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Shakedown Deflections. A Finite Element Approach

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1. Introduction

Wellknown theorems of Melan and W. T. Koiter [1] allow to estimate the ranges of variation of loads allowing for adaptation (shake-down) of an elastic-plastic structure. The theorems have been demonstrated under assumptions of geometrical linearity of the structure and unlimited deformability of the material. Yet the hitherto results of the shakedown theory do not give any bounds on deformations of a structure which has been shaken down to a given load program. Therefore some doubts may arise concerning validity of the theorems when they are applied to actual structures.

The paper presents a procedure which reduces the problem of the displacements estimation to a problem of mathematical programming and thus projects some light on the validity of the shakedown theorems in practical problems. The discrete description of structures is employed, following papers by G. Maier [2, 3].

In our considerations we assume that the solution of the classical shakedown problem is given. Namely, that, for a given structure and a given load program, we know the safety factor s against inadaptation as well as an appropriate field of time-independent residual generalized stresses as required by Melan's theorem are known.

The loading program is assumed to be possible to be described by a finite set of linear load parametrs μ_i [6], such that

(1.1) $$\mu_i^- < \mu_i < \mu_i^+, \quad i=1, 2, \ldots, z$$

and such that all the elastic generalized stress states $\mathbf{Q}^E$ admitted by this program may be presented as linear combinations of the partial stress states $\mathbf{Q}^{E_i}$, $i=1, 2, \ldots, z$ associated with the respective load parameters:

(1.2) $$\mathbf{Q}^E = \sum_{i=1}^{z} \mu_i \mathbf{Q}^{E_i}.$$

Analogously, the state of elastic displacements $\mathbf{u}^E$ is the linear combination

$$\mathbf{u}^E = \sum_{i=1}^{z} \mu_i \mathbf{u}^{E_i}.$$

Now let us construct the following 2^z generalized stress states:

$$\mathbf{Q}^{E\alpha} = \sum_{i=1}^{z} \beta_i \mathbf{Q}^{E_i} \tag{1.3}$$

where $\beta_i = \mu_i^-$ or $\beta_i = \mu_i^+$ arbitrarily and let the integer variable $\alpha = 1, 2, \ldots, 2^z$ numerates all these states. This assembly of $\mathbf{Q}^{E\alpha}$ suffices to describe all the possible states of elastic stresses $\mathbf{Q}^E$.

2. Basic Relations

We consider a structure as a finite assembly of n elements connected and interacting at special points called nodes. The elements are sufficiently small to assume their stress and strain states to be homogeneous within each element. The states are thus described by generalized stresses Q_i^k and generalized strains q_i^k, $k=1, 2, \ldots, n$; $i=1, 2, \ldots, v$. The external loads F_s, $s=1, 2, \ldots, m$ act exclusively at nodes and the respective displacements of the nodes in the directions of the loads are u_s. Every element is assumed to remain completely in one of the two admissible states: elastic or plastic.

To make formulas as short as possible let us introduce the following quantities:

μ, μ^-, μ^+ — vectors of current, minimal and maximal load parameters $[z \times l]$;
$\mathbf{F}$ — vector of nodal loads $[m \times l]$;
$\mathbf{u}$ — vector of nodal displacements $[m \times l]$;
$\mathbf{Q}$ — vector of generalized stresses $[vn \times l]$;
$\mathbf{q}$ — vector of generalized strains $[vn \times l]$;
$\mathbf{Q}^E$ — vectors of extremal elastic generalized stresses $[vn \times l]$;
$\mathfrak{a}$ — rectangular matrix of compatibility $[m \times vn]$;
$\mathbf{A}$ — nonsingular, positively defined, square matrix of elastic moduli $[vn \times vn]$;
$\mathbf{K}$ — vector of plastic constants $[pn \times l]$;
$\mathbf{L}$ — rectangular matrix $[pn \times vn]$;
$\boldsymbol{\lambda}$ — vector of modes of plastic strains $[pn \times l]$.

Note: numbers in brackets show the appropriate numbers of columns and rows.

Now the basic relations may be written as follows:

equilibrium equations

$$\mathbf{F} = \mathfrak{a}\mathbf{Q}^T \tag{2.1}$$

compatibility conditions

$$\mathbf{q} = \mathfrak{a}^T \mathbf{u} \tag{2.2}$$

Prandtl-Reuss assumption

$$\mathbf{q} = \mathbf{q}^E + \mathbf{q}^P \tag{2.3}$$

Hooke's law

$$\mathbf{q}^E = \mathbf{A}\mathbf{Q} \text{ or } \mathbf{Q} = \mathbf{A}^{-l}\mathbf{q}^E \tag{2.4}$$

linear yield condition

$$\mathbf{L}\mathbf{Q} - \mathbf{K} \leq \mathbf{O} \tag{2.5}$$

associated flow rule

(2.6) $$\dot{q}^P = L^T \dot{\lambda}$$

active loading criterion

(2.7) $$[LQ - K]\dot{\lambda} = O.$$

3. Displacements Decomposition

The generalized stresses may be presented as

(3.1) $$Q = Q^E + Q^R$$

where $\alpha Q^R = O$ and Q^E is the solution appropriate for the perfectly elastic structure.

Thus the generalized strain may be decomposed in the following (generally not unique) way:

(3.2) $$q = q^{EE} + q^R + q^N + q^M$$

where

(3.3) $$q^{EE} = AQ^E; \quad q^R = AQ^R$$

and there exists such u^M that

(3.4) $$q^M = \alpha^T u^M.$$

As Q^E is a solution of a problem of elasticity there exists a unique vector u^E such that

(3.5) $$q^{EE} = \alpha^T u^E.$$

Thus there exists a vector u^{RN} such that

(3.6) $$q^R + q^N = \alpha^T u^{RN}.$$

According to (2.4), (3.2), (3.3), (3.6) we can write

(3.7) $$\alpha A^{-1} \alpha^T u^{RN} = \alpha A^{-1} q^N.$$

This system is uniquely solvable with respect to u^{RN}. The solution is

(3.8) $$u^{RN} = [\alpha A^{-1} \alpha^T]^{-1} \alpha A^{-1} q^N = C q^N, \quad C\,[m \times vn].$$

The uniqueness of this solution results from the uniqueness of the elastic solution of the structure.

Then the respective residual stress is

(3.9) $$Q^R = A^{-1} q^R = A^{-1}[\alpha^T u^{RN} - q^N] = A^{-1}[\alpha^T C - 1] q^N = D q^N, \quad D\,[m \times vn].$$

4. Melan's Theorem

The static shakedown theorem may be now formulated as follows: if there exists a safety factor $s > 1$ and a time-independent vector $\bar{Q}^R$ of residual generalized stresses such that for every $a = 1, 2, \ldots, 2^z$

(4.1) $$LQ^E + L\bar{Q}^R \leq \frac{1}{s} K$$

then the structure shakes down and the total energy dissipated may be estimated as

$$(4.2) \qquad W_p = \int_0^\infty \int_V \sigma_{ij} \dot{\varepsilon}_{ij}^P dV dt = \int_0^\infty \mathbf{Q}_{ij}^T \dot{\mathbf{q}}^P dt \leq \frac{s}{2(s-1)} (\bar{\mathbf{Q}}^R)^T \mathbf{A} \bar{\mathbf{Q}}^R = a.$$

According to (2.6) the energy dissipation may be written down as

$$(4.3) \qquad \mathbf{Q}^T \dot{\mathbf{q}}^P = (\dot{\mathbf{q}}^P)^T \mathbf{Q} = (\mathbf{L}^T \dot{\boldsymbol{\lambda}})^T \mathbf{Q} = \dot{\boldsymbol{\lambda}}^T \mathbf{L} \mathbf{Q} = \dot{\boldsymbol{\lambda}}^T \mathbf{K}$$

and the total energy dissipated as

$$(4.4) \qquad \int_0^t \mathbf{Q}^T \dot{\mathbf{q}}^P \, dt = \boldsymbol{\lambda}^T \mathbf{K}.$$

Since demonstration of the theorem follows from the wellknown considerations of the energy functional $\frac{1}{2}(\mathbf{Q}^R - \bar{\mathbf{Q}}^R)\mathbf{A}(\mathbf{Q}^R - \bar{\mathbf{Q}}^R)$ there is no idea in repeating it here [1], [4].

The fact that the total generalized stresses $\mathbf{Q}$ must satisfy the yield condition (2.5) imposes some additional limitations on the range of possible actual residual generalized stresses $\mathbf{Q}^R$. Namely the inequalities (2.5) give

$$(4.5) \qquad \mathbf{L}[\mathbf{Q}^E + \mathbf{Q}^R] \leq \mathbf{K}.$$

Thus, for all $\alpha = 1, 2, \ldots, 2^z$:

$$(4.6) \qquad \mathbf{L}\mathbf{Q}^R \leq \mathbf{K} - \mathbf{L}\mathbf{Q}^{E\alpha},$$

If the load program is taken with the safety factor s then, instead of (4.6) the following must be taken:

$$(4.7) \qquad \mathbf{L}\mathbf{Q}^R \leq \mathbf{K} - \frac{1}{s} \mathbf{L}\mathbf{Q}^{E\alpha}.$$

5. Displacements Bounds

According to the formulas (3.4), (3.5), (3.9) a total displacement at a node is the following sum:

$$(5.1) \qquad u_s = u_s^E + u_s^M + u_s^{RN}.$$

The problem of bounds of the u_s component is a problem of maximum (minimum) of the u_s as of a linear combination of the following variables: $\mu_i, (i = 1, 2, \ldots, z)$, u_i^M, $(i = 1, 2, \ldots, m)$, q_i^N, $(i = 1, 2, \ldots, n)$, λ_i, $(i = 1, 2, \ldots, vn)$. However, the q_i^N variables may be expressed by λ_i, (2.3), (2.6), (3.4):

$$(5.2) \qquad \mathbf{q}^N = \mathbf{L}^T \boldsymbol{\lambda} - \boldsymbol{\alpha}^T \mathbf{u}^M.$$

Thus the problem under consideration may be formulated as follows:

find $$\max(\min)\, u_s = \sum_{i=1}^{z} \mu_i u_s^{Ei} + u_s^M + \mathbf{C}[\mathbf{L}^T \boldsymbol{\lambda} - \boldsymbol{\alpha}^T \mathbf{u}^M]|_s$$

subject to the following constraints:

$$\boldsymbol{\mu} \leq \boldsymbol{\mu}^+,$$
$$\bar{\boldsymbol{\mu}} \leq \boldsymbol{\mu},$$
$$\boldsymbol{\lambda}^T \mathbf{K} \leq a,$$
$$\boldsymbol{\lambda} \geq \mathbf{O},$$

(5.3) $$\mathbf{LD}\,[\mathbf{L}^T\boldsymbol{\lambda}-\alpha^T\mathbf{u}^M]\leq\mathbf{K}-\frac{1}{s}\,\mathbf{LQ}^{E\alpha}.$$

Standard procedures applied in solving linear programming problems usually contain requireing all the variables to be non-negative. To satisfy the above condition it suffices to introduce the following new variables:

(5.4) $$\bar{\mu}=\mu-\mu^-;\quad \mathbf{u}^M=\mathbf{u}^+-\mathbf{u}^-$$

where $\mathbf{u}^+\geq\mathbf{O}$, $\mathbf{u}^-\geq\mathbf{O}$, $\bar{\mu}\geq\mathbf{O}$.

Now the problem will take the form of:

find $$\max(\min)\,\mathbf{u}_s=\sum_{i=1}^{z}\bar{\mu}_i u_s^{Ei}-\sum_{i=1}^{z}\mu_i^- u_s^{Ei}+u_s^+-u_s^-+\mathbf{C}[\mathbf{L}^T\boldsymbol{\lambda}-\alpha^T(\mathbf{u}^+-\mathbf{u}^-)]|_s$$

with the constrains:

$$\bar{\mu}\geq\mathbf{O};\quad \mathbf{u}^+\geq\mathbf{O};\quad \mathbf{u}^-\geq\mathbf{O};\quad \boldsymbol{\lambda}\geq\mathbf{O};\quad \boldsymbol{\lambda}^T\mathbf{K}\leq a;$$

(5.5) $$\bar{\mu}\leq\mu^+-\mu^-;\quad \mathbf{LD}[\mathbf{L}^T\boldsymbol{\lambda}-\alpha^T(\mathbf{u}^+-\mathbf{u}^-)]\leq\mathbf{K}-\frac{1}{s}\,\mathbf{LQ}^{E\alpha}.$$

Now the problem fits into the classical scheme of linear programming and any standard procedure may be used to solve it.

6. Remarks

The main idea of decomposition and estimation of deflections as applied to framed structures was presented in [5].

A problem of exactness of the results is a problem of applicability of the finite element method to elastic-plastic continua. This problem, however, has not yet been solved.

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