + \bar{\sigma}_{12}^2\right)$$

the invariants take the simple form:

(38) $$\mathrm{II}_{\bar{s}}^2 = \frac{1}{2}r^2, \quad \mathrm{II}_{s^{\mu}}^2 = \frac{1}{2}r^{\mu^2} \quad \Sigma_{\mathrm{III}} = rr^{\mu}\cos\theta$$

where

$$\theta = \vartheta - \vartheta^{\mu}, \quad \operatorname{tg}\vartheta^{\mu} = \frac{\sqrt{3}\sigma_{12}^{\mu}}{\varphi_{11}^{\mu}}, \quad r^{\mu^2} = 2\left(\frac{1}{3}\sigma_{11}^{\mu^2} = \sigma_{12}^{\mu^2}\right).$$

The factor χ is then

(39) $$\chi = 2\cos\theta.$$

The yield condition takes the form:

(40) $$r = \frac{2\sqrt{2}\,[\varphi(\chi) + \overset{*}{\tau}_p]}{\sqrt{4 + Ar^{\mu^2}\chi^2}}.$$

In the (ξ_1, ξ_2) plane (fig. 1), where

(41) $$\bar{s}_1 = \xi_1\cos\vartheta^{\mu} - \xi_2\sin\vartheta^{\mu}, \quad \bar{s}_2 = \xi_1\sin\vartheta^{\mu} + \xi_2\cos\vartheta^{\mu}$$

the yield condition reads:

(42) $$F = \frac{1}{2}(1 + Ar^{\mu^2})\xi_1^2 + \frac{1}{2}\left(1 - \frac{2}{3}A\frac{s_1^{\mu^2}s_2^{\mu^2}}{r^{\mu^2}}\right)\xi_2^2 - \varphi^{*2} = 0$$

where

$$\varphi^*(\chi) = \varphi(\chi) + \overset{*}{\tau}_p.$$

At $\varphi^* = \mathrm{const}$ eqn. (42) is the central equation of an ellipse. Its diameters are given by the expressions:

(43) $$a_1^2 = \frac{2\varphi^{*2}}{1 + Ar^{\mu^2}}, \quad a_2^2 = \frac{2\varphi^{*2}}{1 - \frac{2}{3}A\frac{s_1^{\mu^2}s_2^{\mu^2}}{r^{\mu^2}}}.$$

The ratio of both diameters is

(44)
$$k^2=\frac{a_2^2}{a_1^2}=\frac{1+A^{r^{\mu^2}}}{1-\frac{2}{3}A\frac{s_1^{\mu^2}s_2^{\mu^2}}{r^{\mu^2}}}.$$

At a given preliminary plastic deformation, described by s_1^μ and s_2^μ, this ratio depends on the material characteristic A. The requirement that (42) should be an equation of an ellipse yields the restriction for A:

(45)
$$\frac{3r^{\mu^2}}{2s_1^{\mu^2}s_2^{\mu^2}}\geq A\geq-\frac{1}{r^{\mu^2}}.$$

5. The one-dimensional stress state

In the case of a one-dimensional stress state:

(46)
$$\begin{aligned}&\sigma_{11}\neq 0,\quad \sigma_{22}=\sigma_{33}=\sigma_{12}=\sigma_{23}=\sigma_{13}=0\\&\sigma_{11}^\mu\neq 0,\quad \sigma_{22}^\mu=\sigma_{33}^\mu=\sigma_{12}^\mu=\sigma_{23}^\mu=\sigma_{13}^\mu=0\end{aligned}$$

the yield condition reads:

(47)
$$F=\frac{1}{3}\left(1+\frac{2}{3}A\sigma_{11}^{\mu^2}\right)\bar{\sigma}_{11}^2-[\varphi(\chi)+\tau_p^*]^2=0$$

where

$$\chi=2\,\mathrm{sign}\,(\bar{\sigma}_{11}\sigma_{11}^\mu)$$

and according to (29) $\varphi=\varphi_1$ at $\chi=2$ and $\varphi=\varphi_2$ at $\chi=-2$. The yield condition (34) takes the form:

(48)
$$\bar{\sigma}_{11}=\frac{\sqrt{3}(\varphi+\tau_p^*)}{1+\frac{2}{3}A\sigma_{11}^{\mu^2}}\,\mathrm{sign}\,(\bar{\sigma}_{11}\sigma_{11}^\mu).$$

6. Determining the material characteristics

The method of determining the dependence of the material characteristics A, R, Q, φ_1^*, φ_2^*, and $\varphi^*(\chi)$ on π in the case of statical processes ($\beta_h=\beta=\beta_s$) will be given. Experiments with thin walled tubes under tension and internal pressure at different values of π are necessary.

Assume that the microstress tensor is coaxial with the plastic strain tensor. As a result it follows that $R=\frac{dQ}{d\gamma}$ at fixed values of γ_h. The equation of evolution (14) leads to:

(49)
$$\begin{aligned}\dot{s}_{11}^\mu&=\dot{\gamma}Re_{11}^p+Q\dot{e}_{11}^p\\\dot{s}_{12}^\mu&=\dot{\gamma}Re_{12}^p+Q\dot{e}_{12}^p.\end{aligned}$$

The experiments must be realized at a constant ratio $m=\frac{s_{11}^\mu}{s_{12}^\mu}$. The integration of (49) yields:

$$s^{\mu}_{11}-s^{\mu}_{11_h}=\int_{\gamma_h}^{\gamma}\left(\frac{dQ}{d\gamma}e^{p}_{11}d\gamma+Qde^{p}_{11}\right)$$

(50)

$$s^{\mu}_{12}-s^{\mu}_{12_h}=\int_{\gamma_h}^{\gamma}\left(\frac{dQ}{d\gamma}e^{p}_{12}d\gamma+Qde^{p}_{12}\right).$$

Substituting in $(50)_2$ the relation $s^{\mu}_{12}=\frac{1}{m}s^{\mu}_{11}$ we obtain:

(51) $$s^{\mu}_{11}-s^{\mu}_{11_h}=m\int_{\gamma_h}^{\gamma}\left(\frac{dQ}{d\gamma}e^{p}_{12}d\gamma+Qde^{p}_{12}\right).$$

The comparison of (51) and $(50)_1$ shows that $e^{p}_{11}=me^{p}_{12}$. The plastic strain intensity is then $\gamma=\sqrt{6}\sqrt{e^{p^2}_{11}+4e^{p^2}_{12}}=le^{p}_{11}$ where $l=\sqrt{6\left(1+\frac{4}{m_2}\right)}$. The relation (51) takes now the form:

(52) $$s^{\mu}_{11}-s^{\mu}_{11_h}=\int_{\gamma_h}^{\gamma}\gamma\frac{dQ}{d\gamma}\frac{1}{l}d\gamma+\int_{\gamma_h}^{\gamma}Q\frac{1}{l}d\gamma=\frac{1}{l}\int_{\gamma_h}^{\gamma}d(\gamma Q)=\frac{1}{l}(Q\gamma-Q_n\gamma_n)$$

or

(53) $$s^{\mu}_{11}(\gamma_h,\gamma)=Q(\gamma_h,\gamma)\gamma\frac{1}{l}.$$

The function s^{μ}_{11} must be experimentally determined. Then function Q and $R=\frac{dQ}{d\gamma}$ are obtained by means of (53).

The axis of both ellipses, a_1 and b_1, as well as the ratio $k=\frac{a_2}{a_1}=\frac{b_2}{b_1}$ and r^{μ} (fig. 1) are to be obtained by means of the least square method, using experimental points in the $(\sigma_{11}, \sigma_{12})$ plane. Then we determine

(54) $$\varphi^{*}_{1}=\frac{ka_1}{\sqrt{2}},\quad \varphi^{*}_{2}=\frac{kb_1}{\sqrt{2}},\quad A=\frac{k^2-1}{r^{\mu^2}},\quad s^{\mu}_{11}=r^{\mu}\cos\vartheta^{\mu}$$

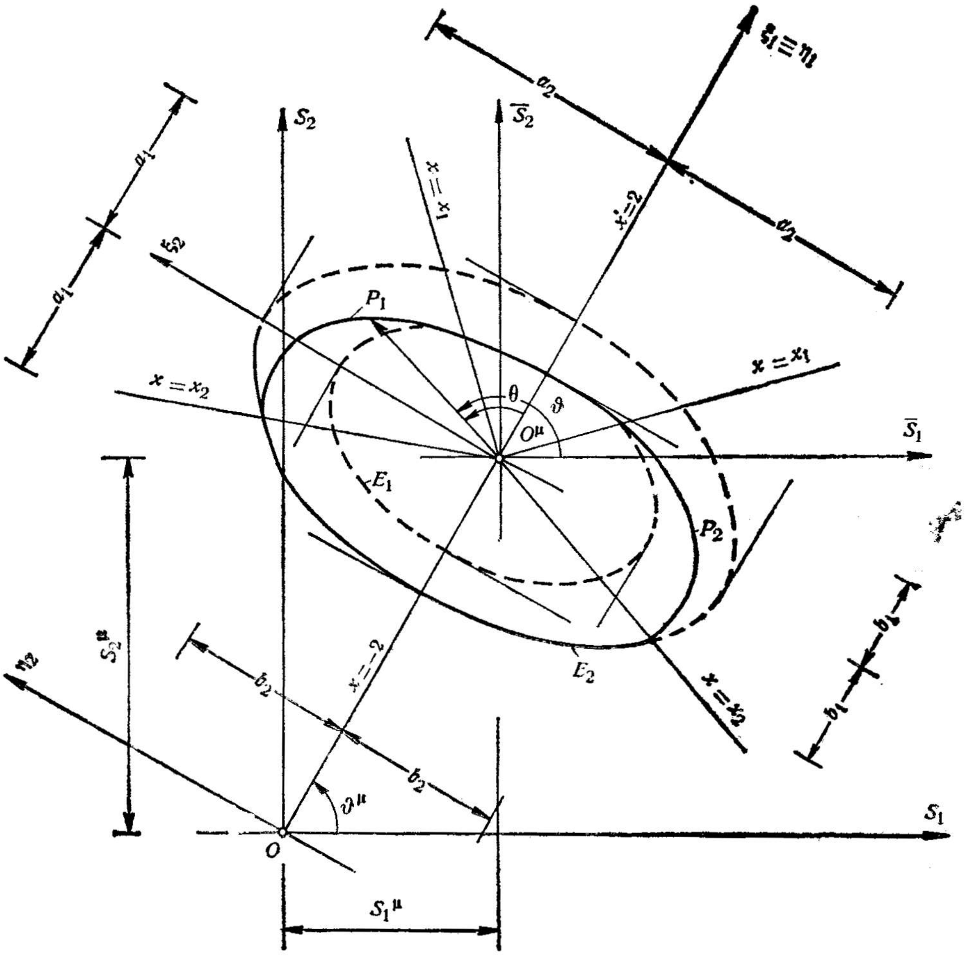
where

$$\theta^{\mu}=\operatorname{arc\,tg}\frac{\sqrt{3}\,\sigma^{\mu}_{12}}{\varphi^{\mu}_{11}}.$$

We assume the transition curve of region III to be a parabola in the (ξ_1, ξ_2) plane (fig. 1):

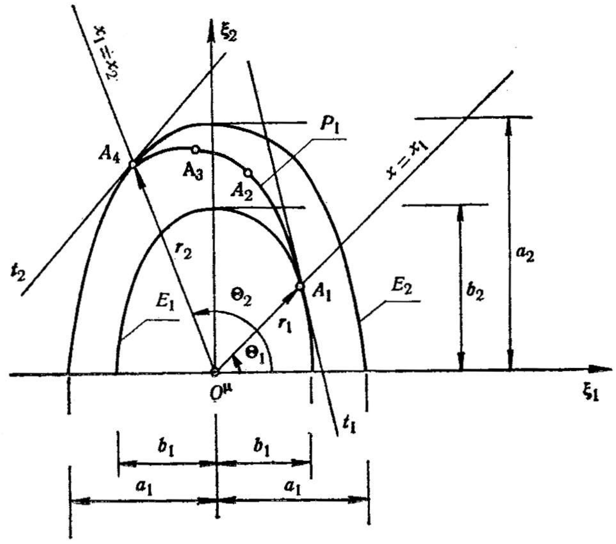
(55) $$(\xi_1+a\xi_2)^2+2b\xi_1+2c\xi_2+d=0.$$

The coefficients a, b, c, d are to be determined by the condition that this parabola passes through two experimental points A_2 and A_3 of region III and is in the same time tangential to both ellipses at A_1 and A_4 (fig. 2). An iterative procedure may be applied, prescribing point $A_1(\xi^{(1)}_1, \xi^{(0)}_2)$ and moving it along the first ellipse E_1 till the tangent to the second ellipse E_2 at point $A_4(\xi^{(4)}_1, \xi^{(4)}_2)$ coincides with the tangent to the parabola at the same point.



S_2
$\bar{S}_2$
$\xi_1 \equiv \eta_1$
a_2
a_1
ξ_2
P_1
$x = x_1$
$x = x_2$
θ
ϑ
O^μ
$\bar{S}_1$
E_1
P_2
E_2
b_1
b_2
S_2^μ
η_2
$x = -2$
$x = 2$
ϑ^μ
S_1
O
S_1^μ

Fig. 1



ξ_2
$x = x_2$
$x = x_1$
P_1
A_4
A_3
A_2
A_1
t_2
t_1
r_2
r_1
Θ_2
Θ_1
E_1
E_2
a_2
b_2
O^μ
ξ_1
b_1
a_1

Fig. 2

Point A_4 is determined as a point of intersection of ellipse E_2 and the parabola. The regions are determined by

$$\chi_1=\frac{2\xi_1^{(1)}}{r_1}, \quad \chi_2=\frac{2\xi_1^{(4)}}{r_2} \tag{56}$$

where

$$r_1=\sqrt{\xi_1^{(1)2}+\xi_2^{(1)2}}, \quad r_2=\sqrt{\xi_1^{(4)2}+\xi_2^{(4)2}}.$$

Using (40), (55) and (39) we obtain the function $\varphi^*(\chi)$:

$$\varphi^*(\chi)=\frac{\sqrt{4+Ar^{u2}\chi^2}\left(-B_\chi\pm\sqrt{B_\chi^2-A_\chi C_\chi}\right)}{2\sqrt{2}\,A_\chi} \tag{57}$$

where

$$A_\chi=\frac{1}{4}[\chi^2(1-a^2)+2a\chi\sqrt{4-\chi^2}+4a^2]$$

$$B_\chi=\frac{1}{2}(b\chi+c\sqrt{4-\chi^2})$$

$$C_\chi=d$$

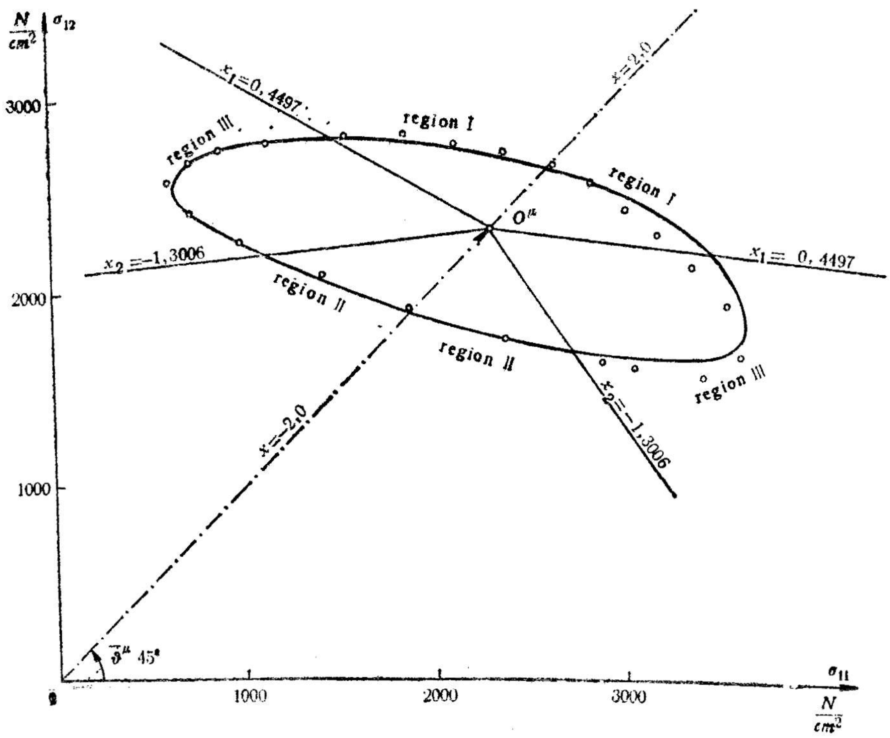


Fig. 3

From both signs in (57) that one is valid, for which

$$A_\chi r_1+B_\chi\leq\pm\sqrt{B_\chi^2-A_\chi C_\chi}\leq A_\chi r_\chi B_\chi \tag{58}$$

is fulfilled.

The values of φ_1^*, φ_2^*, A, s_{11}^u, R, Q, χ_1, χ_1 as well as the coefficients a, b, c, d are to be determined for different values of the process parameters γ_u and γ. Then by means of the least square method the functions $\varphi_1^*(\gamma_h, \gamma)$, $\varphi_2^*(\gamma_h, \gamma)$, $A(\gamma_h, \gamma)$, $s_{11}^u(\gamma_h, \gamma)$, $R(\gamma_h, \gamma)$, $Q(\gamma_h, \gamma)$, $\chi_1(\gamma_h, \gamma)$, $\chi_2(\gamma_h, \gamma)$, $a(\gamma_h, \gamma)$, $b(\gamma_h, \gamma)$, $c(\gamma_h, \gamma)$, $d(\gamma_h, \gamma)$ are to be found.

7. Numerical example.

In order to illustrate the proposed nonsymmetric anisotropic hardening rule, experimental data published by Phillips [5] were used. Allu-

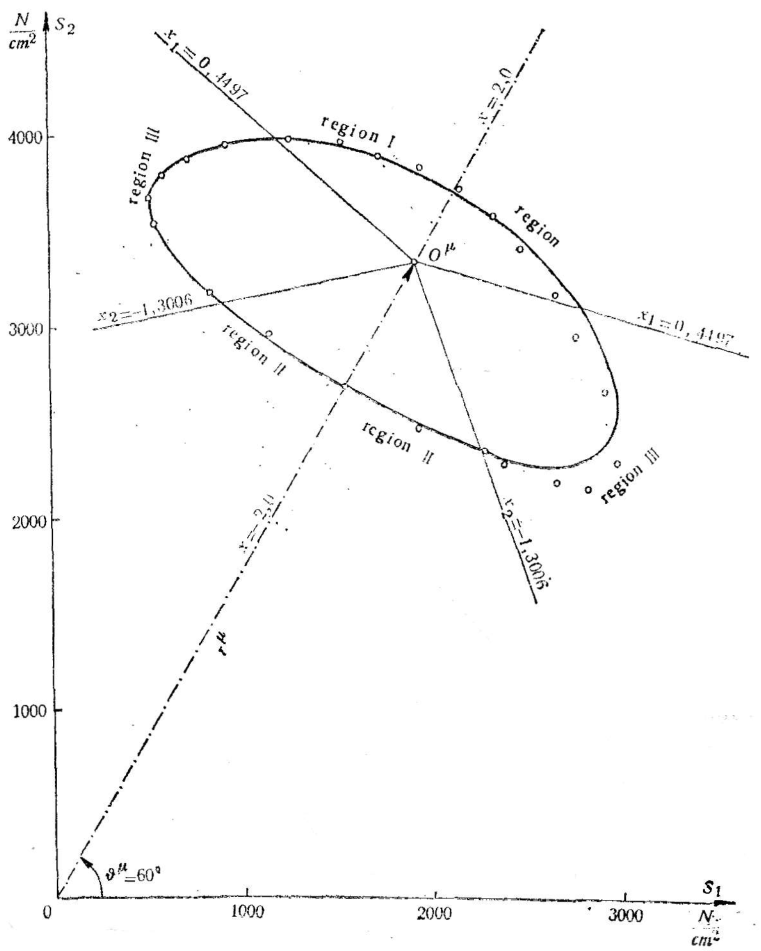


Fig. 4

minium specimens were plastically deformed at $\sigma_{11}^h=\sigma_{12}^h=5{,}6$ kN/cm² and $\varepsilon_{11_h}^p=0{,}001763$, $\varepsilon_{12_h}^p=0{,}009769$ and then partially unloaded. The following parameters were obtained for the subsequent yield surface at the constant values $\gamma_h=0{,}009848$ and $\gamma=\gamma_h+0{,}002=0{,}011848$:

$$\vartheta^{\mu}=60^{\circ}, \quad A=0{,}351\,.\,10^{-6}\ \mathrm{cm}^4/\mathrm{N}^2$$

$$s_{11}^{\mu}=0{,}1905\,.\,10^4\ \mathrm{N/cm}^2,$$

$$\varphi_1^*=763\ \mathrm{N/cm}^2 \quad \varphi_2^*=1310\ \mathrm{N/cm}^2, \quad \chi_1=0{,}4497, \quad \chi_2=-1{,}3006,$$

$$\vartheta_1=137^{\circ}, \quad \vartheta_2=190^{\circ}, \quad a=0{,}2277, \quad b=0{,}6385,$$

$$c=206{,}75, \quad d=-5{,}70\,.\,10^5\ \mathrm{N/cm}^2.$$

Fig. 3 shows the yield surface, as well as the experimental points. Fig. 4 shows the same surface in the (s_1, s_2) plane. According to the assumed model, the yield surface in that plane is symmetric with respect to the preliminary loading direction. Both figures show a good approximation of the experimental points by the theoretical curve.

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Получена на 25. IX. 1984 г.