

Inelastic Deformation of Quasihomogeneous Materials (Survey of the author's papers)

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Many technically important materials are homogeneous from the macroscopic point of view, but heterogeneous microscopically, i. e. quasihomogeneous. In quasistatic elastic deformation this fact is not so important as the process is reversible, the response to any change of loading is instantaneous and invariable, it is sufficient to characterize the material by elastic moduli only that are easy to measure. Inelasticity is different, the state of the material and the response to loading change in the process and these changes are closely connected with the microstructure.

If we want to describe such a material more closely than the supposition of homogeneity of the material allows, a straightforward way leads to the supposition of heterogeneity of the material and homogeneity of stress (Reuss solution) or heterogeneity of the material and homogeneity of strain (Voigt solution). The reality lies somewhere between.

In a number of papers the author developed a theoretical conception that characterizes a heterogeneous material by the constitutive equations of the material constituents, by the respective volume fractions v_n and by specific structural parameters η_n, η_n^0 that are non-negative and their change from zero to infinity means the change from the Voigt solution to the Reuss solution.

It is not possible and is not the aim of this short survey article to explain the deduction of the basic set of equations of the concept. Let us only state that in the quoted papers it underwent a development in the formulation of the suppositions and the way of deduction, but the resulting basic equations remained the same with the exception that the structural parameters η_n were in some cases considered as different for the deviatoric and the isotropic parts of the tensors, in some cases a simpler model was taken into account, where this difference was not distinguished. For deviatoric parts the result was the same, the analysis of the isotropic parts is important only in some special cases in inelasticity, usually it is simply considered as elastic.

In any case the deduction of the basic set of equations starts from a simplified description of the distribution of stress and strain in the microstructure. The characteristic features of the hypersurfaces of the distribution depend on the characteristic features of the microstructure and are specified by the structural parameters η_n , which are material constants. These constants are to be determined from simple macroscopic tests on the composite.

Let us confine in what follows to the case of a two-component material only, in which the constitutive equations of the two material constituents (labelled by a and b indices) can be symbolically written in the form

$$\begin{aligned} L_a(e_{ij}, \dot{e}_{ij}, s_{ij}, \dot{s}_{ij}) &= 0 \\ L_b(e_{ij}, \dot{e}_{ij}, s_{ij}, \dot{s}_{ij}) &= 0 \\ L_a^0(\varepsilon, \dot{\varepsilon}, \sigma, \dot{\sigma}) &= 0 \\ L_b^0(\varepsilon, \dot{\varepsilon}, \sigma, \dot{\sigma}) &= 0, \end{aligned}$$

where e_{ij}, s_{ij} (ε, σ) are deviatoric (isotropic) parts of strain and stress respectively, i. e.

$$\begin{aligned} s_{ij} &= \sigma_{ij} - \delta_{ij}\sigma \\ e_{ij} &= \varepsilon_{ij} - \delta_{ij}\varepsilon, \end{aligned}$$

L_a, L_b are linear forms the coefficients of which can depend on macroscopic coordinates and can change in the course of deformation. Then from our conception the following set of equations results

$$\begin{aligned} (1) \quad & v_a \sigma_{ija} + v_b \sigma_{ija} = \bar{\sigma}_{ij} \\ (2) \quad & v_a \varepsilon_{ija} + v_b \varepsilon_{ija} = \bar{\varepsilon}_{ij} \\ (3) \quad & \varepsilon'_{ija} = \varepsilon_{ija} - \bar{\varepsilon}_{ij} \\ (4) \quad & \varepsilon'_{ijb} = \varepsilon_{ijb} - \bar{\varepsilon}_{ij} \\ (5) \quad & s_{ija} - s_{ijb} + \frac{s'_{ija}}{\eta_a} - \frac{s'_{ijb}}{\eta_b} = 0 \\ & \sigma_a - \sigma_b + \frac{\sigma'_a}{\eta_a^0} - \frac{\sigma'_b}{\eta_b^0} = 0 \\ (6) \quad & L_a(e_{ija}, \dot{e}_{ija}, s_{ija}, \dot{s}_{ija}) = 0 \\ & L_a^0(\varepsilon_a, \dot{\varepsilon}_a, \sigma_a, \dot{\sigma}_a) = 0 \\ (7) \quad & L_a(e'_{ija}, \dot{e}'_{ija}, s'_{ija}, \dot{s}'_{ija}) = 0 \\ & L_a^0(\varepsilon'_a, \dot{\varepsilon}'_a, \sigma'_a, \dot{\sigma}'_a) = 0 \\ (8) \quad & L_b(e_{ijb}, \dot{e}_{ijb}, s_{ijb}, \dot{s}_{ijb}) = 0 \\ & L_b^0(\varepsilon_b, \dot{\varepsilon}_b, \sigma_b, \dot{\sigma}_b) = 0 \\ (9) \quad & L_b(e'_{ija}, \dot{e}'_{ija}, s'_{ija}, \dot{s}'_{ija}) = 0 \\ & L_b^0(\varepsilon'_b, \dot{\varepsilon}'_b, \sigma'_b, \dot{\sigma}'_b) = 0, \end{aligned}$$

where v_a is the volume fraction of the a-material, σ_{ija} is the mean value of stress in the a-material, $\bar{\sigma}_{ij}$ is the macroscopic stress, ε'_{ija} is defined by (3), s'_{ija} is the difference between s_{ija} and the deviatoric stress that would exist in the

a-material if by some artificial constraint the deformation of this material constituent was made equal to the macroscopic value \bar{e}_{ij} throughout the process.

For material models of inelastic bodies it is of course not possible to formulate the constitutive equations in terms of stresses and deformations only but in terms of their rates. In any state in the course of the deformation process the values of the tensors are supposed to be known, the rate or the infinitesimal increment of the macroscopic stress ($\dot{\sigma}_{ij}$ or $d\bar{\sigma}_{ij}$) to be given and the other rates are to be calculated. Hence the rates to be determined are: $\dot{\varepsilon}_{ij}, \dot{\varepsilon}_{ija}, \dot{\varepsilon}_{ijb}, \dot{\varepsilon}'_{ija}, \dot{\varepsilon}'_{ijb}, \dot{\sigma}_{ija}, \dot{\sigma}_{ijb}, \dot{\sigma}'_{ija}, \dot{\sigma}'_{ijb}$, i. e. 9 unknown tensor quantities.

Equations (1) to (4) can be decomposed for deviatoric and isotropic parts separately and eqns. (1) to (5) being valid throughout the deformation process, can be written for the rates of the respective tensors. The number of the tensorial equations that are available is thus also 9: (1) to (9).

From this set of equations some of the variable tensors can be eliminated and the macroscopic constitutive equation deduced. Depending on the complexity of the linear forms L_a and L_b the resulting equation can comprise none, one or more latent variables.

The advantage of this approach compared with the phenomenological description, the mechanical models and the self-consistent method lies in the possibility of describing the stresses and strains in the material constituents (including the description of the non-uniformity of their distribution), without limitation to some very simple constitutive equations and to the case of inclusions with simple shape and with dilute distribution.

A special choice of the forms L_a and L_b leads to a special mathematical model that is able to describe the behaviour of a special group of real materials. These materials can be elastic, viscoelastic, elastic-plastic, viscoplastic, etc. In this sense the outlined approach is a unifying concept that covers the above material kinds and the resulting macroscopic equations represent a concretization of the general theory of latent variables.

In what follows we will go shortly through the main author's papers that are devoted to the problems of inelasticity of quasihomogeneous materials.

The first author's paper [1] in this line takes into consideration an unlimited stratified medium composed of two different materials of the same type (ideal elastic-plastic Prandtl-Reuss materials with von Mises' yield criterion). The equations for stress- and strain-distribution at uniaxial loading are derived and especially the qualitative influence of the heterogeneity is discussed. In consequence of the heterogeneity the following phenomena appear: strain-hardening, Bauschinger effect, permanent compressibility and the dependence of the yield criterion and of the ultimate strength criterion on the first invariant of the macroscopic stress tensor. Priority of this paper in the description of the last important phenomenon is sometimes overlooked.

In another paper [2] a similar analysis is performed for the case of a layered system consisting of a cohesive and a brittle material. The criteria for brittle fracture or plastic yielding are formulated and discussed.

The third paper [3] is the first one, where the irregular general statistically homogenous microscopic configuration of the material constituents is studied and eqn. (5) derived. The microscopic heterogeneity due to the different plastic, as well as elastic properties and the ensuing inhomogeneities of stress and strain are taken to represent the fundamental characteristics of the material as regard the deformation properties. It is assumed that the elastic-plastic deformation properties of a macroscopically isotropic polycrystalline medium can well be described by such a mathematical model with microparticles that have

the properties of the perfectly elastic-plastic Prandtl-Reuss body. The micro-particles with higher resistance to plastic deformation represent in the mathematical model the barriers to easy plastic glide. Experimental evidence of other investigators, as well as of the author's own were shown to indicate a good qualitative agreement.

The fourth paper [4] is a straightforward continuation to paper [3]. The theoretical basis is analysed in detail and its consequences are compared with experimental data. Correspondence was found among others in the following points:

- The plastic limit of a virgin material is given by the Huber-Hencky-Mises criterion;

- Under uniaxial stress elastic deformation is followed by nonlinear strain-hardening with the typical features of the curve;

- Under uniaxial stress the course of the microstresses agrees well with the course found by the X-ray technique;

- In plastic deformation there occurs a very small permanent increase in macroscopic volume;

- The changes of the microstress-state in plastic deformation lead to Bauschinger effect;

- The direction of the plastic strain-rate lies between the direction implied by the "total" theory and that indicated by the classical incremental theory, closer to the latter;

- A new plastic deformation "erases" the influence of the older plastic deformation with regard to the deformation properties (i. e. diminishes the respective deviatoric residual microstress state), but the influence with regard to the exhaustion of strength accumulates (i. e. the isotropic parts of the microstresses grow continually larger);

- After a plastic deformation process there remain in the material residual microstresses that can cause a new plastic deformation if the temperature is raised.

In the following work [5] the identification problem, i. e. the problem of determining the parameters of the mathematical model, is solved. The determination is based upon the mathematical analysis of the macroscopic stress-strain diagram. The procedure is illustrated by an example showing theoretical and experimental stress-strain diagrams for an aluminum-alloy in tension, unloading and a new tension in another direction.

In article [6] the author's approach is interpreted as a concretization of the general theory of latent variables. The role of the variables is played by microstresses.

A specific application of the conception is contained in paper [7], where the material under study is reinforced with unidirectional continuous fibers and macroscopically is the material transversal-isotropic. The theoretical analysis is here simplified by the supposition of elastic incompressibility of both material constituents. All kinds of macrostress-loading are taken into account and the changes of stress in the fibers and the matrix are demonstrated on examples. In the second part of the paper the problem of determining the structural parameters from macroscopic tests is solved.

This line is followed also in paper [8], where the analysis is generalized for elastically compressible materials with macroscopic transversal isotropy and the method of calculating stresses caused by thermal changes is shown. The comparison with paper [7] leads to the conclusion that the yield criterion of a fiber-reinforced material composed of elastically incompressible constituents coincides with the Hill's yield criterion proposed for metals with transversal

isotropy that does not depend on the hydrostatic stress. However, in the case of elastic compressibility the macroscopic criterion is different, with hydrostatic stress involved.

In [9] the concept is confronted with the general relations of thermodynamics and some formal relations of thermodynamics acquire a clear physical meaning. Those from this approach issuing theoretical values of the stored energy in the course of plastic deformation are compared with the general experimental evidence and a good agreement is found.

A continuation to this line is paper [10], where the problem of determining stored energy from a given stress-strain curve is studied more in detail; an older method is proved to give an upper bound to the real values of stored energy and the results of the newly proposed method are compared with calorimetric measurements on several materials. A good quantitative agreement is found.

In [11] the fundamental equations of the conception are derived in a new way and applied to the description of the inelastic deformation of quasihomogeneous composite materials allowing for a broad class of constitutive equations and a general geometry of the microscopic composition. It is shown how normality on the macroscale results in this conception from the normality on the microscale and uniqueness of the boundary value problems is proved.

Paper [12] is an application of the conception to the problem of plastic deformation of polycrystalline materials under varying temperature. Two problems are studied in detail — plastic deformation caused by the change of temperature in a previously plastically strained material and the influence of the temperature of the preceding plastic deformation process upon a new process.

The last paper [13] differs substantially from the others. No deformation process is studied and no use of the eqns. (3) to (9) is made. The upper bounds for the strength of heterogeneous materials consisting of an arbitrary number of constituents are deduced under very broad assumptions — no restriction on constitutive equations and on the initial state of stress or strain is needed. The bounds are proved both for stresses and strains, which is of importance for materials with limited possibilities of deformation. Some older criteria of other investigators derived on the basis of the classical limit design theorems result as special cases.

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(the author of all of them is the author of this survey paper)

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