

INFLUENCE OF POINT MASS RELATIVE MOTION ON DELAMINATION IN MULTILAYERED BEAMS

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ABSTRACT: This paper deals with analysis of delamination in moving multilayered beams in the case when a point mass performs non-uniform relative motion along a straight line on the beam upper surface. The beam itself undergoes rotation in horizontal plane. The study is focussed mainly on the influence of the relative motion of the point mass on the strain energy release rate (SERR) in the beam. One of the peculiarities of the problem under consideration is that some of the inertia forces are directed perpendicular to the plane of the beam structure which loads the beam in double bending and torsion. The analysis yields results clarifying the effect of the parameters of the motion law and layers inhomogeneity on the SERR.

KEY WORDS: dynamics, delamination, acceleration, multilayered beam, strain energy release rate, material gradient.

1 INTRODUCTION

Application of multilayered structures in different branches of engineering increases rapidly [1–3]. During the operation of various multilayered structural systems, the stresses induced by the external static or dynamic loading in many cases are accompanied by irreversible destructive processes in the multilayered material such as delamination fracture [4–10]. Appearance of delamination cracks between layers of the multilayered structures may lead to local failure. Depending on the type and intensity of the external loading and material parameters, delamination may also cause sudden collapse of the entire engineering structure. The delamination problem is especially important for the operational reliability and safety of the structural members and components that are subjected to various dynamic influences in the period of their exploitation. This fact indicates that one of the ways for enhancing the performance of the multilayered engineering structures is to improve their delamination behaviour which, among other things, requires developing of analyses of the effects of a variety of factors on delamination.

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This paper deals with dynamic delamination in moving multilayered beam structures. The main goal of the paper is to evaluate the effect of a point mass that moves non-uniformly along a straight line on the top of the beam structure. Since the beam itself performs rotation in horizontal plane, the point mass practically is involved in a relative motion. Analyzing the effect of the relative motion of a point mass on the dynamic delamination behaviour is an important task since moving point-like objects that can be treated as point masses are used as parts of different mechanisms in various machines, devices, manipulators, apparatuses and robotised manufacturing systems in the modern engineering. The multilayered beam whose delamination is analyzed in this paper is made by different layers that are continuously inhomogeneous in longitudinal direction. It should be mentioned that using of continuously inhomogeneous (functionally graded) layers is one of the ways for development and modernization of multilayered structural systems [11–13]. As known, the functionally graded engineering materials are very efficient and promising continuously inhomogeneous composites that undergo a rapid development nowadays [14, 15]. They have undisputed advantages in comparison with the traditional fibre reinforced composite materials [16, 17]. In the first place, the microstructure of functionally graded materials can be tailored during production process to achieve desired smooth distribution of material properties along certain directions (coordinates) in a functionally graded structural component [18]. Namely this makes the functionally graded materials a very attractive replacement to the fibre reinforced composites in such important areas of up-to-date engineering as aeronautics and aerospace, nuclear reactors, robotics, optics, microelectronics, car industry, etc. [19]. That is why we are focussed here on analyzing the dynamic delamination in multilayered beams with continuously inhomogeneous layers. Another important moment in the present paper is that the inertia loading is more complex since some of the inertia forces are directed perpendicular to the plane of the beam structure. This loads the beam in dynamic double bending and torsion (it should be noted here that usually the delamination analyses are concerned with beams subjected to special bending in the beam plane [4] or to pure torsion under static loading [20]). In this paper, solution of the dynamic problem for the SERR is derived by analyzing the strain energy stored in beam due to the normal stresses and shear stresses as a result of double bending and torsion, respectively. The solution is checked-up by applying an approach published previously in the research literature. The solution derived is used for evaluating the influence of the parameters of the law for relative motion of the mass point on the SERR. The effects of inhomogeneity of layers and the law for rotation of the multilayered beam are evaluated too. One can sum up that the novelty of this paper is not only in analyzing the influence of the relative motion of a point mass on dynamic delamination but also in the fact that the multilayered beam is under a complex inertia loading that generates both double bending and torsion.

2 DELAMINATION ANALYSIS

The horizontal multilayered beam, D_1D_3 , depicted in Fig. 1 has n different layers. The geometrical parameters of the beam are as follows: length, l , thickness, h , and width, b . The beam hosts a delamination crack of length, a , in portion, D_1D_2 . The upper and the lower arms of the delamination crack have thicknesses, h_{up} and h_{lw} ,

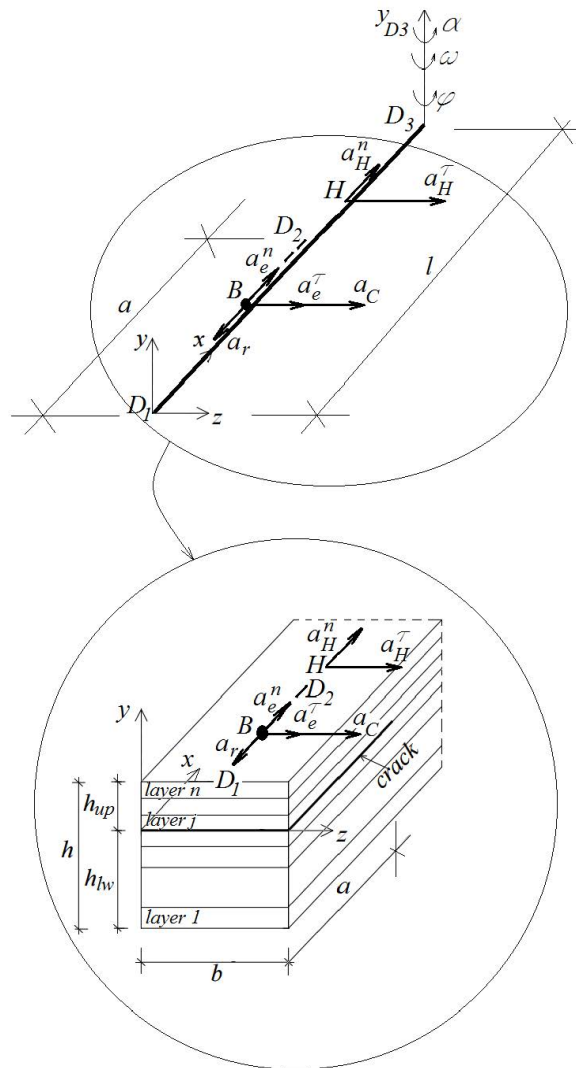


Fig. 1. Model of a multilayered beam with delamination.

respectively. The beam undergoes rotational motion around vertical axis, y_{D_3} , in the law written in Eq. (1).

$$(1) \quad \phi = \beta t^\mu ,$$

where ϕ is the angle of rotation, t is time, β and μ are parameters. ϕ is depicted in Fig. 1. It should be specified that the axis of rotation, y_{D_3} , is motionless.

A point mass, B , performs relative motion along a straight line on the beam upper surface from point, D_2 , to point, D_1 , in the law written in Eq. (2).

$$(2) \quad BD_2 = \delta \left(e^{\lambda t} - 1 \right) ,$$

where δ and λ are parameters. The trajectory of relative motion of B is depicted by a dashed line in Fig. 1.

The first step in the current analysis is to derive the accelerations of the beam and the point mass. These accelerations generate inertia loading that acts upon the beam structure. The inertia loading is used when determining the SERR in the beam.

The angular velocity and acceleration of the beam, ω and α , are defined in Eqs. (3) and (4), respectively.

$$(3) \quad \omega = \frac{d\phi}{dt} ,$$

$$(4) \quad \alpha = \frac{d^2\phi}{dt^2} .$$

We obtain from Eqs. (1), (3) and (4) that

$$(5) \quad \omega = \beta \mu t^{\mu-1} ,$$

$$(6) \quad \alpha = \beta \mu (\mu - 1) t^{\mu-2} .$$

The quantities, ω and α , are depicted in Fig. 1.

The normal and tangential accelerations, a_H^n and a_H^τ , of a random point, H , in the j -th layer of the beam are defined as

$$(7) \quad a_H^n = \omega^2 (l - x) ,$$

$$(8) \quad a_H^\tau = \alpha (l - x) ,$$

where

$$(9) \quad 0 \leq x \leq l .$$

The axis, x , is defined in Fig. 1. Form Eqs. (5), (6), (7) and (8), we obtain

$$(10) \quad a_H^n = (\beta \mu t^{\mu-1})^2 (l - x) ,$$

$$(11) \quad a_H^\tau = \beta \mu (\mu - 1) t^{\mu-2} (l - x) .$$

The accelerations, a_H^n and a_H^τ , are depicted in Fig. 1. These accelerations generate distributed inertia loads that act upon the beam. The densities of the distributed inertia loads, q_{Hj}^n and q_{Hj}^τ , are found by the next two equations, i.e.

$$(12) \quad q_{Hj}^n = -m_j a_H^n ,$$

$$(13) \quad q_{Hj}^\tau = -m_j a_H^\tau ,$$

where m_j is the mass per unit length of the beam in the j -th layer. The densities, q_{Hj}^n and q_{Hj}^τ , are directed opposite to accelerations, a_H^n and a_H^τ . It should be specified here that since the layers are continuously inhomogeneous along the beam length, the mass per unit length varies continuously in longitudinal direction. This variation is defined by the law in Eq. (14)

$$(14) \quad m_j = m_{D1j} e^{\rho_j \frac{x}{l}} ,$$

where

$$(15) \quad 0 \leq x \leq l .$$

Here, m_{D1j} is the mass per unit length in end, D_1 , of the beam, ρ_j is a parameter whose value depends on the distribution of the mass per unit length. When the mass per unit length increases in direction from end, D_1 , towards end, D_3 , of the beam, then $\rho_j > 0$. If the mass per unit length decreases in direction from end, D_1 , towards end, D_3 , of the beam, then $\rho_j < 0$. If the beam is homogeneous, then $\rho_j = 0$.

The acceleration, a_B , of the point mass, B , is found by using Eq. (16)

$$(16) \quad \vec{a}_B = \vec{a}_r + \vec{a}_e + \vec{a}_C ,$$

where a_r is the relative acceleration, a_e is the acceleration due to rotation of the beam around y_{D3} -axis, a_C is the Coriolis acceleration. The accelerations involved in Eq. (16) are defined by Eqs. (17), (18) and (19)

$$(17) \quad a_r = \frac{d^2(BD_2)}{dt^2} ,$$

$$(18) \quad \vec{a}_e = \vec{a}_e^n + \vec{a}_e^\tau ,$$

$$(19) \quad a_C = 2\omega \cdot v_r ,$$

where

$$(20) \quad a_e^n = \omega^2 (l - a + BD_2) ,$$

$$(21) \quad a_e^\tau = \alpha (l - a + BD_2) ,$$

$$(22) \quad v_r = \frac{d(BD_2)}{dt} .$$

Here, BD_2 , ω and α are determined by Eqs. (2), (5) and (6), respectively. The accelerations, a_r , a_e^n , a_e^τ and a_C , of the point mass, B , are depicted in Fig. 1. The concentrated forces of inertia, Φ_r , Φ_e^n , Φ_e^τ and Φ_C , which act upon the point, B , are determined by the next four equations, i.e.

$$(23) \quad \Phi_r = -m_B a_r ,$$

$$(24) \quad \Phi_e^n = -m_B a_e^n ,$$

$$(25) \quad \Phi_e^\tau = -m_B a_e^\tau ,$$

$$(26) \quad \Phi_C = -m_B a_C ,$$

where m_B is the mass of point, B . The forces of inertia, Φ_r , Φ_e^n , Φ_e^τ and Φ_C , are directed opposite to accelerations, a_r , a_e^n , a_e^τ and a_C , respectively.

The SERR, G , for the beam is determined by Eq. (27).

$$(27) \quad G = \frac{dU}{bda} ,$$

where U is the strain energy. U is derived by Eq. (28)

$$(28) \quad U = U_{D_1B} + U_{BD_2} + U_{D_1D_2} + U_{D_2D_3} ,$$

where U_{D_1B} , U_{BD_2} , $U_{D_1D_2}$ and $U_{D_2D_3}$ are the strain energies in portions, D_1B and BD_2 , of the upper arm of the delamination crack, the lower arm of the delamination crack and the portion, D_2D_3 , of the beam, respectively.

The strain energy, U_{D_1B} , is found by Eq. (29)

$$(29) \quad U_{D_1B} = \sum_{j=1}^{i=n_1} \iiint_{V_j} (u_{0\sigma j} + u_{0\tau j}) dV ,$$

where n_1 is the layers number in the upper arm, V_j is the volume of the j -layer, $u_{0\sigma j}$ is the strain energy density due the normal stresses, $u_{0\tau j}$ is the strain energy density due to the shear stresses induced by torsion of the beam.

The strain energy densities, $u_{0\sigma j}$ and $u_{0\tau j}$, are obtained by Eqs. (30) and (31), respectively

$$(30) \quad u_{0\sigma j} = \frac{\sigma_j^2}{2E_j},$$

$$(31) \quad u_{0\tau j} = \frac{\tau_j^2}{2G_j},$$

where σ_j is the normal stress, E_j is the modulus of elasticity, τ_j is the shear stress due to torsion, G_j is the shear modulus.

Due to material inhomogeneity, E_j and G_j vary continuously along the beam length. This variation is presented by Eqs. (32) and (33), respectively

$$(32) \quad E_j = E_{D_1j} e^{\gamma_j \frac{x}{l}},$$

$$(33) \quad G_j = G_{D_1j} e^{\eta_j \frac{x}{l}},$$

where E_{D_1j} and G_{D_1j} are the values of E_j and G_j in end, D_1 , of the beam, γ_j and η_j are parameters. The values of γ_j and η_j depend on the distribution of E_j and G_j . When E_j and G_j increase in direction from end, D_1 , towards end, D_3 , of the beam, then $\gamma_j > 0$ and $\eta_j > 0$. If E_j and G_j decrease in direction from end, D_1 , towards end, D_3 , of the beam, then $\gamma_j < 0$ and $\eta_j < 0$. If the beam is homogeneous, then $\gamma_j = 0$ and $\eta_j = 0$.

It should be pointed-out here that the developed methodology can be applied also for other types of material gradients, such as quadratic or other distributions.

The shear stress, τ_j , involved in Eq. (31) is found by using the approach developed in [21]. This approach is presented here through Eqs. (34) – (42)

$$(34) \quad \tau_j = \frac{T}{S} \left\{ \left[\sum_{k=1,3,\dots}^{\infty} (P_{k,j} \cosh \alpha_k y + Q_{k,j} \sinh \alpha_k y) \alpha_k \sinh \alpha_k z \right]^2 + \left[\sum_{k=1,3,\dots}^{\infty} \left(P_{k,j} \sinh \alpha_k y + Q_{k,j} \cosh \alpha_k y + \frac{8G_j b^2}{k^3 \pi^3} \right) \alpha_k \cos \alpha_k z \right]^2 \right\}^{\frac{1}{2}},$$

where y and z are depicted in Fig. 1, T is the torsion moment in a random cross-section of portion, D_1B , of the upper arm of the delamination crack, S is the stiffness in torsion. The parameter, α_k , depends on the beam width and is obtained by Eq. (35) [21]

$$(35) \quad \alpha_k = k\pi/b,$$

where $k = 1, 3, \dots$. The recurrent expressions for determining the coefficients, $P_{k,j}$ and $Q_{k,j}$, in the infinite series are written in Eqs. (36) and (37), respectively [21]

$$(36) \quad P_{k,j} = \frac{2}{(g_{j,j+1} - 1) \sinh 2\alpha_k h_j} \left[Q_{k,j+1} g_{j,j+1} + Q_{K,j} (\sinh^2 \alpha_k h_j - g_{j,j+1} \cosh^2 \alpha_k h_j) + g_{j,j+1} (r_{k,j+1} - r_{k,j}) \cosh \alpha_k h_j \right],$$

$$(37) \quad P_{k,j+1} = \frac{2}{(g_{j,j+1} - 1) \sinh 2\alpha_k h_j} \left[Q_{K,j+1} (\cosh^2 \alpha_k h_j - g_{j,j+1} \sinh^2 \alpha_k h_j) - Q_{k,j} + (r_{k,j+1} - r_{k,j}) \cosh \alpha_k h_j \right],$$

$$(38) \quad g_{j,j+1} = \frac{G_j}{G_{(j+1)}},$$

$$(39) \quad r_{k,j} = \frac{8G_j b^2}{k^3 \pi^3},$$

where h_j is the distance between the upper surface of the j -layer and the lower surface of the delamination crack upper arm. Besides [21],

$$(40) \quad P_{k,1} = - \frac{Q_{k,1} \cosh \alpha_k h_0 + r_{k,1}}{\sinh \alpha_k h_0},$$

$$(41) \quad P_{k,n} = - \frac{Q_{k,n} \cosh \alpha_k h_n + r_{k,n}}{\sinh \alpha_k h_n}.$$

The stiffness in torsion, S , involved in Eq. (34) is determined by Eq. (42) [21]

$$(42) \quad S = \frac{8}{\pi^2} \sum_{k=1,3,\dots}^{\infty} \frac{1}{k^2} \left[\frac{\alpha_k}{2} \sum_{j=1}^{n_1} r_{k,j} (h_j - h_{j-1}) + \sum_{j=1}^{n_1} P_{k,j} \sinh \frac{\alpha_k (h_j + h_{j-1})}{2} \sinh \frac{\alpha_k (h_j - h_{j-1})}{2} + \sum_{j=1}^{n_1} Q_{k,j} \sinh \frac{\alpha_k (h_j - h_{j-1})}{2} \cosh \frac{\alpha_k (h_j + h_{j-1})}{2} \right].$$

The torsion moment, T , in a random cross-section of portion, $D_1 B$, of the upper arm of the delamination crack located at distance, ψ , from end, D_1 , of the beam is derived by Eq. (43)

$$(43) \quad T = \sum_{j=1}^{j=n_1} \int_0^{\psi} \left[q_{Hj}^T \left(\frac{h_{up}}{2} - \frac{h_j + h_{j-1}}{2} \right) \right] dx,$$

where h_{j-1} is the distance between the lower surface of the j -layer and the lower surface of the delamination crack upper arm. It should be noted that ψ can change between 0 and $a - BD_2$ where BD_2 is defined by Eq. (2). In this way, one can obtain the torsion moment in any cross-section of portion, D_1B , of the upper arm of the delamination crack. The torsion moment obtained by Eq. (43) is used in Eq. (34). Concerning the convergence of infinite series in the above expressions, it should be noted that the analysis performed reveals that at $k \geq 9$ the results practically do not change, i.e. the infinite series are convergent.

The normal stress, σ_j , involved in Eq. (30) is found by the Hook's law, i.e.

$$(44) \quad \sigma_j = E_j \varepsilon,$$

where ε is the strain. The distribution of ε in a random cross-section is presented by Eq. (45)

$$(45) \quad \varepsilon = \varepsilon_{C_1} + \kappa_y \left(\frac{h_{\text{up}}}{2} - y \right) + \kappa_z \left(z - \frac{b}{2} \right),$$

where ε_{C_1} is the strain in the centre, κ_y and κ_z are the curvatures.

Equations (46), (47) and (48) are used for determining of ε_{C_1} , κ_y and κ_z

$$(46) \quad N = \sum_{j=1}^{j=n_1} \iint_{(A_j)} \sigma_j dA,$$

$$(47) \quad M_{yC} = \sum_{j=1}^{j=n_1} \iint_{(A_j)} \sigma_j \left(z - \frac{b}{2} \right) dA,$$

$$(48) \quad M_{zC} = \sum_{j=1}^{j=n_1} \iint_{(A_j)} \sigma_j \left(\frac{h_{\text{up}}}{2} - y \right) dA,$$

where N , M_{yC} and M_{zC} are the axial force and the bending moments in the cross-section.

The quantities, N , M_{yC} and M_{zC} , in a random cross-section of portion, D_1B , of the upper arm of the delamination crack located at distance, ψ , from end, D_1 , of the beam are determined by Eqs. (49), (50) and (51)

$$(49) \quad N = \sum_{j=1}^{j=n_1} \int_0^{\psi} q_{Hj}^n dx,$$

$$(50) \quad M_{yC} = \sum_{j=1}^{j=n_1} \int_0^{\psi} q_{Hj}^T (\psi - x) dx ,$$

$$(51) \quad M_{zC} = \sum_{j=1}^{j=n_1} \int_0^{\psi} \left[q_{Hj}^n \left(\frac{h_{up}}{2} - \frac{h_j + h_{j-1}}{2} \right) - m_j g (\psi - x) \right] dx ,$$

where g is the gravity acceleration. The quantities, ε_{C1} , κ_y and κ_z , determined from Eqs. (46), (47) and (48) are used to derive the strain energy density, $u_{0\sigma j}$. Then, $u_{0\sigma j}$ and $u_{0\tau j}$ are substituted in Eq. (29) for obtaining U_{D_1B} (the integration is performed by the MatLab). The approach presented by Eqs. (29) – (51) is applied also for determining of U_{BD_2} , $U_{D_1D_2}$ and $U_{D_2D_3}$. For this purpose, the axial force, the bending and the torsion moments in the corresponding portion of the beam are obtained and used to determine the strain energy densities which are needed to derive U_{BD_2} , $U_{D_1D_2}$ and $U_{D_2D_3}$. At the end, G is found by using Eq. (27).

A check-up is performed by applying an approach for deriving G taken from [22]. According to [22], G can be written as given below.

$$(52) \quad G = \sum_{j=1}^{j=n_1} \int_{h_{j-1}}^{h_j} (u_{0\sigma j} + u_{0\tau j}) dy + \sum_{j=1}^{j=n-n_1} \int_{h_{j-1}}^{h_j} (u_{0lw\sigma j} + u_{0lw\tau j}) dy - \sum_{j=1}^{j=n} \int_{h_{j-1}}^{h_j} (u_{0ct\sigma j} + u_{0ct\tau j}) dy ,$$

where $u_{0\sigma j}$ and $u_{0\tau j}$ are the strain energy densities in the j -th layer of the upper arm of the delamination crack behind the delamination tip, $u_{0lw\sigma j}$ and $u_{0lw\tau j}$ are the strain energy densities in the j -th layer of the lower arm of the delamination crack behind the delamination tip, $u_{0ct\sigma j}$ and $u_{0ct\tau j}$ are the strain energy densities in the j -th layer of the beam ahead of the delamination tip. The SERR found by Eq. (52) matches the SERR determined by Eq. (27) which indicates that the current analysis is correct.

3 PARAMETRIC ANALYSIS

As already stated, the main goal of this paper is to study the influence of the relative motion of the point mass, B , on the delamination in the multilayered beam structure in Fig. 1. This is done here by assessing the effects of parameters, δ , λ and m_B , on the SERR. The data used in the study are $h = 0.060$ m, $l = 0.700$ m, $a = 0.450$ m,

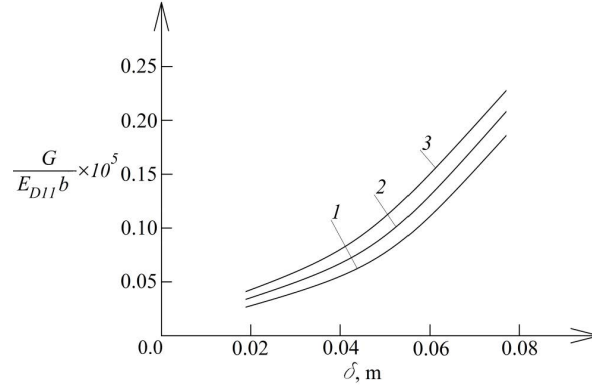


Fig. 2. Change of the SERR induced by varying parameter, δ (graph 1 – at $\lambda = 0.01$ 1/s, graph 2 – at $\lambda = 0.02$ 1/s and graph 3 – at $\lambda = 0.03$ 1/s).

$b = 0.060$ m, $n = 5$ and $n_1 = 2$. The results of the study are depicted in the form of various graphs.

Figure 2 illustrates the change of the SERR induced by varying parameter, δ . This change is studied at three values of parameter, λ . The SERR in Fig. 2 is presented in normalized form by the formula $G / (E_{D11} b)$. The inspection of the graphs depicted in Fig. 2 indicates that both parameters have significant influence on the SERR. The growth of the SERR with increase of the values of δ and λ that is observed in Fig. 2 is caused by increase of the acceleration of the beam structure and the point mass which leads to rise of the densities of distributed inertia load acting upon the beam and the concentrated inertia forces in the point mass.

The influence of the mass of the point, B , and the position of B along the upper surface of the beam on the SERR is analyzed too. The mass and the position of B are characterized by $m_B / (m_{D11} l)$ and $BD_2 / D_1 D_2$ ratios, respectively. The graphs depicted in Fig. 3 illustrate how the SERR varies as a result of increase of $m_B / (m_{D11} l)$ and $BD_2 / D_1 D_2$ ratios. One can observe in Fig. 3 that the SERR grows with rise of $m_B / (m_{D11} l)$ ratio. Analogical behaviour, i.e. growth of the SERR is observed also with rise of $BD_2 / D_1 D_2$ ratio (Fig. 3).

The influence of the beam length on the SERR can be evaluated based on the graphs depicted in Fig. 4. The beam length is presented by the ratio, l/h , along the abscissa in Fig. 4. The influence of the continuous variation of the shear modulus, G_1 , along the beam length on the SERR is also evaluated (this variation is characterized by the parameter, η_1). For this purpose, the graph SERR - l/h is plotted at three values of η_1 in Fig. 4. It can be seen in Fig. 4 that rise of the ratio, l/h , leads to significant growth of the SERR (this growth is induced by the rise of the acceleration

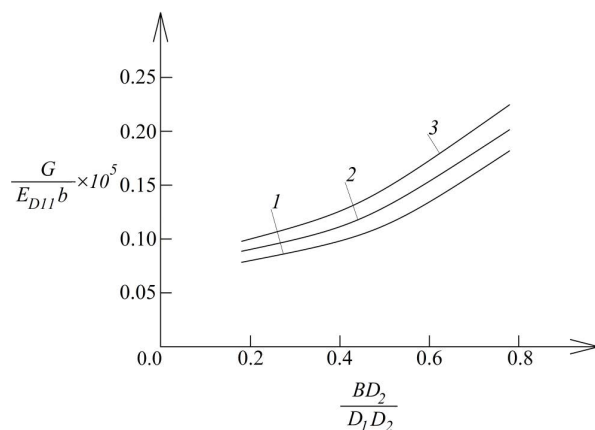


Fig. 3. Change of the SERR induced by varying ratio, BD_2/D_1D_2 (graph 1 – at $m_B/(m_{D11}l) = 0.15$, graph 2 – at $m_B/(m_{D11}l) = 0.30$ and graph 3 – at $m_B/(m_{D11}l) = 0.45$).

and the inertia loading). It can be observed that the SERR reduces with growth of η_1 (Fig. 4).

It is analyzed also how the SERR is influenced by the rotation of the beam around axis, y_{D3} . The law of rotation has two parameters (β and μ). Figure 5 shows how the SERR changes as a result of growth of parameter, β . The change of the SERR as a result of the continuous variation of the mass per unit length of the beam is also analyzed. For this purpose, the SERR is obtained at three values of parameter, ρ_1 ,

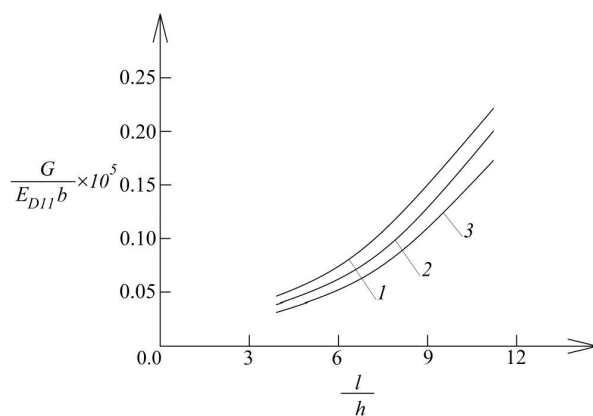


Fig. 4. Change of the SERR induced by varying ratio, l/h (graph 1 – at $\eta_1 = 0.25$, graph 2 – at $\eta_1 = 0.50$ and graph 3 – at $\eta_1 = 0.75$).

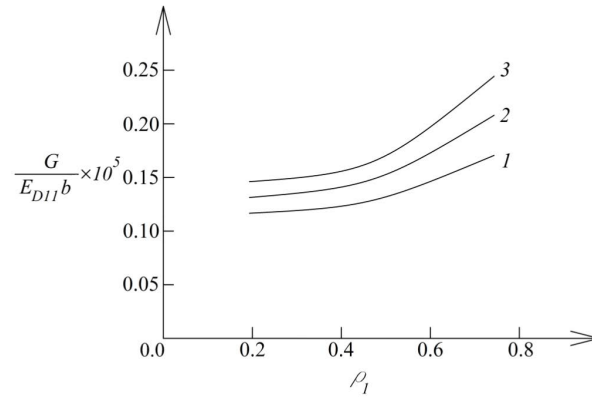


Fig. 5. Change of the SERR induced by varying parameter, ρ_1 (graph 1 – at $\beta = 0.2$, graph 2 – at $\beta = 0.4$ and graph 3 – at $\beta = 0.6$).

and the corresponding graphs are depicted in Fig. 5. It can be seen in Fig. 5 that rise of both β and ρ_1 has significant effect on the SERR that presents growth of the latter.

The role of the other parameter of the law for rotation of the beam around axis, y_{D3} , is also clarified. The change of the SERR caused by rise of parameter, μ , is reported in Fig. 6. The effect of the continuous variation of the modulus of elasticity, E_1 , along the beam length on the SERR is clarified too. The variation of E_1 is characterized by the parameter, γ_1 . The graphs reported in Fig. 6 reveal that increase of μ brings about growth of the SERR. Increase of γ_1 leads to reduction of the SERR as can be seen in Fig. 6.

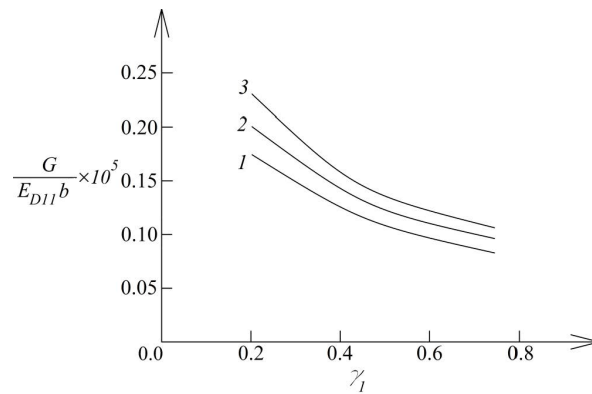


Fig. 6. Change of the SERR induced by varying parameter, γ_1 (graph 1 – at $\mu = 2.3$, graph 2 – at $\mu = 2.6$ and graph 3 – at $\mu = 2.9$).

4 CONCLUSIONS

The analysis performed in this paper yields results which give reason to conclude that delamination in the multilayered beam structure under consideration is influenced in a high degree by the relative motion of the point mass. In particular, it is found that the SERR changes significantly as a result of variation of the two parameters, δ and λ , of the law for relative motion of the point mass. Increase of the values of these parameters causes a quick growth of the SERR. This means that increase of the magnitude of the relative velocity and acceleration of the point mass has a destabilizing effect on the delamination in the beam since the SERR increases. Other two parameters connected to the relative motion are the mass and the position of the moving point. The analysis performed indicates that increase of the mass of the point leads to destabilization of the delamination behaviour since the SERR rises. Concerning the effect of the position of the moving point, it is observed that increase of the distance between the point and the delamination tip results in destabilization of the delamination behaviour of the beam due to growth of the SERR. The influence of the parameters, β and μ , of the law for rotation of the beam on the delamination is evaluated too. It is observed that increase of the magnitude of β and μ results in growth of the SERR. The effect of the continuous change of the mass per unit length of the beam on the delamination is also studied. It is found that growth of the parameter, ρ_1 , that controls the change of the mass per unit length has destabilizing effect on the delamination behaviour since the SERR quickly grows. Increase of the magnitudes of the parameters, γ_1 and η_1 , which control the continuous change of the modulus of elasticity and the shear modulus has stabilizing effect on the beam delamination behaviour since the SERR reduces. The findings reported in this paper can serve as reference for safety design of beams with point-like objects performing relative motion. At the end, two main limitations of the proposed approach are underlined. The first limitation is that the approach holds for beams which perform rotational motion. The second limitation consists in the fact that the beams must have rectangular cross-section. Concerning delamination in beams with cross-section other than rectangular, some interesting results for the case of static loading are obtained in Ref. [9]. Analysis of delamination in beams with non-rectangular cross-sections under dynamic loading can be pointed-out as a future research task.

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