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Elastic Waves in an Infinite Elliptical Cylinder with Micropolar Structure

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I. Introduction

The propagation of harmonic waves in infinite circular cylinder with stressfree surface is investigated on the basis of the classical linear theory of elasticity for the cases of anisotropy and different kinds cross section of the cylinder. Simultaneously a new direction of divelopment is seen, connected with the asymmetric theory of elasticity for the Cosserat medium — medium with micropolar structure [1], the deformation of which is described by six independent functions: three components of the displacement vector and three components of the rotation vector. Upon the basis of this theory W. Nowacki and W. K. Nowacki [2, 3] have obtained the cases of propagation of longitudinal and torsional waves in infinite circular cylinder. The general case of harmonic wave propagation is investigated in [4].

In this paper on the basis of micropolar theory of elasticity the propagation of harmonic waves in an infinite cylinder of elliptical cross section is considered. Through the introduction of potential functions and the separation of variables, the solution of the problem is reduced to the solution of Mathieu's equations. The boundary condition that the cylindrical surface is stressfree is obtained in the form of an infinite determinant permitting the phase velocities of wave propagation to be determined.

The derived results allow a possibility of obtaining the particular case of harmonic wave propagation in circular cylinder.

II. Basic Equations and Their Solution

The basic equations of motion for isotropic, homogeneous and centrosymmetric medium with micropolar structure in Cartesian coordinate system $x_i'(i=1,2,3)$ have the form [2];

(2.1) $$(\mu+\alpha)\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}+2\alpha\,\mathrm{rot}\,\boldsymbol{\omega}=\varrho\ddot{\mathbf{u}}$$

(2.2) $$(\gamma+\varepsilon)\nabla^2\boldsymbol{\omega}+(\beta+\gamma-\varepsilon)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega}+2\alpha\,\mathrm{rot}\,\mathbf{u}-4\alpha\boldsymbol{\omega}=J\ddot{\boldsymbol{\omega}},$$

where the following notations are introduced:

$\mathbf{u}=(u_1', u_2', u_3')$ — the displacement vector;

$\boldsymbol{\omega}=(\omega'_1, \omega'_2, \omega'_3)$ — the rotation vector;
ϱ — the density;
J — the rotational inertia;
$\lambda, \mu, \beta, \gamma, \alpha, \varepsilon$ — the characteristic constants of material;
$\nabla^2=\frac{\partial}{\partial x'_i}\frac{\partial}{\partial x'_i}$ — the Laplacian operator.

The repeating of the subscripts denotes summing up from 1 to 3. The functions $\mathbf{u}$ and $\boldsymbol{\omega}$ depend on the position $\mathbf{x}=(x'_1, x'_2, x'_3)$ and the time t. With dot upon the functions the time derivatives are marked.

The state of strain is described by two asymmetric tensors: tensor of strain γ'_{ij} and the curvature twist tensor $\varkappa'_{ij}$. They have the form

$$\gamma'_{ij}=\frac{\partial u'_j}{\partial x'_i}-\varepsilon_{kij}\omega'_k, \quad \varkappa'_{ij}=\frac{\partial \omega'_j}{\partial x'_i}, \tag{2.3}$$

where ε_{kij} is the tensor of Levi-Civita. The subscripts can take values from 1 to 3. The state of stress is defined by two asymmetric tensors: tensor of stress σ'_{ij} and the couple stress tensor μ'_{ij}. The stress-strain relation is described by the equations

$$\begin{aligned}\sigma'_{ij}&=(\mu+\alpha)\gamma'_{ij}+(\mu-\alpha)\gamma'_{ij}+\lambda\gamma'_{kk}\delta_{ij};\\ \mu'_{ij}&=(\gamma+\varepsilon)\varkappa'_{ij}+(\gamma-\varepsilon)\varkappa'_{ji}+\beta\varkappa'_{kk}\delta_{ij},\end{aligned} \tag{2.4}$$

where δ_{ij} is Kronecker deltas.

The solution of the problem is being sought in the form

$$\begin{aligned} u'_1&=\left(\frac{\partial\varphi}{\partial x'_1}+\frac{\partial\psi}{\partial x'_2}\right)\cos(kx'_3+\omega t); & \omega'_1&=\left(\frac{\partial f}{\partial x'_1}+\frac{\partial g}{\partial x'_2}\right)\sin(kx'_3+\omega t),\\ u'_2&=\left(\frac{\partial\varphi}{\partial x'_2}-\frac{\partial\psi}{\partial x'_1}\right)\cos(kx'_3+\omega t); & \omega'_2&=\left(\frac{\partial f}{\partial x'_2}-\frac{\partial g}{\partial x'_1}\right)\sin(kx'_3+\omega t),\\ u'_3&=p\varphi\sin(kx'_3+\omega t); & \omega'_3&=qf\cos(kx'_3+\omega t),\end{aligned} \tag{2.5}$$

where φ, ψ, f and g are functions of x'_1 and x'_2, $\frac{2\pi}{k}$ is the wavelength, $\frac{\omega}{2\pi}$ — its frequency; $c=\frac{\omega}{k}$ denotes the phase velocity of propagation in x'_3— direction; p and q are constants which will be determined later on.

Equations (2.1) and (2.2) will be satisfied on condition that $\alpha k \neq 0$, if the following equations are satisfied [4]:

$$\Delta\varphi_i+\xi_i^2\varphi_i=0, \quad \Delta f_i+\eta_i^2 f_i=0 \tag{2.6}$$

where i is equal to 1, to 2 or to 3. $\varphi_i(x'_1, x'_2)$ and $f_i(x'_1, x'_2)$ are potential functions corresponding to the quantities ξ_i^2 and η_i^2 respectively and

$$\Delta=\frac{\partial^2}{\partial x_1'^2}+\frac{\partial^2}{\partial x_2'^2}, \tag{2.7}$$

$$\xi_1^2=\frac{\varrho\omega^2}{\lambda+2\mu}-k^2, \quad \eta_1^2=\frac{J\omega^2-4\alpha}{\beta+2\gamma}-k^2. \tag{2.8}$$

The quantities $\xi_i^2=\eta_i^2 (i=2, 3)$ are roots of the equation

(2.9) $$(\mu-\alpha)(\gamma-\varepsilon)(\xi^2+k^2)^2-[(\gamma+\varepsilon)\varrho\omega^2+(\mu+\alpha)(J\omega^2-4\alpha)+4\alpha^2](\xi^2+k^2)+\varrho\omega^2(J\omega^2-4\alpha)=0.$$

They are always real numbers. The functions ψ_i and g_i have the form

(2.10) $$\psi_i=\tau_i f_i,\quad g_i=\sigma_i\varphi_i,$$

where

(2.11) $$\tau_1=0,\quad 2\alpha k\tau_i=-(\gamma+\varepsilon)(\eta_i^2+k^2)+(J\omega^2-4\alpha),\quad (i=2,3),$$
$$\sigma_1=0,\quad 2\alpha k\sigma_i=(\mu+\alpha)(\xi_i^2+k^2)-\varrho\omega^2,\quad (i=2,3).$$

The general solution for the displacements and rotations is given by (2.5), where φ, ψ, f, g, p and q are replaced by φ_i, ψ_i f_i, g_i, p_i and q_i respectively and the obtained expressions are summed up from 1 to 3. The quantities p_i and q_i are

(2.12) $$p_1=-q_1=-k,\quad kp_i=-kq_i=\xi_i^2,\quad (i=2,3).$$

When $k=0$, but $\alpha=0$, the problem can be reduced again to a system of the same kind as (2.6). It is shown in [4].

If $\alpha=0$, the equations (2.1) and (2.2) become independent one from the other, as the equations (2.1) correspond to the classical theory of elasticity. Sinse the two systems have the same structure, the solution of the equations (2.2) is obtained in the same way, as the solution of (2.1) in the classical theory.

III. Determination of the Displacements, Rotations and Stresses

The solution of Eqs. (2.6) will be sought by separation the variables in an elliptical coordinate system $x_i (i=1,2,3)$, where

(3.1) $$x_1'=l\operatorname{ch}x_1\cos x_2;\quad x_2'=l\operatorname{sh}x_1\sin x_2;\quad x_3'=x_3$$

and $2l$ is the focal length of the ellipse corresponding to $x_1=\text{const}$. The operator Δ has the form

(3.2) $$\Delta=\frac{1}{h^2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right),\quad 2h^2=l^2(\operatorname{ch}2x_1-\cos 2x_2).$$

Using representation

(3.3) $$\varphi_i(x_1,x_2)=\varphi_{i1}(x_1)\varphi_{i2}(x_2),\quad (i=1,2,3),$$
$$f_i(x_1,x_2)=f_{i1}(x_1)f_{i2}(x_2),\quad (i=1,2,3)$$

is obtained

(3.4) $$\frac{d^2\varphi_{i2}}{dx_2^2}+(a_i-2\lambda_i\cos 2x_2)\varphi_{i2}=0,\quad \frac{d^2f_{i2}}{dx_2^2}+(\beta_i-2\mu_i\cos 2x_2)f_{i2}=0,$$

(3.5) $$\frac{d^2\varphi_{i1}}{dx_1^2}-(a_i-2\lambda_i\operatorname{ch}2x_1)\varphi_{i1}=0,\quad \frac{d^2f_{i1}}{dx_1^2}-(\beta_i-2\mu_i\operatorname{ch}2x_i)f_{i1}=0,$$

where $4\lambda_i = l^2\xi_i^2$ and $4\mu_i = l^2\eta_i^2 (i=1, 2, 3)$; α_i and β_i are the separation constants. The equations (3.4) are Mathieu's equations and the equations (3.5) are the modified equations of Mathieu. Since the solution has to be periodic in x_2 with period 2π, the separation constants must be chosen equal to the characteristic numbers of the corresponding equations. All information on Mathieu's equations and also for the notation used can be found in the book of McLachlan [5].

We will consider only the case when $\lambda_i > 0$, $\mu_i > 0 (i=1, 2, 3)$. The other cases can be considered in an analogous way.

The periodic solutions of the equations (3.4) and the two linearly independent solutions of the equations (3.5) are respectively

$$(3.6)\qquad \varphi_{i2}(x_2) = \begin{cases} ce_{2n}(x_2, \lambda_i) = \sum_{r=0}^{\infty} A_{2r}^{2n}(\lambda_i) \cos 2rx_2 \\ ce_{2n+1}(x_2, \lambda_i) = \sum_{r=0}^{\infty} A_{2r+1}^{2n+1}(\lambda_i) \cos (2r+1)x_2 \\ se_{2n+1}(x_2, \lambda_i) = \sum_{r=0}^{\infty} B_{2r+1}^{2n+1}(\lambda_i) \sin (2r+1)x_2 \\ se_{2n+2}(x_2, \lambda_i) = \sum_{r=0}^{\infty} B_{2r+2}^{2n+2}(\lambda_i) \sin (2r+2)x_2, \end{cases}$$

$$(3.7)\qquad \varphi_{i1}(x_1) = \begin{cases} Ce_{2n}(x_1, \lambda_i); & Fe_{2n}(x_1, \lambda_i) \\ Ce_{2n+1}(x_1, \lambda_i); & Fe_{2n+1}(x_1, \lambda_i) \\ Se_{2n+1}(x_1, \lambda_i); & Ge_{2n+1}(x_1, \lambda_i) \\ Se_{2n+2}(x_1, \lambda_i); & Ge_{2n+2}(x_1, \lambda_i) \end{cases}$$

where the corresponding characteristic numbers are

$$(3.8)\qquad \alpha_i = a_{2n}(\lambda_i), \quad \alpha_i = a_{2n+1}(\lambda_i), \quad \alpha_i = b_{2n+1}(\lambda_i), \quad \alpha_i = b_{2n+2}(\lambda_i)$$

and analogously for $f_{i2}(x_2)$ and $f_{i1}(x_1)$.

It is possible to satisfy the boundary conditions for four types of harmonic waves, for which the following conditional designations are used, analogous to those introduced for a circular cylinder.

A. Flexural waves of the Ist type

$$(3.9)\qquad \begin{aligned} \varphi_i(x_1, x_2) &= \sum_{n=0}^{\infty} [C_{2n+1}^{(i)} Ce_{2n+1}(x_1, \lambda_i) + F_{2n+1}^{(i)} Fe_{2n+1}(x_1, \lambda_i)] ce_{2n+1}(x_2, \lambda_i); \\ f_i(x_1, x_2) &= \sum_{n=0}^{\infty} [S_{2n+1}^{(i)} Se_{2n+1}(x_1, \mu_i) + G_{2n+1}^{(i)} Ge_{2n+1}(x_1, \mu_i)] se_{2n+1}(x_2, \mu_i). \end{aligned}$$

For this solution, when $x_1 = \text{const}$, the displacements u_1 and u_2 are symmetric with respect to the major axis and antisymmetric with respect to the minor axis and the displacement u_2 is antisymmetric with respect to the major axis and symmetric with respect to the minor axis and the opposite for the rotation $\boldsymbol{\omega}$.

B. Flexural waves of the IInd type

$$
(3.10)\quad
\begin{aligned}
\varphi_i(x_1, x_2) &= \sum_{n=0}^{\infty} [S_{2n+1}^{(i)} Se_{2n+1}(x_1, \lambda_i) + G_{2n+1}^{(i)} Ge_{2n+1}(x_1, \lambda_i)] se_{2n+1}(x_2, \lambda_i);\\
f_i(x_1, x_2) &= \sum_{n=0}^{\infty} [C_{2n+1}^{(i)} Ce_{2n+1}(x_1, \mu_i) + F_{2n+1}^{(i)} Fe_{2n+1}(x_1, \mu_i)] ce_{2n+1}(x_2, \mu_i).
\end{aligned}
$$

For this solution, when $x_1 = \text{const}$, the displacements u_1 and u_3 are antisymmetric with respect to the major axis and symmetric with respect to the minor axis and the displacement u_2 is symmetric with respect to the major axis and antisymmetric with respect to the minor axis and the opposite for the rotation ω.

C. Longitudinal waves

$$
(3.11)\quad
\begin{aligned}
\varphi_i(x_1, x_2) &= \sum_{n=0}^{\infty} [C_{2n}^{(i)} Ce_{2n}(x_1, \lambda_i) + F_{2n}^{(i)} Fe_{2n}(x_1, \lambda_i)] ce_{2n}(x_2, \lambda_i);\\
f_i(x_1, x_2) &= \sum_{n=0}^{\infty} [S_{2n+2}^{(i)} Se_{2n+2}(x_1, \mu_i) + G_{2n+2}^{(i)} Ge_{2n+2}(x_1, \mu_i)] se_{2n+2}(x_2, \mu_i).
\end{aligned}
$$

For this solution, when $x_1 = \text{const}$, the displacements u_1 and u_3 are symmetric with respect to the major and minor axis, while u_2 is antisymmetric and the opposite for the rotation ω.

D. Torsional waves

$$
(3.12)\quad
\begin{aligned}
\varphi_i(x_1, x_2) &= \sum_{n=0}^{\infty} [S_{2n+2}^{(i)} Se_{2n+2}(x_1, \lambda_i) + G_{2n+2}^{(i)} Ge_{2n+2}(x_1, \lambda_i)] se_{2n+2}(x_2, \lambda_i);\\
f_i(x_1, x_2) &= \sum_{n=0}^{\infty} [C_{2n}^{(i)} Ce_{2n}(x_1, \mu_i) + F_{2n}^{(i)} Fe_{2n}(x_1, \mu_i)] ce_{2n}(x_2, \mu_i).
\end{aligned}
$$

For this solution, when $x_1 = \text{const}$, the displacements u_2 and u_3 are antisymmetric with respect to the major and minor axis, while u_1 is symmetric and the opposite for the rotation ω.

In the equations (3.9) $\div$ (3.12) $C_m^{(i)}$, $F_m^{(i)}$, $S_m^{(i)}$ and $G_m^{(i)}$ are constants, which will be determined by the boundary conditions.

The components of the displacement and the rotation in an elliptical coordinate system have the form

$$
(3.13)\quad
\begin{aligned}
hu_1 &= \sum_{i=i}^{3} \left(\frac{\partial \varphi_i}{\partial x_1} + \frac{\partial \psi_i}{\partial x_2} \right) \cos(kx_3 + \omega t); & h\omega_1 &= \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x_1} + \frac{\partial g_i}{\partial x_2} \right) \sin(kx_3 + \omega t),\\
hu_2 &= \sum_{i=1}^{3} \left(\frac{\partial \varphi_i}{\partial x_2} - \frac{\partial \psi_i}{\partial x_1} \right) \cos(kx_3 + \omega t); & h\omega_2 &= \sum_{i=1}^{3} \left(\frac{\partial f_i}{\partial x_2} - \frac{\partial g_i}{\partial x_1} \right) \sin(kx_3 + \omega t),\\
u_3 &= \sum_{i=1}^{3} p_i \varphi_i \sin(kx_3 + \omega t); & \omega_3 &= \sum_{i=1}^{3} q_i f_i \cos(kx_3 + \omega t),
\end{aligned}
$$

where to φ_i, ψ_i, f_i and g_i is looked upon now as functions of x_1 and x_2. The state of strain is described in an elliptical coordinates by the tensors:

$$
(3.14)\quad
\begin{aligned}
&\gamma_{11}=\frac{1}{h}\frac{\partial u_1}{\partial x_1}+\frac{1}{h^2}\frac{\partial h}{\partial x_2}u_2; && \varkappa_{11}=\frac{1}{h}\frac{\partial \omega_1}{\partial x_1}+\frac{1}{h^2}\frac{\partial h}{\partial x_2}\omega_2,\\
&\gamma_{22}=\frac{1}{h}\frac{\partial u_2}{\partial x_2}+\frac{1}{h^2}\frac{\partial h}{\partial x_1}u_1; && \varkappa_{22}=\frac{1}{h}\frac{\partial \omega_2}{\partial x_2}+\frac{1}{h^2}\frac{\partial h}{\partial x_1}\omega_1,\\
&\gamma_{33}=\frac{\partial u_3}{\partial x_3}; && \varkappa_{33}=\frac{\partial \omega_3}{\partial x_3},\\
&\gamma_{23}=\frac{1}{h}\frac{\partial u_3}{\partial x_2}-\omega_1; && \varkappa_{23}=\frac{1}{h}\frac{\partial \omega_3}{\partial x_2},\\
&\gamma_{32}=\frac{\partial u_2}{\partial x_3}+\omega_1; && \varkappa_{32}=\frac{\partial \omega_2}{\partial x_3},\\
&\gamma_{13}=\frac{1}{h}\frac{\partial u_3}{\partial x_1}+\omega_2; && \varkappa_{13}=\frac{1}{h}\frac{\partial \omega_3}{\partial x_1},\\
&\gamma_{31}=\frac{\partial u_1}{\partial x_3}-\omega_2; && \varkappa_{31}=\frac{\partial \omega_1}{\partial x_3},\\
&\gamma_{12}=\frac{1}{h}\frac{\partial u_2}{\partial x_1}-\frac{1}{h^2}\frac{\partial h}{\partial x_2}u_1-\omega_3; && \varkappa_{12}=\frac{1}{h}\frac{\partial \omega_2}{\partial x_1}-\frac{1}{h^2}\frac{\partial h}{\partial x_2}\omega_1,\\
&\gamma_{21}=\frac{1}{h}\frac{\partial u_1}{\partial x_2}-\frac{1}{h^2}\frac{\partial h}{\partial x_1}u_2+\omega_3; && \varkappa_{21}=\frac{1}{h}\frac{\partial \omega_1}{\partial x_2}-\frac{1}{h^2}\frac{\partial h}{\partial x_1}\omega_2.
\end{aligned}
$$

By using the stress-strain relations, which are the same as in Cartesian coordinate system and (3.14), is obtained

$$
(3.15)\quad
\begin{aligned}
&\sigma_{11}=(\lambda+2\mu)\left(\frac{1}{h}\frac{\partial u_1}{\partial x_1}+\frac{1}{h^2}\frac{\partial h}{\partial x_2}u_2\right)+\lambda\left(\frac{1}{h}\frac{\partial u_2}{\partial x_2}+\frac{1}{h^2}\frac{\partial h}{\partial x_1}u_1+\frac{\partial u_3}{\partial x_3}\right);\\
&\sigma_{12}=(\mu+\alpha)\left(\frac{1}{h}\frac{\partial u_2}{\partial x_1}-\frac{1}{h^2}\frac{\partial h}{\partial x_2}u_1\right)+(\mu-\alpha)\left(\frac{1}{h}\frac{\partial u_1}{\partial x_2}-\frac{1}{h^2}\frac{\partial h}{\partial x_1}u_2\right)-2\alpha\omega_3;\\
&\sigma_{13}=(\mu+\alpha)\frac{1}{h}\frac{\partial u_3}{\partial x_1}+(\mu-\alpha)\frac{\partial u_1}{\partial x_3}+2\alpha\omega_2;\\
&\mu_{11}=(\beta+2\gamma)\left(\frac{1}{h}\frac{\partial \omega_1}{\partial x_1}+\frac{1}{h^2}\frac{\partial h}{\partial x_2}\omega_2\right)+\beta\left(\frac{1}{h}\frac{\partial \omega_2}{\partial x_2}+\frac{1}{h^2}\frac{\partial h}{\partial x_1}\omega_1+\frac{\partial \omega_3}{\partial x_3}\right);\\
&\mu_{12}=(\gamma+\varepsilon)\left(\frac{1}{h}\frac{\partial \omega_2}{\partial x_1}-\frac{1}{h^2}\frac{\partial h}{\partial x_2}\omega_1\right)+(\gamma-\varepsilon)\left(\frac{1}{h}\frac{\partial \omega_1}{\partial x_2}-\frac{1}{h^2}\frac{\partial h}{\partial x_1}\omega_2\right);\\
&\mu_{13}=(\gamma+\varepsilon)\frac{1}{h}\frac{\partial \omega_3}{\partial x_1}+(\gamma-\varepsilon)\frac{\partial \omega_1}{\partial x_3}.
\end{aligned}
$$

IV. Boundary Conditions and Equations for Phase Velocities

The boundary condition that cylindrical surface is stressfree has the form

$$
(4.1)\qquad \sigma_{11}=\sigma_{12}=\sigma_{13}=\mu_{11}=\mu_{12}=\mu_{13}=0
$$

for $x_1=\xi_0$ and ξ in the case of hollow elliptical cylinder ($\xi_0\leq x_1\leq\xi$), where ξ_0 and ξ are constants. The solutions (3.9)÷(3.12) can be applied in this case.

In the case of a solid elliptical cylinder the boundary condition (4.1) must be satisfied for $x_1=\xi(0\leq x_1\leq\xi)$. For this case we set $F_m^{(i)}=G_m^{(i)}=0$ $(i=1, 2, 3)$ in the solutions (3.9)÷(3.12), as it is shown in [6]. It should be noted here that these solutions lead to continuous displacement and rotation expressions across the interfocal line. In the same time they are correct for the case of an infinite space with elliptical cavity ($\xi \leq x_1 < \infty$).

Since the following calculations are very long we will consider only one of the possible cases: propagation of flexural waves of the Ist type where $F^{(i)}_m = G^{(i)}_m = 0$ $(i=1, 2, 3)$. For the purpose of simplification of the notations we write ${}^iA^{2n-1}_{2r-1}$, ${}^iB^{2n-1}_{2r-1}$, $a^{(i)}_{2n-1}$ and $b^{(i)}_{2n-1}$ $(i=1, 2, 3)$ instead of $A^{2n+1}_{2r+1}(\lambda_i)$, $B^{2n+1}_{2r+1}(\mu_i)$, $a_{2n-1}(\lambda_i)$ and $b_{2n-1}(\mu_i)$.

When we use the equations (3.6), we obtain for the potential functions

(4.2)
$$\varphi_i(x_1, x_2) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} C^{(i)}_{2n-1} Ce_{2n+1}(x_1, \lambda_i)\, {}^iA^{2n+1}_{2r+1} \cos(2r+1)x_2;$$
$$f_i(x_1, x_2) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} S^{(i)}_{2n-1} Se_{2n-1}(x_1, \mu_i)\, {}^iB^{2n+1}_{2r+1} \sin(2r+1)x_2.$$

By substituting (4.2) in (3.13) and the obtained result in (3.15) is seen that in order to satisfy (4.1) we must have

(4.3)
$$\begin{vmatrix} {}^1U^1_1(\xi) & {}^2U^1_1(\xi) & {}^3U^1_1(\xi) & {}^1X^1_1(\xi) & {}^2X^1_1(\xi) & {}^3X^1_1(\xi) & {}^1U^3_1(\xi) & \dots \\ {}^1V^1_1(\xi) & {}^2V^1_1(\xi) & {}^3V^1_1(\xi) & {}^1Y^1_1(\xi) & {}^2Y^1_1(\xi) & {}^3Y^1_1(\xi) & {}^1V^3_1(\xi) & \dots \\ {}^1W^1_1(\xi) & {}^2W^1_1(\xi) & {}^3W^1_1(\xi) & {}^1Z^1_1(\xi) & {}^2Z^1_1(\xi) & {}^3Z^1_1(\xi) & {}^1W^3_1(\xi) & \dots \\ {}^4U^1_1(\xi) & {}^5U^1_1(\xi) & {}^6U^1_1(\xi) & {}^4X^1_1(\xi) & {}^5X^1_1(\xi) & {}^6X^1_1(\xi) & {}^4U^3_1(\xi) & \dots \\ {}^4V^1_1(\xi) & {}^5V^1_1(\xi) & {}^6V^1_1(\xi) & {}^4Y^1_1(\xi) & {}^5Y^1_1(\xi) & {}^6Y^1_1(\xi) & {}^4V^3_1(\xi) & \dots \\ {}^4W^1_1(\xi) & {}^5W^1_1(\xi) & {}^6W^1_1(\xi) & {}^4Z^1_1(\xi) & {}^5Z^1_1(\xi) & {}^6Z^1_1(\xi) & {}^4W^3_1(\xi) & \dots \\ {}^1U^1_3(\xi) & {}^2U^1_3(\xi) & {}^3U^1_3(\xi) & {}^1X^1_3(\xi) & {}^2X^1_3(\xi) & {}^3X^1_3(\xi) & {}^1U^3_3(\xi) & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{vmatrix} = 0$$

where

$${}^iU^{2n+1}_{2r+1}(x_1) = \Big\{\frac{l^2}{8}\lambda(kp_i - \xi^2_i)\,[(2\,\mathrm{ch}^2\,2x_1+1)\,{}^iA^{2n+1}_{2r+1} - 4\,{}^iK^{2n+1}_{2r+1}\,\mathrm{ch}\,2x_1 + {}^iL^{2n+1}_{2r+1}]$$
$$+\mu(a^{(i)}_{2n+1} - 2\lambda_i\,\mathrm{ch}\,2x_1)({}^iA^{2n+1}_{2r+1}\,\mathrm{ch}\,2x_1 - {}^iK^{2n+1}_{2r+1}) - \mu\,{}^iM^{2n+1}_{2r+1}\Big\}$$
$$\times Ce_{2n+1}(x_1, \lambda_i) - \mu\,{}^iA^{2n+1}_{2r+1}\,\mathrm{sh}\,2x_1\, Ce'_{2n+1}(x_1, \lambda_i),$$

$${}^iX^{2n-1}_{2r-1}(x_1) = \mu\tau_i\Big\{[(2r+1)\,\mathrm{ch}\,2x_1\,{}^iB^{2n+1}_{2r+1} - {}^i\bar{N}^{2n+1}_{2r+1}]Se'_{2n+1}(x_1, \mu_i)$$
$$-(2r-1)\,{}^iB^{2n-1}_{2r-1}\,\mathrm{sh}\,2x_1\, Se_{2n-1}(x_1, \mu_i),$$

(4.4)
$${}^iV^{2n-1}_{2r-1}(x_1) = \mu[{}^iN^{2n-1}_{2r-1} - (2r-1)\,{}^iA^{2n-1}_{2r-1}\,\mathrm{ch}\,2x_1]\,Ce'_{2n-1}(x_1, \lambda_i)$$
$$-\mu(2r-1)\,{}^iA^{2n-1}_{2r-1}\,\mathrm{sh}\,2x_1\, Ce_{2n-1}(x_1, \lambda_i),$$

$${}^iY^{2n-1}_{2r-1}(x_1) = \Big\{\frac{1}{2}\mu\tau_i[({}^i\bar{K}^{2n-1}_{2r-1} - {}^iB^{2n-1}_{2r-1}\,\mathrm{ch}\,2x_1)(b^{(i)}_{2n-1} - 2\mu_i\,\mathrm{ch}\,2x_1) + {}^i\bar{R}^{2n+1}_{2r+1}$$

$$-(2r+1)^2\,{}^iB_{2r+1}^{2n+1}\operatorname{ch}2x_1]+\alpha(\tau_i\eta_i^2-2q_i)\frac{l^2}{8}[(2\operatorname{ch}^2 2x_1+1){}^iB_{2r+1}^{2n+1}$$

$$-4\,{}^iK_{2r+1}^{2n+1}\operatorname{ch}2x_1$$

$$+{}^i\bar{L}_{2r+1}^{2n+1}]\Big\}Se_{2n+1}(x_1,\mu_i)+\mu\tau_i\,{}^iB_{2r+1}^{2n+1}\operatorname{sh}2x_1Se'_{2n+1}(x_1,\mu_i),$$

$${}^iW_{2r+1}^{2n+1}(x_1)=[(\mu+\alpha)p_i-(\mu-\alpha)k-2\alpha\sigma_i]\,{}^iA_{2r+1}^{2n+1}Ce'_{2n+1}(x_1,\lambda_i),$$

$${}^iZ_{2r+1}^{2n+1}(x_1)=[-(\mu-\alpha)k\tau_i+2\alpha](2r+1)\,{}^iB_{2r+1}^{2n+1}Se_{2n+1}(x_1,\mu_i),$$

$${}^{3+i}U_{2r+1}^{2n+1}(x_1)=-\gamma\sigma_i\{[(2r+1)\operatorname{ch}2x_1\,{}^iA_{2r+1}^{2n+1}-{}^iN_{2r+1}^{2n+1}]\,Ce'_{2n+1}(x_1,\lambda_i)$$

$$-(2r+1)\,{}^iA_{2r+1}^{2n+1}\operatorname{sh}2x_1Ce_{2n+1}(x_1,\lambda_i)\},$$

$${}^{3+i}X_{2r+1}^{2n+1}(x_1)=\Big\{-\frac{l^2}{8}\beta(kq_i+\eta_i^2)[(2\operatorname{ch}^2 2x_1+1)\,{}^iB_{2r+1}^{2n+1}-4\,{}^iK_{2r+1}^{2n+1}\operatorname{ch}2x_1+{}^i\bar{L}_{2r+1}^{2n+1}]$$

$$+\gamma(b_{2n+1}^{(i)}-2\mu_i\operatorname{ch}2x_1)({}^iB_{2r+1}^{2n+1}\operatorname{ch}2x_1-{}^i\bar{K}_{2r+1}^{2n+1})-\gamma\,{}^i\bar{M}_{2r+1}^{2n+1}\Big\}$$

$$\times Se_{2n+1}(x_1,\mu_i)-\gamma\,{}^iB_{2r+1}^{2n+1}\operatorname{sh}2x_1Se'_{2n+1}(x_1,\mu_i),$$

$${}^{3+i}V_{2r+1}^{2n+1}(x_1)=\Big\{\frac{1}{2}\gamma\sigma_i\,[({}^iK_{2r+1}^{2n+1}-{}^iA_{2r+1}^{2n+1}\operatorname{ch}2x_1)(a_{2n+1}^{(i)}-2\lambda_i\operatorname{ch}2x_1)+{}^iR_{2r+1}^{2n+1}$$

$$-(2r+1)^2\,{}^iA_{2r+1}^{2n+1}\operatorname{ch}2x_1]+\alpha\sigma_i\xi_i^2\frac{l^2}{8}[(2\operatorname{ch}^2 2x_1+1)\,{}^iA_{2r+1}^{2n+1}$$

$$-4\,{}^iK_{2r+1}^{2n+1}\operatorname{ch}2x_1+{}^iL_{2r+1}^{2n+1}]\Big\}Ce_{2n+1}(x_1,\lambda_i)$$

$$+\gamma\sigma_i\,{}^iA_{2r+1}^{2n+1}\operatorname{sh}2x_1Ce'_{2n+1}(x_1,\lambda_i),$$

$${}^{3+i}Y_{2r+1}^{2n+1}(x_1)=-\gamma\,[{}^i\bar{N}_{2r+1}^{2n+1}-(2r+1)\,{}^iB_{2r+1}^{2n+1}\operatorname{ch}2x_1]\,Se'_{2n+1}(x_1,\mu_i)$$

$$-\gamma(2r+1)\,{}^iB_{2r+1}^{2n+1}\operatorname{sh}2x_1Se_{2n+1}(x_1,\mu_i),$$

$${}^{3+i}W_{2r+1}^{2n+1}(x_1)=-(\gamma-\varepsilon)k\sigma_i\,(2r+1)\,{}^iA_{2r+1}^{2n+1}\,Ce_{2n+1}(x_1,\lambda_i)$$

$${}^{3+i}Z_{2r+1}^{2n+1}(x_1)=[(\gamma+\varepsilon)q_i+(\gamma-\varepsilon)k]\,{}^iB_{2r+1}^{2n+1}Se'_{2n+1}(x_1,\mu_i),\quad(i=1,2,3)$$

where the prime denotes the derivative of x_1.

We have used the notations

(4.5)

$$2\,{}^iM_1^{2n+1}=3\,{}^iA_3^{2n+1}+{}^iA_1^{2n+1};\qquad 2\,{}^iM_{2r+1}^{2n+1}=(2r+3)\,{}^iA_{2r+3}^{2n+1}-(2r-1)\,{}^iA_{2r-1}^{2n+1}$$

$$2\,{}^i\bar{M}_1^{2n+1}=3\,{}^iB_3^{2n+1}-{}^iB_1^{2n+1};\qquad 2\,{}^i\bar{M}_{2r+1}^{2n+1}=(2r+3)\,{}^iB_{2r+3}^{2n+1}-(2r-1)\,{}^iB_{2r-1}^{2n+1},$$

$${}^iN_1^{2n+1}=2\,{}^iA_3^{2n+1}-{}^iA_1^{2n+1};\qquad {}^iN_{2r+1}^{2n+1}=(r+2)\,{}^iA_{2r+3}^{2n+1}+(r-1)\,{}^iA_{2r-1}^{2n+1}$$

$${}^i\bar{N}_1^{2n+1}=2\,{}^iB_3^{2n+1}+{}^iB_1^{2n+1};\qquad {}^i\bar{N}_{2r+1}^{2n+1}=(r+2)\,{}^iB_{2r+3}^{2n+1}+(r-1)\,{}^iB_{2r-1}^{2n+1},$$

$$2\,{}^iR_1^{2n+1}=15\,{}^iA_3^{2n+1}+3\,{}^iA_1^{2n+1};\qquad 2\,{}^iR_{2r+1}^{2n+1}=(2r+3)(2r+5)\,{}^iA_{2r+3}^{2n+1}+(2r-1)(2r-3)\,{}^iA_{2r-1}^{2n+1},$$

$$2\,{}^i\bar{R}_1^{2n+1}=15\,{}^iB_3^{2n+1}-3\,{}^iB_1^{2n+1};\qquad 2\,{}^i\bar{R}_{2r+1}^{2n+1}=(2r+3)(2r+5)\,{}^iB_{2r+3}^{2n+1}+(2r-1)(2r-3)\,{}^iB_{2r-1}^{2n+1},$$

$$2\,{}^iK_1^{2n+1}={}^iA_3^{2n+1}+{}^iA_1^{2n+1};\qquad 2\,{}^iK_{2r+1}^{2n+1}={}^iA_{2r+3}^{2n+1}+{}^iA_{2r-1}^{2n+1},$$

$$2\,{}^{i}\overline{K}_{1}^{2n-1}={}^{i}B_{3}^{2n-1}-{}^{i}B_{1}^{2n+1}; \qquad 2\,{}^{i}\overline{K}_{2r+1}^{2n+1}={}^{i}B_{2r+3}^{2n+1}+{}^{i}B_{2r-1}^{2n+1}, \quad r\geqq 1$$

$$2\,{}^{i}L_{1}^{2n-1}={}^{i}A_{5}^{2n-1}-{}^{i}A_{3}^{2n-1}; \qquad 2\,{}^{i}L_{3}^{2n+1}={}^{i}A_{7}^{2n+1}+{}^{i}A_{1}^{2n+1};$$

$$2\,{}^{i}L_{2r-1}^{2n-1}={}^{i}A_{2r+5}^{2n+1}-{}^{i}A_{2r-3}^{2n-1}; \qquad 2\,{}^{i}\overline{L}_{1}^{2n+1}={}^{i}B_{5}^{2n+1}-{}^{i}B_{3}^{2n+1};$$

$$2\,{}^{i}\overline{L}_{3}^{2n-1}={}^{i}B_{7}^{2n-1}-{}^{i}B_{1}^{2n+1}; \qquad 2\,{}^{i}\overline{L}_{2r+1}^{2n+1}={}^{i}B_{2r+5}^{2n+1}+{}^{i}B_{2r-3}^{2n+1}, \quad r\geqq 2.$$

The equation (4.3) is the frequency equation, from which the possible velocities of wave propagation can be determined. An approximate solution we will have when we put equal to zero the subdeterminant which is obtained from the determinant in (4.3), taking $6m$ $(m=1, 2, 3, \ldots)$ rows and columns, counted from the left upper corner. The mechanical interpretation of this approximation is given in [6].

The case of wave propagation in cylinder with a circular cross-section, investigated in [4], can be obtained from the present formulae as l tends to zero and x_1 tends to infinity in such a way, that $l\,\mathrm{ch}\,x_1$ should tend to r, this means that the ellipse with the semimajor axis r tends to a circle with the same radius. Then the determinant in (4.3) where ${}^{i}U_{2r+1}^{2n+1}$, ${}^{i}X_{2r+1}^{2n+1}$, ${}^{i}V_{2r+1}^{2n+1}$ and ${}^{i}Y_{2r+1}^{2n+1}$ $(i=1, 2, \ldots, 6)$ are previously multiplied with l^2, reduces to the product of infinite number of finite determinants with six rows and columns, every one of which can be put equal to zero, in order that the boundary condition (4.1) be satisfied.

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