

An Application of Volterra-Wiener Series in Mechanics of Composite Materials

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1. Introduction

In this paper we consider the effective thermal conductivity for a random suspension of spheres or cylinders in the limit of small volume fraction. The proposed method can be successfully employed, however, not only to the thermal conductivity, but to the problem of effective behaviour for heterogeneous materials with any other physical background, e. g., electrostatics, magnetostatics, elasticity, etc.

We note first of all that the random differential equation of heat conduction in the medium defines implicitly a nonlinear operator which transforms the random field of thermal conductivity into the random temperature field. According to the general idea of Volterra and Wiener, this operator can be represented as a Volterra series and, in order to get approximate solutions, it is natural to truncate this series afterwards. This method, previously used for identification of nonlinear systems with Gaussian input (cf. [1]), was recently employed to analyze some nonlinear stochastic processes (e. g. turbulence), making use of the Poissonian process as basic in the Volterra-Wiener series [2,3]. Later on the method proposed in [3] was applied in [4] to the problem of thermal conduction through a "Perfectly Disordered" suspension of spheres for which the sphere centers are distributed according to a Poissonian-like law such that overlapping of the spheres is impossible. Here, making use of the methods of [2-4] we consider general distributions of the inclusion centers, but only for dilute suspensions. As an illustration we show in the last section how the obtained results allow to account explicitly for the two-clustering of the inclusions when predicting the bulk conductivity of the medium.

2. Volterra Series

We consider an infinite medium which consists of an array of spheres, each of radius a and conductivity λ_f , randomly distributed in a matrix of conductivity λ_m . The volume fractions of the matrix and of the spheres are c_m and c_f , respectively, $c_m + c_f = 1$.

The equations governing the temperature field, $\theta(\mathbf{x})$, in the medium are

$$(2.1) \quad \nabla \cdot \mathbf{q}(\mathbf{x}) = 0, \quad \mathbf{q}(\mathbf{x}) = \lambda(\mathbf{x}) \nabla \theta(\mathbf{x}),$$

where \mathbf{q} is the heat flux vector and $\lambda(\mathbf{x})$ is a random function of position which takes the values λ_f or λ_m depending on whether \mathbf{x} lies in a sphere or in the matrix, respectively. The well-known problem consists then in finding the statistical properties of the temperature field, $\theta(\mathbf{x})$, in the medium provided the statistical properties of $\lambda(\mathbf{x})$ are known and a constant heat flux is impressed upon the medium [5]. In what follows, the distribution of the spheres is taken homogeneous and thus the random fields are to be strongly homogeneous in statistical sense.

Consider the random set of points $\{\xi_j\}_{j=1}^{\infty}$ —the centers of the spheres given. The statistical properties of the set $\{\xi_j\}$ are fully described by the sequence of its probability distribution functions $f_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$, $n=1, 2, \dots$. By definition, [6],

$$(2.2) \quad dP = f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) dV_1 \dots dV_n$$

is the probability to find simultaneously at least one point of the set $\{\xi_j\}$ within every neighbourhood dV_i of the locations \mathbf{x}_i , $i=1, \dots, n$. As the medium is statistically homogeneous, $f_n(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = f_n(\mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1)$.

The given set of points $\{\xi_j\}$ generates the basic "random density function"

$$(2.3) \quad F(\xi) = V \sum_j \delta(\xi - \xi_j),$$

[7], where $V = \frac{4}{3} \pi a^3$ is the volume of a single sphere and δ is the Dirac delta function. The statistical properties of the random function F are fully described through those of the random set $\{\xi_j\}$, i. e. through the probability distribution functions (2.2). For instance, the first three moments of F are

$$(2.4) \quad \begin{aligned} \overline{F(x)} &= c_f, \\ \overline{F(\mathbf{x}_1)F(\mathbf{x}_2)} &= c_f V \delta(\mathbf{x}_1 - \mathbf{x}_2) + f_2(\mathbf{x}_1 - \mathbf{x}_2), \\ \overline{F(\mathbf{x}_1)F(\mathbf{x}_2)F(\mathbf{x}_3)} &= V^2 \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}_3) c_f \\ &\quad + 3\{\delta(\mathbf{x}_1 - \mathbf{x}_2) f_2(\mathbf{x}_1 - \mathbf{x}_3)\}_s + f_3(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3), \end{aligned}$$

cf. [6], p. 142, where the overbar means ensemble average and s stands for the symmetrization with respect to the indices 1, 2, 3.

The random field of conductivity $\lambda(\mathbf{x})$ has a simple representation through the random density function

$$(2.5) \quad \lambda(\mathbf{x}) = \bar{\lambda} + \lambda'(\mathbf{x}), \quad \lambda'(\mathbf{x}) = \frac{[\lambda]}{V} \int h(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y}) d^3 \mathbf{y},$$

where $\bar{\lambda} = c_m \lambda_m + c_f \lambda_f$, $[\lambda] = \lambda_f - \lambda_m$, $h(\mathbf{x})$ is the characteristic function for the sphere $|\mathbf{x}| < a$, and

$$(2.6) \quad \psi(\xi) = F(\xi) - c_f$$

is the centered random density function.

The governing equations (2.1) define implicitly a nonlinear operator which transforms the random field of conductivity, $\lambda(\mathbf{x})$, (the input) into the random field of temperature, $\theta(\mathbf{x})$, (the output). This operator, following the general idea of Volterra and Wiener, can be represented as a Volterra series

$$(2.7) \quad \theta(\mathbf{x}) = \sum_{p=0}^{\infty} T_p[F(\mathbf{x})],$$

where T_p is the p th order Volterra operator excited by the input, i. e. by the random density function $F(\mathbf{x})$,

$$(2.8) \quad T_p[F(\mathbf{x})] = \int \dots \int T_p(\mathbf{x}-\mathbf{y}_1, \dots, \mathbf{x}-\mathbf{y}_p) F(\mathbf{y}_1) \dots F(\mathbf{y}_p) d^3\mathbf{y}_1 \dots d^3\mathbf{y}_p;$$

each integral is taken over the whole space.

Before proceeding with the examination of the Volterra series (2.7), a remark of a more general nature should be inserted.

As is known, the Volterra series proved to be a powerful tool when investigating nonlinear stochastic systems. It was first acknowledged by Wiener [7] who chose the input $F(\mathbf{x})$ in (2.7) to be the Brownian motion, i. e. the white Gaussian noise, and developed the needed techniques (Wiener G-functionals) for an optimal identification of nonlinear systems. A thorough and comprehensive analysis of this trend of application of Volterra series is given in the book [1]. A rigorous mathematical basis for the Wiener method was built up by Cameron and Martin [8] who introduced the name Wiener-Hermite expansion. Later on the Wiener-Hermite expansion was applied to the Burgers turbulence, but it was faced there with some difficulties (cf. [3] for a short survey of the related results). As argued in [3], it is a result of the fact that the behaviour of the real nonlinear systems is in general well apart of Gaussianity. The general theory of Volterra-Wiener expansions of functionals, generated by a Poisson process, was developed by Ogura [9], who introduced the name Poisson-Wiener expansion. However, the privileged position of the Poisson process in modelling nonlinear stochastic process has been only recently revealed by Christov [2,3]. A detailed study of the conductivity for a Perfectly Disordered suspension of spheres was carried out in [4] by means of the Volterra-Wiener method.

Reverting to the composite materials under consideration we note that, as far as the Volterra series application is concerned, the corresponding stochastic problem (2.1) is in a more favorite position than the above mentioned problems (system identification, turbulence, etc). Namely, the basic process in the Volterra series (2.7) is prescribed in advance: this is the random density function $F(\mathbf{x})$ given in (2.3). Thus the profound problem about the choice of the basic process in the Volterra expansion is avoided in mechanics of composite materials.

To find an approximate solution for the temperature field, in the medium we truncate the Volterra series (2.7) after the N th term

$$(2.9) \quad \theta^{(N)}(\mathbf{x}) = \sum_{p=0}^N T_p[F(\mathbf{x})],$$

$N=0, 1, \dots$. The technique employed in [3, 4] allows then to specify, in principle, the kernels T_p : we insert (2.9) into (2.1), multiply both sides of the obtained equation for $\theta(\mathbf{x})$ by $\theta(\mathbf{x}_1) \dots \theta(\mathbf{x}_n)$, $n=1, \dots, N$, and average the result. Keeping in mind the relations (2.4) for the moments of the random field F as well as the representation (2.5) for the random conductivity $\lambda(\mathbf{x})$, we get a system of N integro-differential equations for the unknown kernels T_p , $p=1, \dots, N$, which contains the first $N+2$ probability distribution functions (2.2).

A very important feature of such an approach is that the problem we are faced with is closed for each N , i. e. we have enough equations to specify the unknown kernels. Thus, there is no need to devise additional conditions for closing the infinite systems of cascade equations for the moments of

the temperature field, cf. [5]. Moreover, the corresponding systems, once solved, give not only the effective properties of the composite, but full statistical information for the sought solution within the frame of the adopted approximation (2.9). This unique feature of the Volterra-Wiener series approach in stochastic problems was underlined in [2, 3] where the respective systems for the kernels were derived in the cases for the Burgers turbulence, and later on in [4] for a Perfectly Disordered suspension of spheres, characterized by a Poissonian-like system $\{\xi_j\}$ such that intersection of inclusions is forbidden. We consider here an arbitrary random system $\{\xi_j\}$, but confine the study to the first-order approximation only, i. e. to the case of dilute suspension.

3. First-order Approximation

According to (2.8), the first-order approximation for the temperature field in the medium reads

$$(3.1) \quad \theta(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x} + \int T_1(\mathbf{x} - \mathbf{y}) \psi(\mathbf{y}) d^3\mathbf{y},$$

the second term on the right-hand side being the fluctuation from the constant temperature gradient, \mathbf{G} , impressed upon the medium.

Note that the first three moments for the centered random density function, $\psi(\mathbf{x})$, are

$$(3.2) \quad \begin{aligned} \overline{\psi(\mathbf{x})} &= 0, \quad \overline{\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)} = V\delta(\mathbf{x}_1 - \mathbf{x}_2) c_f + f_2(\mathbf{x}_1 - \mathbf{x}_2) - c_f^2, \\ \overline{\psi(\mathbf{x}_1)\psi(\mathbf{x}_2)\psi(\mathbf{x}_3)} &= c_f V^2 \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}_3) \\ &+ f_3(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3) + 3c_f \{V\delta(\mathbf{x}_1 - \mathbf{x}_2) f_2(\mathbf{x}_1 - \mathbf{x}_3)\}_s \\ &- 3c_f \{f_2(\mathbf{x}_1 - \mathbf{x}_2)\}_s - 3c_f^2 V \{\delta(\mathbf{x}_1 - \mathbf{x}_2)\}_s + 2c_f^3. \end{aligned}$$

Instead of the function f_2 it seems more visual to employ the conditional probability $P_2(\mathbf{x}) = P(\mathbf{x} | \mathbf{0})$ which represents the probability density of finding a second point at \mathbf{x} knowing that a point is located at the origin. The definition of the conditional probability shows that

$$(3.3) \quad f_2(\mathbf{x}) = c_f P_2(\mathbf{x}).$$

Similarly, we introduce the function $P_3(\mathbf{x}, \mathbf{y}) = P(\mathbf{x}, \mathbf{y} | \mathbf{0})$ which represents the probability density of finding two spheres with centers at x and y , respectively, knowing that a sphere is located at the origin. Obviously,

$$(3.4) \quad f_3(\mathbf{x}, \mathbf{y}) = c_f P_3(\mathbf{x}, \mathbf{y}).$$

Note that in [10] another characteristic of the random set $\{\xi_j\}$ is employed, namely the conditional probability density function $P_1(\mathbf{x}) = P(\mathbf{x} | \mathbf{0})$, which locates the center of the nearest neighbour sphere at the point \mathbf{x} knowing that a sphere is positioned at the origin. In case we know only the first three probability distribution functions (2.2), we have

$$(3.5) \quad \int_{|\mathbf{y}| > |\mathbf{x}|} P_3(\mathbf{x}, \mathbf{y}) d^3\mathbf{y} = P_1(\mathbf{x}).$$

As noted in [10], the functions P_1 and P_2 represent different statistical characteristics of the random medium. The obtained relations (3.3) to (3.5) clarify the reason — P_1 and P_3 are connected with distribution functions f_n with different n .

As argued in [3], employing the first-order approximation (3.1) we are able to cover adequately only the case for which the random inhomogeneities in the medium (eddies in a turbulent flow, inclusions in a composite material, etc.) are far one from other. For the random medium under study we thus consider (3.1) to be exact for dilute suspensions, i. e. to yield rigorous results to order c_f . That is why all the relations hereafter will be tacitly linearized with respect to c_f .

Let us first insert (2.5) and (3.1) into the Fourier law (2.1)₂ and average the result. Making use of (3.2)₂ and (3.3) we get for the average heat flux

$$(3.6) \quad \overline{\mathbf{q}(\mathbf{x})} = \overline{\lambda(\mathbf{x})\theta(\mathbf{x})} = \mathbf{A}^* \cdot \mathbf{G} = \overline{\lambda} \mathbf{G} + c_f [\lambda] \left\{ \frac{1}{V} \int \nabla T_1(\mathbf{x}-\mathbf{y}) h(\mathbf{x}-\mathbf{y}) d^3 \mathbf{y} + \frac{1}{V^2} \int \int \nabla T_1(\mathbf{x}-\mathbf{y}_1) h(\mathbf{x}-\mathbf{y}_2) P_2(\mathbf{y}_1, \mathbf{y}_2) d^3 \mathbf{y}_1 d^3 \mathbf{y}_2 \right\},$$

where \mathbf{A}^* is the 2nd rank tensor of the effective conductivity for the medium, thus obtainable explicitly once the kernel T_1 is known.

To get an equation for T_1 , we first rewrite the differential equation for the temperature field, $\theta(\mathbf{x})$, in the form

$$\overline{\lambda} \nabla \theta + \nabla \cdot (\lambda' \nabla \theta) = 0$$

(cf. (2.1)), insert there the representations (2.5) and (3.1) for $\lambda(\mathbf{x})$ and $\theta(\mathbf{x})$, multiply by $\psi(\mathbf{0})$ and average the result. This procedure, previously employed in [2, 3] and later on in [4], yields the required equation for the kernel $T_1 = T$

$$(3.7) \quad \overline{\lambda} (\Delta T + \Delta T * P_2) + [\lambda] \nabla \cdot \{ \mathbf{G}(h + h * P_2 + (h \nabla T) * P_2 + h(\nabla T + \nabla T * P_2) + \nabla T(h * P_2) + \mathbf{J}) \} = 0$$

where $(a * b)(\mathbf{x}) = \frac{1}{V} \int a(\mathbf{x}-\mathbf{y}) b(\mathbf{y}) d^3 \mathbf{y}$ is the convolution integral, and

$$(3.8) \quad \mathbf{J} = \frac{1}{V^2} \int \int \nabla T(\mathbf{x}-\mathbf{y}_1) h(\mathbf{x}-\mathbf{y}_2) P_3(\mathbf{y}_1, \mathbf{y}_2) d^3 \mathbf{y}_1 d^3 \mathbf{y}_2.$$

The equations (3.6) to (3.8) show that for dilute suspensions the effective conductivity \mathbf{A}^* of the composite depends both on two- and three-points probability distribution functions f_2 and f_3 .

4. Some Particular Cases

The equation (3.7) is, in general, a very complicated integro-differential equation. (This is no wonder because, once solved, it gives not only the bulk conductivity \mathbf{A}^* but also full statistical information about the random field $\theta(\mathbf{x})$, correct to the order c_f .) That is why we confine ourselves to some particular distributions of the random points $\{\xi_j\}$.

Let us introduce the functions of distribution correlation for the random set $\{\xi_j\}$ ([6], I.6.1). The latter characterize the degree of correlation between the points $\{\xi_j\}$; the connection between g_n and f_n is the same as that between the cumulants and the moments for a random variable

$$(4.1) \quad \begin{aligned} f_1 &= g_1, f_2(\mathbf{x}_1, \mathbf{x}_2) = g_2(\mathbf{x}_1, \mathbf{x}_2) + g_1(\mathbf{x}_1)g_1(\mathbf{x}_2), \\ f_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= g_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + 3\{g_1(\mathbf{x}_1)g_2(\mathbf{x}_1, \mathbf{x}_2) + g_1(\mathbf{x}_1)g_1(\mathbf{x}_2)g_1(\mathbf{x}_3)\} \dots \end{aligned}$$

Let the points $\{\xi_j\}$ be distributed according to a Poisson law, i. e. there is no correlations between them and thus $\{\xi_j\}$ is a Poisson random field. In this case

$$(4.2) \quad g_1(\mathbf{x}) = c_f, \quad g_2 = g_3 = \dots = 0$$

and (3.1) becomes the first-order approximation in the Poisson-Wiener expansion investigated in detail in [4].

According to (3.3), (3.4) and (4.1) we have

$$(4.3) \quad P_2(\mathbf{x}) = c_f, \quad P_3(\mathbf{x}) = c_f^2.$$

Thus the convolution integrals in (3.7) are of the order c_f and \mathbf{J} is of the order c_f^2 ; they all should be then omitted in (3.7) leaving only the terms of order 1, that is

$$(4.4) \quad \bar{\lambda} \Delta S + [\lambda] \nabla \cdot \{h(\mathbf{x})S(\mathbf{x})\} = 0, \quad S(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x} + T(\mathbf{x}).$$

The equation (4.4) was derived in [4]; its solution $S(\mathbf{x})$ is the temperature field in an unbounded medium of conductivity $\bar{\lambda}$, containing at the origin a spherical inclusion of conductivity $\bar{\lambda} + [\lambda]$ and undergoing a temperature gradient \mathbf{G} at infinity. The solution of this problem is well-known to be

$$(4.5) \quad S(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x} \begin{cases} \frac{3\lambda_m}{\lambda_f + 2\lambda_m}, & r \leq a \\ 1 - \beta_3 \left(\frac{a}{r}\right)^3, & r > a \end{cases},$$

$r = |\mathbf{x}|$, $\beta_3 = [\lambda]/(\lambda_f + 2\lambda_m)$. Inserting (4.5) into (3.6) and keeping in mind (4.3), we find the bulk conductivity for the Poisson dilute suspension of spheres to be

$$(4.6) \quad \lambda^* = \lambda_m(1 + 3\beta_3 c_f).$$

It is important to note after [4], that (4.6) coincides with the linearization of the known Maxwell formula

$$(4.7) \quad \frac{\lambda^*}{\lambda_m} = \frac{1 + 2\beta_3 c_f}{1 - \beta_3 c_f}$$

for dilute concentration $c_f \ll 1$.

Consider next the so-called "non-falling closely" points, for which the functions of distribution correlation are

$$(4.8) \quad g_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = (-1)^{n-1} (n-1)! c_f^n \{R(\mathbf{x}_2, \mathbf{x}_1) \dots R(\mathbf{x}_n, \mathbf{x}_1)\}_s$$

(cf. [6], I.6.4, for details). That means that there is no clustering, i. e. the points $\{\xi_j\}$ cannot fall close one to other. Such an assumption resembles the "well-separated" medium discussed in [10]. As seen from (4.9), the same equation (4.4) holds and therefore the same relations (4.5) and (4.6) remain valid. Thus, the Maxwell formula (4.7) for the bulk conductivity holds true to the order c_f , if there is no clustering of the inclusions. The same conclusion was come to in [10].

To investigate the effect of clustering, consider the so-called "pair-correlated" points $\{\xi_j\}$, characterized by the assumption that all the functions of distribution correlation g_n except for the first two vanish — $g_3 = g_4 = \dots = 0$

(cf. [6], I.6.4). In this case, according to (4.1) and (3.4), the integral \mathbf{J} from (3.8) should be omitted and the basic equation (3.7) simplifies ($P=P_2$)

$$(4.9) \quad \bar{\lambda}(\triangle T + \triangle T * P) + [\lambda] \nabla \cdot \{\mathbf{G}(h + h * P) + h(\nabla T + \nabla T * P) + (h \nabla T) * P + \nabla T(h * P)\} = 0.$$

5. Bulk Conductivity for a Medium with Second-Order Clustering of Inclusions

In this section, as an illustration of the applicability of the equation (4.9), we investigate the influence of the spatial sphere positioning on the bulk conductivity of the random medium for the particular case of second-order clustering of the spheres.

According to the definition adopted in [10], there is n th order clustering, if the random points $\{\mathfrak{X}_j\}$ fall in groups (clusters) comprising n points. If $n=2$ — the case we are interested in — the points fall in pairs, so that we choose

$$(5.1) \quad P(\mathbf{x}) = P_2(\mathbf{x}) = V\delta(\mathbf{x} - \mathbf{A}) + c_f,$$

where \mathbf{A} is the vector, connecting the sphere centers in the two-clusters. The adopted form (5.1) for the conditional probability implies, in our opinion, that

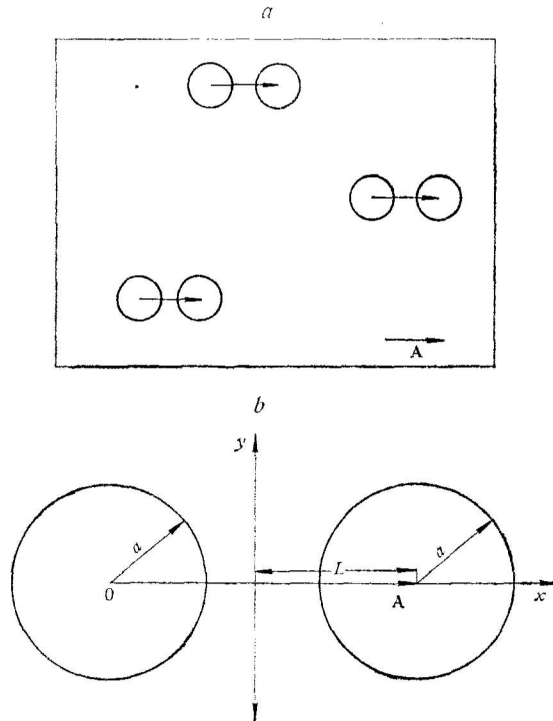


Fig. 1

all the two-clusters are identically oriented (along the vector \mathbf{A}). Thus the considered random medium could be visualized as a dilute suspension of aligned dumb-bell-shaped inclusions ((Fig. 1a). The geometrical parameters for a single "dumb-bell" are shown in Fig. 1b, where $2L = A$.

Let us insert now (5.1) into (3.7). Noting that $R * \delta(\mathbf{x} - \mathbf{A}) = R_{\mathbf{A}}$, where $R_{\mathbf{A}}(\mathbf{x}) = R(\mathbf{x} - \mathbf{A})$ is the shifting of the function R , we rewrite (3.7) in the form

$$(5.2) \quad \bar{\lambda}(\Delta T + \Delta T_{\mathbf{A}}) + [\lambda] \nabla \cdot \{\mathbf{G}(h + h_{\mathbf{A}}) + h \nabla T + h \nabla T_{\mathbf{A}} + h_{\mathbf{A}} \nabla T_{\mathbf{A}} + \nabla T h_{\mathbf{A}}\} = 0.$$

Let

$$(5.3) \quad h_{\mathbf{w}}(\mathbf{x}) = h(\mathbf{x}) + h(\mathbf{x} - \mathbf{A}), \quad S(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x} + T(\mathbf{x}) + T(\mathbf{x} - \mathbf{A}).$$

Then (5.2) can be recast to the form

$$(5.4) \quad \bar{\lambda} \nabla S + [\lambda] \nabla \cdot \{h_{\mathbf{w}}(\mathbf{x}) \nabla S(\mathbf{x})\} = 0, \quad \nabla S \xrightarrow{\mathbf{x} \rightarrow \infty} \mathbf{G},$$

which represents the differential equation governing the temperature field $S(\mathbf{x})$ in an infinite medium of conductivity $\bar{\lambda}$, containing a dumb-bell shaped inclusion of conductivity $\bar{\lambda} + [\lambda]$ (since $h_{\mathbf{w}}$ is just the characteristic function for the dumb-bell shown in Fig. 1b).

For the function $P(\mathbf{x})$ chosen, cf. (5.1), the second term in (3.6) becomes $\frac{1}{V} \int \nabla T(\mathbf{x} - \mathbf{A} - \mathbf{y}) h(\mathbf{x} - \mathbf{y}) d^3 \mathbf{y}$ so that (3.6) can be rewritten as

$$(5.5) \quad \mathbf{\Lambda}^* \cdot \mathbf{G} = \lambda_m \mathbf{G} + c_f [\lambda] \langle \nabla S \rangle_V,$$

where $\langle \nabla S \rangle_V = \frac{1}{V} \int_V \nabla S d^3 v$ is the average of the field ∇S , introduced in (5.3),

over one (the left-hand in Fig. 1b) of the spheres in the dumb-bell.

Note that the result for the effective conductivity, obtained in [4] for a Poisson distribution of spheres, coincides formally with (5.5). In the latter case, however, $S(\mathbf{x})$ is the temperature field in a medium containing a single spherical inclusion (cf. (4.5)); here, in (5.5), the medium contains already two identical inclusions.

To make the obtained results more tangible, we shall numerically illustrate the influence of the second-order clustering upon the bulk conductivity. For simplicity we confine ourselves to the two-dimensional counterpart of the random medium discussed, i. e. to a suspension of aligned (along the axis z) infinite circular cylinders of radius a and conductivity λ_f , impressing upon the medium a temperature gradient perpendicular to z . The solution of the corresponding plane problem (5.4) can be obtained more or less straightforwardly, making use of bipolar coordinates. The final result for the bulk conductivity tensor for the medium is

$$(5.6) \quad \mathbf{\Lambda}^* = \lambda_m \{\mathbf{I} + 2\beta_2 c_f (\mathbf{I} + \alpha_1 \mathbf{e}_x \mathbf{e}_x + \alpha_2 \mathbf{e}_y \mathbf{e}_y)\},$$

$$\alpha_{1,2} = 2\beta_2 \frac{1 - \delta^2}{\delta^2} \sum_{n=1}^{\infty} \frac{n d^{2n}}{1 \mp \beta_2 d^n}; \quad d = \frac{1 - \sqrt{1 - \delta^2}}{1 + \sqrt{1 - \delta^2}}, \quad \beta_2 = \frac{[\lambda]}{\lambda_f + \lambda_m},$$

where $\mathbf{e}_x = \mathbf{A}/A$ and \mathbf{e}_y are the orts of the coordinate axes x and y perpendicular to the cylinder axis z , thus x being directed along the axis \mathbf{A} of the dumb-bells, and $\delta = a/L$.

For prolate dumb-bells, $\delta \ll 1$, we have $d \approx \frac{1}{4} \delta^2$ and thus

$$(5.7) \quad \alpha_1 = \alpha_2 = \frac{1}{8} \beta_2 \delta^2 + O(\delta^4).$$

Therefore, a suspension of prolate aligned dumb-bells does not exhibit anisotropy to order δ^2 ; its effective conductivity, according to (5.6) and (5.7), is

$$(5.8) \quad \frac{\lambda^*}{\lambda_m} = 1 + 2\beta_2 c_f \left(1 + \frac{1}{4} \beta_2 \delta^2 \right) + O(\delta^4), \quad \delta \ll 1.$$

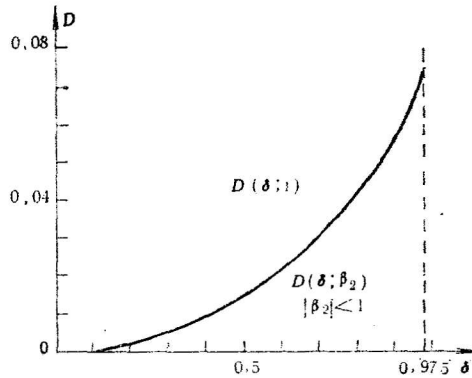


Fig. 2

Let the dumb-bells be randomly oriented in the plane (x, y) , so that the medium is isotropic in the latter plane. Averaging (5.6) over all directions of e_x then gives

$$(5.9) \quad \frac{\lambda^*}{\lambda_m} = 1 + 2\beta_2 c_f (1 + 2\beta_2 D); \quad D = \frac{1 - \delta^2}{\delta^2} \sum_{n=1}^{\infty} \frac{nd^{2n}}{1 - \beta_2^2 d^{2n}},$$

which reduces to (5.8) for prolate dumb-bells, $\delta \ll 1$.

If $\delta = 0$, i. e. the inclusions are well apart, we have $D = 0$ and (5.9) turns into the two-dimensional analogy of (4.6)

$$(5.10) \quad \lambda^* = \lambda_m (1 + 2\beta_2 c_f).$$

The parameter $D = D(\delta; \beta_2)$ introduced in (5.9) can thus serve as a quantitative characteristic of the influence of the two-clustering upon the overall conductivity in the limit of small concentration, $c_f \ll 1$. From the definition (5.9) it is clear that $D(\delta; \beta_2)$ is a positive and monotonically increasing function of β_2^2 , so that

$$(5.11) \quad 0 < D(\delta; \beta_2) < D(\delta; 1) = \sum_{n=1}^{\infty} \frac{nd^{2n}}{1 - d^{2n}},$$

since $|\beta_2| \leq 1$; the right-hand side corresponds to the case of rigid or pore-like inclusions for which $|\beta_2| = 1$. The variation of $D(\delta; 1)$ with δ is shown in Fig. 2 and thus the region within which the parameter D could vary is established (for $0 \leq \delta \leq 0.975$), cf. (5.11).

6. Conclusions

As obvious conclusions of the performed analysis one can state the following:

a) The second order clustering of inclusions always increases the effective conductivity of the suspension when compared to the case of well separated inclusions.

b) Such an increase in effective conductivity becomes noticeable only if the inclusions in the clusters are close enough, say, $\delta > 0.5$.

c) Even if these inclusions almost touch one another, for $\delta = 0.975$, the change in the coefficient of c_f in the formula for the effective conductivity does not exceed 16 per cent.

These conclusions hold, let be repeated, for dilute suspension only. The nondilute case could be treated by means of the second-order approximation, $N=2$, for the temperature field $\theta(\mathbf{x})$, cf. (2.9), taking into account the finite size of the inclusions and employing some of the ideas of [4].

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