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Nonparallel instability of a liquid capillary jet

S. Radev

1. Introduction

As is known, linear analysis of the stability of a liquid capillary jet is based on the investigation of growth or decay of small disturbances imposed on the jet. Most often the latter are taken to be periodic downstream (see [1], [2], [3], etc.). In precise form such waves occur only if the jet basic (undisturbed) flow does not change in axial direction, i. e. the jet surface is to be cylindrical and velocity is considered to be uniform downstream. However, for conditions similar to the real ones, two tendencies are observed in the jet flow: — a jet contraction due to acceleration, and a transformation of the axial velocity profile into a plane (homogeneous) one.

The first tendency manifests itself along the whole flow region while the second one — at a distance from the nozzle of an order of several nozzle diameters.

Furthermore, for large velocities of outflow (large Froude numbers), contraction occurring at a distance of an order of a single wave length, stands considerably small. This makes possible a local investigation of the linear stability of a capillary jet.

It is clear that such an approach corresponds to the popular quasiparallel approximation in the linear stability theory. However, it has been employed in [4], [5], while a local analysis of the stability of a liquid capillary jet has been proposed in [6], and a simplified (onedimensional) form of the equation of motion has been taken as a basis, instead of the complete Navier-Stokes equations.

To account for the nonparallel effect of the undisturbed jet flow, the present analysis is somewhat developed and the method of multiple scales [7] is used for solving the onedimensional equations of the perturbated flow. Furthermore, the local solution presented in [6], is obtained as a zero order approximation. The first order approximation includes the effects of axial variation of the disturbance amplitude, wave number, jet velocity and jet radius. Numerical results for the wave number variation and growth rate of disturbances downstream are obtained as well. Results concerning the length of the jet continuous region are presented.

2. Statement of the problem

Let heavy axisymmetric liquid jet of viscosity μ and density ρ emanate vertically downwards (with velocity U_N) from a nozzle of circular cross section of diameter D_N. Space, surrounding the jet, is considered to be occupied by an immovable nonviscous gas of density ρ_1. Cylindrical coordinate system Orz is introduced, its origin coinciding with the centre of the nozzle cross section and the axis Oz directed along the jet axis. As is known, flow within the jet exhibits a wave nature and is a result of the combined action of the forces of surface tension, viscosity and gravity.

The wavy flow of the jet, as shown in [8, 9], can be described by using a simplified form of the equations of motion. Briefly, these simplifications consist in (see [6]) neglecting the dependence of the flow parameters on the radial coordinate. The jet radius h and velocity u become functions of a single axial coordinate z (and of time resp.), and satisfy a system of two onedimensional equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}=\frac{1}{2Fr}-\frac{\partial p}{\partial z} \tag{1}$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial z}=-\frac{h}{2}\frac{\partial u}{\partial z} \tag{2}$$

where $Fr=U_N^2/gD_N$ is the Froude number. Eqn. (1) represents in a simplified and nondimensional form the conservation of momenta in axial direction. Eqn. (2) is an equation of motion of a liquid surface, following one and the same particles. The pressure term in eqn. (1) can be eliminated from the equilibrium condition, valid for the normal stresses and surface tension on the jet surface:

$$p-p_0=\frac{2}{We}\left\{\frac{1}{h}-\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{-1}\frac{\partial^2 h}{\partial z^2}\right\}\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{-1/2}, \tag{3}$$

where $We=\rho D_N U_N^2/\sigma$ is the Weber number, p_0 — the nondimensional external pressure.

To study the stability of the jet flow persented, we shall follow the usual procedure of the linear stability theory, providing the solution of eqns. (1)—(3) in the form

$$u=U+\tilde{u},\ h=H+\tilde{h},\ p=P+\tilde{p}. \tag{4}$$

However, U, H, P denote the steady undisturbed parts of the unknown flow parameters while $\tilde{u}$, $\tilde{h}$, $\tilde{p}$ — the unsteady ones.

The undisturbed flow can be obtained as a solution of the steady onedimensional equations of motion (1)–(3). Using a single step iteration, eqn. (1) and eqn. (3) can be reduced to

$$U^2=\frac{z}{F_r}-2p+\mathrm{const}=1+\frac{z}{Fr}-\frac{4}{We}(1-H^{-1}), \tag{5}$$

$$U^{-1}=H^2. \tag{6}$$

The second equality expresses that the mass flow ($U.H^2=1$) remains constant for an arbitrary jet cross section (including the origin $z=0$, $U=H=1$). Eqn. (5) together with eqn. (6) determine the jet radius H as an implicit function of the axial coordinate z. Introducing eqns. (4)—(6) into eqns. (1)—(3) and neglecting the nonlinear terms that contain disturbed values, it is easily obtained that

(7) $$\frac{\partial \tilde{u}}{\partial t}+U\frac{\partial \tilde{u}}{\partial z}+U'\tilde{u}=-\frac{\partial \tilde{p}}{\partial z},$$

(8) $$\frac{\partial \tilde{h}}{\partial t}+U\frac{\partial \tilde{h}}{\partial z}+\frac{1}{2}H\frac{\partial \tilde{u}}{\partial z}+H'.\tilde{u}+\frac{1}{2}U'.\tilde{h}=0,$$

(9) $$\tilde{p}=-\frac{2}{We}\left(\frac{\tilde{h}}{H^2}+\frac{H'}{H}\frac{\partial \tilde{h}}{\partial z}+\frac{\partial^2 \tilde{h}}{\partial z^2}\right),$$

where the upper prime denotes the derivatives of the undisturbed values with respect to z.

Eqns. (7)—(9) are linear differential equations with variable coefficients, depending on the axial coordinate z. This holds true because the undisturbed flow comprises the acceleration/contraction effects of the gravity forces. Neglecting these effects, eqns. (7)—(9) reduce to the form, obtained in [10]. Formally this result corresponds to the case of $Fr=\infty$, when the jet exhibits cylindrical shape and uniform velocity. Usually $Fr\gg 1$ and $We\gg 1$ within the range of the outflow velocities. Moreover, although variable, coefficients in eqns. (7)—(9) vary very slowly downstream. This enables to apply the method of multiple scales with $\varepsilon=Fr^{-1}$, when looking for the solution of the equations.

3. Method of multiple scales

Let us introduce a new independent variable in axial direction:

(10) $$z_1=\varepsilon z,$$

that will serve as a slow scale along with the fast scale z. Then the solution of eqns. (7)—(9) can be written in the form

(11) $$\tilde{u}=e^{i\theta}(u_0+\varepsilon u_1+\cdots),$$

(12) $$\tilde{h}=e^{i\theta}(h_0+\varepsilon h_1+\cdots),$$

where

(13) $$\frac{\partial \theta}{\partial z}=k=k_r+ik_i,$$

(14) $$\frac{\partial \theta}{\partial t}=-\omega=\text{Const.}$$

It has to be noted that, according to eqns. (11)—(14), spatially growing disturbances are to be introduced with k_r as a wave number and k_i as a spatial growth rate. This means that the fast scale is to be employed to describe the wave disturbances that travel downstream. The case $k_i<0$ corresponds to disturbances whose amplitude grows with the fast scale z. At the same time the wave number, spatial growth rate and amplitude of the disturbances are supposed to be functions of the slow scale z_1. Hence the slow variation of the undisturbed flow along the axis is incorporated in the disturbance set of wave parameters.

So far, as the disturbances are taken to be periodic in time the angular frequency ω in eqn. (14) stands for a known positive constant. Satisfying eqns. (7)—(9) through the solution (11)—(14) and equating the terms of similar powers of ε, we obtain

$$i(kU-\omega)u_0-i\frac{2k}{We}\left(\frac{1}{H^2}-k^2\right)h_0=0, \tag{15}$$

$$i\frac{k}{2}Hu_0+i(kU-\omega)h_0=0 \tag{16}$$

and

$$i(kU-\omega)u_1-i\frac{2k}{We}\left(\frac{1}{H^2}-k^2\right)h_1=\beta, \tag{17}$$

$$i\frac{k}{2}Hu_1+i(kU-\omega)h_1=\alpha. \tag{18}$$

However, the nonhomogeneous parts of eqns. (17)—(18) are denoted for simplicity as follows

$$\alpha=-\frac{1}{2}H\dot{u}_0-\dot{H}u_0-U\dot{h}_0-\frac{1}{2}\dot{U}h_0, \tag{19}$$

$$\beta=-U\dot{u}_0-\dot{U}u_0-\frac{2}{We}\left[\left(3k^2-\frac{1}{H^2}\right)\dot{h}_0+\left(\frac{2\dot{H}}{H^3}+k^2\frac{\dot{H}}{H}+3k\dot{k}\right)h_0\right], \tag{20}$$

where the upper symbol (.) denotes the derivative with respect to z_1.

Accounting for the disturbed jet radius and velocity, eqns. (15)—(18) form two algebraic systems for the determination of the unknown components of the expansions (11)—(12). (eqns. (11)-(12), however, regard the complex amplitudes of the jet radius and velocity under consideration). The first system is homogeneous with respect to zero order amplitudes while the second one includes a nonhomogeneous part, containing derivatives of the zero order amplitudes and of the nondisturbed flow.

The characteristics z_1 is treated as a parameter in the system (15)—(16). The same is with the dispersion relation that follows for k and ω, if one has to look for a nontrivial solution of the system:

$$(kU-\omega)^2=\frac{k^2H}{We}\left(\frac{1}{H^2}-k^2\right). \tag{21}$$

Regarding eqn. (21), k is obtained as a function of the slow scale z_1, but for fixed values of ω and of the nondimensional parameters *Fr* and *We*. Such an approximation of the complex wave number k corresponds to the local (quasiparallel) approach of the linear stability that is described in detail in [6]. In what follows we shall restrict to the nonstable mode of eqn. (21), i. e. $k_i<0$. Each of eqns. (15) and (16), with k defined, represents a relationship between the complex amplitudes u_0 and h_0 but they still remain unidentified. The paragraph that follows provides their identification by solving eqns. (17)—(18).

4. First order solution

The system (17)—(18) has not only the same homogeneous part as in system (15)—(16), but contains an additional nonhomogeneous part, as well. So far, as the determinant of the homogeneous part of the system (17)—(18) is equal to zero, the solution may be obtained if and only if the equations (17) and (18) are linearly dependant. Hence, the following condition has to be introduced

$$
\begin{vmatrix} -\left(\frac{\omega}{k}-U\right) & \beta \\ \frac{H}{2} & \alpha \end{vmatrix} = 0 \tag{22}
$$

which, together with one of eqns. (15) or (16), is to be employed for the determination of the complex amplitudes u_0 and h_0 in zero approximation. It is more convenient to present eqn. (16) in a differential form:

$$
\dot{u}_0 = -\left(\frac{\omega}{k^2}\dot{k}+\dot{U}\right)\frac{2}{H}h_0 - \left(\frac{\omega}{k}-U\right)\frac{2\dot{H}}{H^2}h_0 + \left(\frac{\omega}{k}-U\right)\frac{2}{H}\dot{h}_0. \tag{23}
$$

Then eqns. (22) and (23) may be considered as a system of first order linear differential equations with respect to the complex amplitudes u_0 and h_0. By using eqns. (16) and (23), it seems easy to eliminate the velocity amplitude term u_0 and its derivative $\dot{u}_0$ in eqn. (22):

$$
\alpha = -\frac{\omega}{k}\dot{h}_0 + \frac{\omega}{k}\left(\frac{\dot{k}}{k}-\frac{\dot{H}}{H}\right)h_0, \tag{24}
$$

$$
\begin{aligned}
\beta = & \left[\frac{2}{We}\left(\frac{1}{H^2}-3k^2\right)-\frac{2}{H^3}\left(\frac{\omega}{k}-U\right)\right]\dot{h}_0 \\
& + \left[\frac{\dot{H}}{H}\left(\frac{\omega}{k}\frac{6}{H^3}-\frac{12}{H^5}-\frac{2k^2}{We}-\frac{4}{We}\frac{1}{H^2}\right)+\left(\frac{2}{H^3}\frac{\omega}{k}-\frac{6k^2}{We}\right)\frac{\dot{k}}{k}\right]h_0
\end{aligned} \tag{25}
$$

and from eqns. (5) and (6) and eqn. (21) follows that

$$
\frac{\dot{k}}{k} = -\left\{\frac{\dot{H}}{H}\left[-\frac{4}{H^3}\left(\frac{\omega}{k}-U\right)+\frac{1}{We}\frac{1}{H^2}+\frac{1}{We}k^2\right]\right\}\Big/\left\{\frac{2}{H}\left[\frac{\omega}{k}\left(\frac{\omega}{k}-U\right)+\frac{k^2H}{We}\right]\right\}, \tag{26}
$$

$$
\frac{\dot{H}}{H} = -\frac{1}{4}\frac{1}{U^2+W_e^{-1}U^{1/2}}. \tag{27}
$$

With (α, β) inserted into eqn. (22), the latter can be transformed into a first order ordinary differential equation with respect to h_0:

$$
F\dot{h}_0 + Gh_0 = 0 \tag{28}
$$

or, after integration, into

$$
h_0 = Ce^{-\int\limits_0^{z_1}\frac{G}{F}dz_1}. \tag{29}
$$

A similar expression can be obtained for u_0 as well. C in eqn. (29) is an integration constant that has to be defined through the exit value (at $z_1=0$) of the disturbed jet radius (or jet velocity).

To obtain the series for the disturbances (11)—(12), with accuracy of order ε, we need the first order approximation of the complex amplitudes u_1 and h_1. We obtain from eqn. (18)

$$
\frac{h_1}{h_0} = \frac{h_{1,p}}{h_0} + \frac{1}{2}\left(\frac{\omega}{k}-U\right)^{-1}H\frac{u_1}{h_0}, \tag{30}
$$

where

$$
\frac{h_{1,p}}{h_0} = \frac{i}{k}\frac{\omega}{k}\left(\frac{\omega}{k}-U\right)^{-1}\left(\frac{G}{F}+\frac{\dot{k}}{k}-\frac{\dot{H}}{H}\right). \tag{31}
$$

At this step the last term in eqn. (30) remains unidentified and has to be determined by solving the equations that provide the second order approximation of the amplitudes. Hence, this term may be neglected, since it includes the higher order derivatives of the disturbed and primary flow parameters of the jet.

Finally, what holds for the disturbed jet radius is that

$$\tilde{h}=h_0e^{i\theta}\left(1+\varepsilon\frac{h_{1,p}}{h_0}\right) \tag{32}$$

with an accuracy of order $O(\varepsilon^2)$. However, a similar formula corresponds to the disturbed jet velocity.

4. Results

By assembling eqns. (12) and (29) one can easily define in a first approximation the phase of the disturbance travelling waves:

$$\theta_s=\int_0^z(k+\varepsilon k_1)dz-\omega t, \tag{33}$$

where $k_1=-G/F$. The second term under the integral reflects the influence of the contraction/acceleration effects of the primary flow on the wave phase From eqn. (33) it follows that

$$\frac{\partial\theta_s}{\partial z}=k+\varepsilon k_1=\gamma+i\sigma. \tag{34}$$

Hence, k_1 may be treated as a first order correction of the local complex wave number. The right hand side of eqn. (34) is written in terms of the wave number γ and the growth rate of the disturbances σ as well.

The length of the jet continuous part, known as a break-up length, is the value that is mostly accessible for experimental determination. Theoretically, the break-up point L of the jet is determined as a point where the disturbance amplitude of the jet radius $A=|\tilde{h}|$ attains the nondisturbed value

$$A=H \quad \text{at} \quad z=L. \tag{35}$$

The amplitude A is determined from eqns. (29) and (30) as

$$A=A_0e^{\int_0^z\sigma dz}\left|1+\varepsilon\frac{h_1}{h_0}\right|, \tag{36}$$

where $A_0=|C|$ denotes the value of the amplitude in the zero cross section.

Eqns. (35) and (36) serve as a basis for the determination of the break-up point $z=L$ as a function of the frequency ω. A typical example of such a curve is shown in Fig. 1. To introduce unique value of the break-up length, it seems reasonable to use the value of ω that gives the minimum of the $L-\omega$ curve. Such a frame corresponds to the well known Rayleigh's hypothesis [1], stating that the jet breaks up as a result of the effect of disturbances that increase most fastly in time. In our case, when spatially growing disturbances are considered, the break-up disturbances are taken to be those that produce the shortest jet.

Fig. 2 shows the rate of the disturbance growth as a function of the frequency ω for three chosen cross sections and for fixed values of the nondimensional parameters $We = 148.13$, $Fr = 277.72$. The basic difference between these curves lies in the position of the maximum — along the flow it shifts to

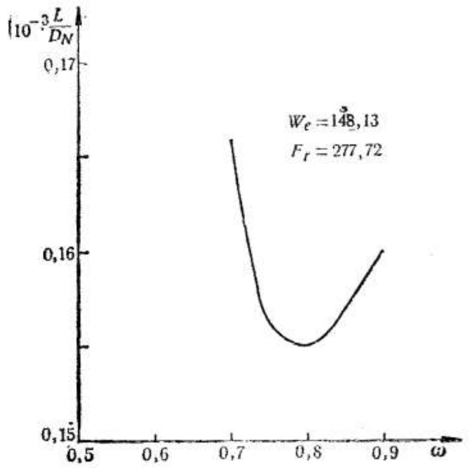

Fig. 1. The break-up length vs. frequency

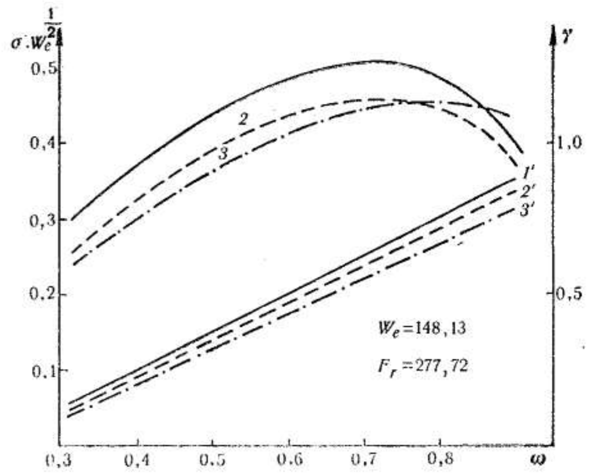

Fig. 2. Growth rate and wave number of the dis turbances vs. frequency at different cross sectioons: $1,1'-z=0$; $2,2'-z=20$; $3,3'-z=50$

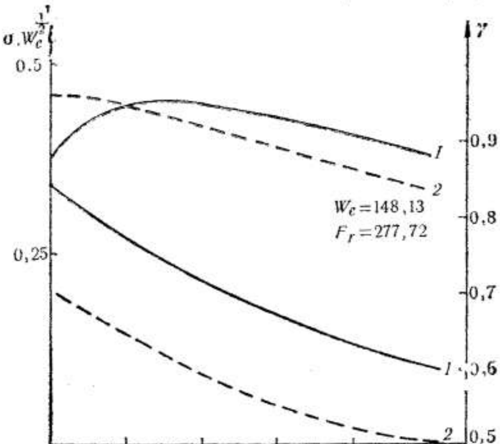

Fig. 3. Growth rate and wave number of disturbances vs. normalized coordinate for various frequencies: $1,1'-\omega=0.85$; $2,2'-\omega=0.7$

greater frequencies or (see curves $1'$, $2'$, $3'$) to shorter wave lengths. More than that, the long and short waves display contradicting tendencies with respect to the rates of the disturbance growth. The rate of the long waves diminishes while that of the short ones grows.

Fig. 3 shows that the rate of the disturbances grows as function of the axial coordinate z, as well as of the corresponding wave number γ. The existance of maximum of the rate growth is confirmed when regarding the range of shorter waves. What follows from these conclusions is that the wave length increases downstream and the short waves grow faster.

Numerical results, concerning the jet break-up length, are shown in Fig. 4 and Fig. 5. Fig. 4 provides the break-up length as a function of the dimension-

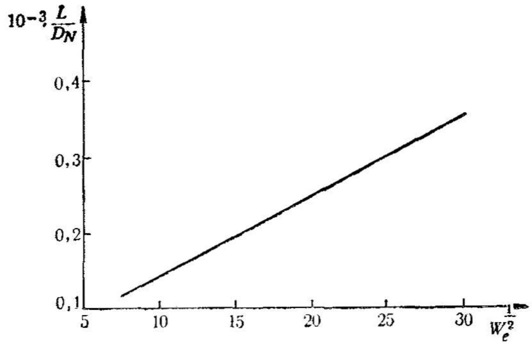

Fig. 4. The break-up length vs. normalized velocity

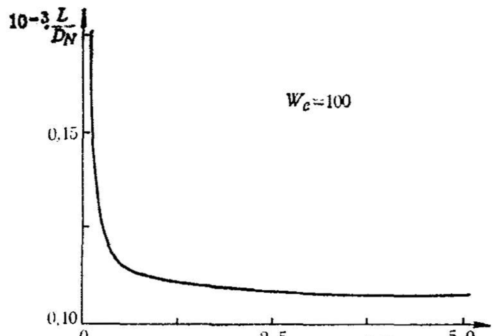

Fig. 5. The break-up length vs. the Froude number

less outflow velocity $U_N/(\sigma/\rho D_N)^{1/2} = We^{1/2}$ ($L-U_N$ curve). The experiments, however, produce under the same conditions the $L-U_N$ curve which goes through a maximum after an almost linear region. As shown in Fig. 4, the nonparallel approximation of the jet break-up length represents only the linear region of the $L-U_N$ curve. This indicates that there is no difference between the character of the $L-U_N$ curve in a local [6] and nonparallel approximation.

The break-up length in Fig. 5 is plotted against the Froude number. at a fixed value of the Weber number. The dash line corresponds to the case of a fully cylindrical jet ($Fr=\infty$), without taking into account the contraction/acceleration effects. These effects provide a considerable correction of the jet length for small Froude numbers.

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