

On a Mathematical Model of the Optimal Reliability Theory and Some Mechanical Applications

Nguyen Van Pho

§1. Introduction

The theory of Reliability for Structures was studied by V. V. Bolotin in [4, 5, ...], which can be summarized as follows.

Suppose that there exist the following relations for a system:

$$(1) \quad Lu = q$$

$$(2) \quad Mu = v$$

where $\vec{u}$ is the state vector, $u \in U$ and $q = q(x, t)$ is the vector of external actions, which is in general a stochastic process. $\vec{q} \in Q$, L , M are differential operators.

Denote by Ω_0 the domain of quality control, $\Omega_0 \in V$.

Let G be the domain of a three dimension space occupied by the system $\vec{x} \in G$.

The probability defined by

$$(3) \quad P(t) = P[\vec{v}(x, \tau) \in \Omega_0, \tau \in [0, t], x \in V]$$

is said to be the reliability of the system.

One follows a procedure of 4 steps for determining $P(t)$:

Step 1: Schematization of the system and the load, i. e. determination of Q , U , L , M .

Step 2: Determining stochastic states of the system, i. e. solving (1).

Step 3: Choosing the quality space V and the domain of quality control Ω_0 .

Step 4: Calculating $P(t)$.

The above-mentioned method cannot be applied to the following cases:

a) Equation (1) has no solution which can be represented as a function and it can only be found approximately.

b) The uniqueness of the solutions of Equation (1) cannot be assured as for the cases of the optimal Design Problem and the Limit analysis of structures.

Moreover, for calculating $P(t)$ there must exist so strong conditions that are difficult to be satisfied.

Note that it is also very difficult to find an algorithm for calculating on a computer for this schema.

In this paper, we shall construct a mathematical model of reliability theory for a general optimal system (a special case of which is the ordinary system) in hope that the above-mentioned defects will be avoided.

The paper consists of the following parts:

- a) Definition for an optimal reliability.
- b) Methods of determining the reliability for an optimal system.
- c) Some applications.

§2. Definition for an Optimal Reliability

Under an optimal system we mean a system where (1) does not have a unique solution. This sense is somewhat more general than the ordinary one, because it includes not only ordinary optimal problems but also problems that are not optimal in the strict sense. We deal with problems the solutions of which can be found only by choosing variants, i. e. using the more optimal algorithm. Problems of the kind raise always in Mechanics, for example, the problem of limit analysis and the problem of shakedown for an elasto-plastic system [3, 4].

1. On the representation of reliability. In [1, 2]

$$P(t) = P[\mathbf{v}(\mathbf{x}, \tau) \in \Omega_0, \quad \forall \tau \in [0, t], \mathbf{x} \in G]$$

has been replaced by

$$P^*(t) = P[\sup_{\tau \in [0, t]} \sup_{\mathbf{x} \in G} v(\mathbf{s}, \tau) \in \Omega_0]$$

Moreover, the author [4, 5] supposes that

$$(4) \quad P(t) = P^*(t).$$

We show by the following counter-example that (4) is not always verified.

In fact, for convenience and without losing the generality, we can represent the quantities $\mathbf{q}$, $\mathbf{u}$ and $\mathbf{v}$ as follows:

$$\mathbf{q} = \mathbf{q}(\mathbf{x}, \theta, t)$$

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, \theta, t)$$

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, \theta, t),$$

where $\theta = \{\theta_i\}$ is a stochastic vector all components of which are random variables.

Consider the process

$$v = -(\tau - \theta)^2 \div 2$$

which is a random variable uniformly distributed on $[0, t]$. We find

$$P[v \leq 1, \quad 0 \leq \tau \leq t]$$

The graphs of v (or the realization of the process v) are parabolas and v attains its maximum at $\tau = \theta$, where $v_{\max} = 2$ (Figure 1).

Extremal point runs on AK when $\tau \in [0, t]$ (t corresponds to B).

If the external point runs on AK then the intersection of the graph of v and the line $\tau = t$ is under the line $v = 1$, and it is over this line when the

external point runs on BK . Since θ is a random variable which is uniformly distributed, we have

$$P(t) = 1 \cdot \frac{AK}{AB}$$

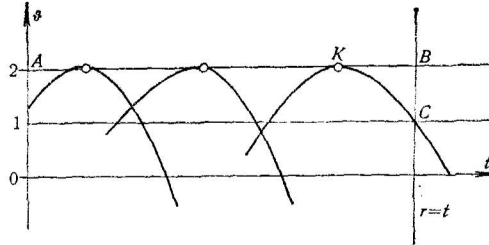


Fig. 1

and the corresponding external point is $-(t-t_k)^2 + 2 = 1 \Rightarrow (t-\theta_k)^2 = 1 \Rightarrow t-\theta_k = 1 \Rightarrow \theta_k = t-1$ and then $P(t) = \frac{t-1}{t} = 1 - \frac{1}{t} (t \geq 1)$ while

$$P[\sup_{\tau \in [0, t]} v(\tau) \leq 1] = 0$$

become all the values of $\sup$ that are greater than $v=1$.

2. Definition: First, let us introduce fundamental relations in the point of view of theory of probability. They are often dealt within mechanics under an equality or an inequality.

In general, for setting up these relations, one must base on theorems, postulates or at least on some experimental constants.

After treating experimental results one finds their mean (expectation) or the regression curve.

For instance, the relation

$$\sigma \leq [\sigma]$$

is replaced by

$$P\{\sigma \leq [\sigma]\} \geq p, \quad p \in [0, 1]$$

and the relation $f(\sigma, x, \dots) = 0$ is replaced by

$$P\{|f(\sigma, x, \dots) - f_0(\sigma, x, \dots)| \leq \delta\} \geq q, \quad q \in [0, 1], \quad \delta = \text{const},$$

where $[\sigma]$ is field stress, f_0 is regressive function.

The probabilities p, q can be considered as control parameters and can be chosen according to observation data and economic-technique indices.

As we can see in the sequel, p and q will be regulated in the calculating course.

For many problems of mechanics, the unknown is a function of θ , i. e. $x = x(\theta)$. For simplicity, we express x in a parameter form [6]

$$x(t, \theta) = x(y, \theta) = \sum_k y_k(t) v_k(\theta, t),$$

where $y_k(t)$ and undetermined functions and v_k are given functions of θ, t .

Without losing generality, from now on the unknown of the problem is chosen as $x = \{x_i\}$ which does not depend on θ .

Denote by X the set defined by the apriori conditions of the problem. Obviously, one must find $\mathbf{x}$ such that $\mathbf{x} \in X$ and the following probability relations are satisfied:

$$(5) \quad P\{\varphi_j(\mathbf{x}, \theta, t) \leq C_j\} \geq p_j, \quad j=1, 2, \dots,$$

where t can be considered as a parameter.

Therefore, the problem of reliability here is to define the following probability

$$(6) \quad P(\mathbf{x}, t) = P \left\{ \begin{array}{l} P[\varphi_j(\mathbf{x}, \theta, \tau) \leq C_j] \geq p_j \\ (j=1, 2, \dots, n) \\ \mathbf{x} \in X, \tau \in [0, t] \end{array} \right\}$$

(6) is the probability of the fact that all the conditions of the problem hold.

In this context, the problem of optimal reliability is stated as follows:

Find $\mathbf{x} = \{x_i\}$ such that $P(\mathbf{x}, t)$ attains its maximum, i. e. one must find

$$(7) \quad P(t) = \max_{\{x\}} P(\mathbf{x}, t)$$

In the case where $\vec{X} = f(t)$ is obtained by solving (1), the relation (7) coincides with the V. V. Bolotin's definition of reliability [4, 5].

For many other problems in mechanics and technics we have to determine the probability

$$(8) \quad \max_{\{x\}} P\{\varphi_0(\mathbf{x}, t) \leq C_0\}$$

subjected to other conditions.

In the case, we need to solve the following problem:

Find

$$(9) \quad \left\{ \begin{array}{l} \max P\{\varphi_0(\mathbf{x}, \tau) \leq C_0\} \\ \text{subject to } \left\{ \begin{array}{l} P\{\varphi_j(\mathbf{x}, \tau) \leq C_j\} \geq p_j \\ j=1, 2, \dots, n \\ \mathbf{x} \in X, \tau \in [0, t]. \end{array} \right. \end{array} \right.$$

It is a standard problem of the stochastic programming [6].

§3. Method for determining the Optimal Reliability

1. General calculation schema.

For determining $P(t)$ according to (7) we will follow a two step course as follows:

Step 1: Suppose $p_1 = \min_{\{j\}} p_j$

Solving this stochastic programming problem

$$(I) \left\{ \begin{array}{l} P[\varphi_1(\mathbf{x}, \theta, \tau) \leq C_1] \rightarrow \max \\ \text{subject to } \left\{ \begin{array}{l} P[\varphi_j \leq C_j] \geq p_j \quad (I)_1 \\ (j=1, 2, \dots, n) \\ \mathbf{x} \in X, \tau \in [0, t] \quad (I)_2 \end{array} \right. \end{array} \right.$$

we can go to one of the following cases:

- a) If $p_1^{\max} < \min_{\{j\}} p_j = p_1$, then the problem is unsolvable,
 b) If $p_1^{\max} = \min_{\{j\}} p_j = p_1$, then $p(t) = p_1^{\max}$
 c) If $p_1^{\max} > \min_{\{j\}} p_j$, then we pass over to step 2 where $p_1^{\max} = \max_{\{x\}} P[\varphi_1 \leq G]$

Step 2: Replace the condition

$$P[\varphi_1 \leq C_1] \geq p_1$$

by this one: $P[\varphi_1 \leq C_1] \geq p_1^{\max}$

choose $p_2 = \min_{j \neq 1} \{p_j, p_1^{\max}\}$

Then solve the following problem

$$(II) \begin{cases} P[\varphi_2(x, \theta, \tau) \leq C_2] \rightarrow \max \\ \text{subject to } (I)_1 \text{ and } (I)_2, \end{cases}$$

where $(I)_1$ is obtained from $(I)_1$ by replacing p_1 by $p_1^{\max}$.

After obtaining $p_2^{\max}$, we consider again the cases a), b), c) as step 1.

The iterate course proceeded in such a way. And it will be stopped in a step S where

$$p_s^{\max} = \min_{\substack{\{j, i\} \\ j \neq k_i}} \{p_j, p_{k_i}^{\max}\} = p_{s-1}^{\max}$$

(I) and (II) are stochastic programming problems the solving algorithms for which are presented in [6, 7].

Now we prove the consistence of the above-mentioned schema. Solving the problem (I) we see that:

a) If $p_1^{\max} < \min_{\{j\}} p_j$ then the problem is obviously unsolvable since there does not exist x satisfying $P[\varphi_1 \leq C_1] \geq p_1$.

b) If $p_1^{\max} = \min_{\{j\}} p_j = p_1$, then the calculating course is stopped because no plan with larger probability can be found.

c) If $p_1^{\max} > \min_{\{j\}} p_j = p_1$ then there exists a vector x satisfying not only all conditions of the problem but also the condition

$$P[\varphi_1 \leq C_1] \geq p_1^{\max} > p_1$$

that is to say we can still increase the probability.

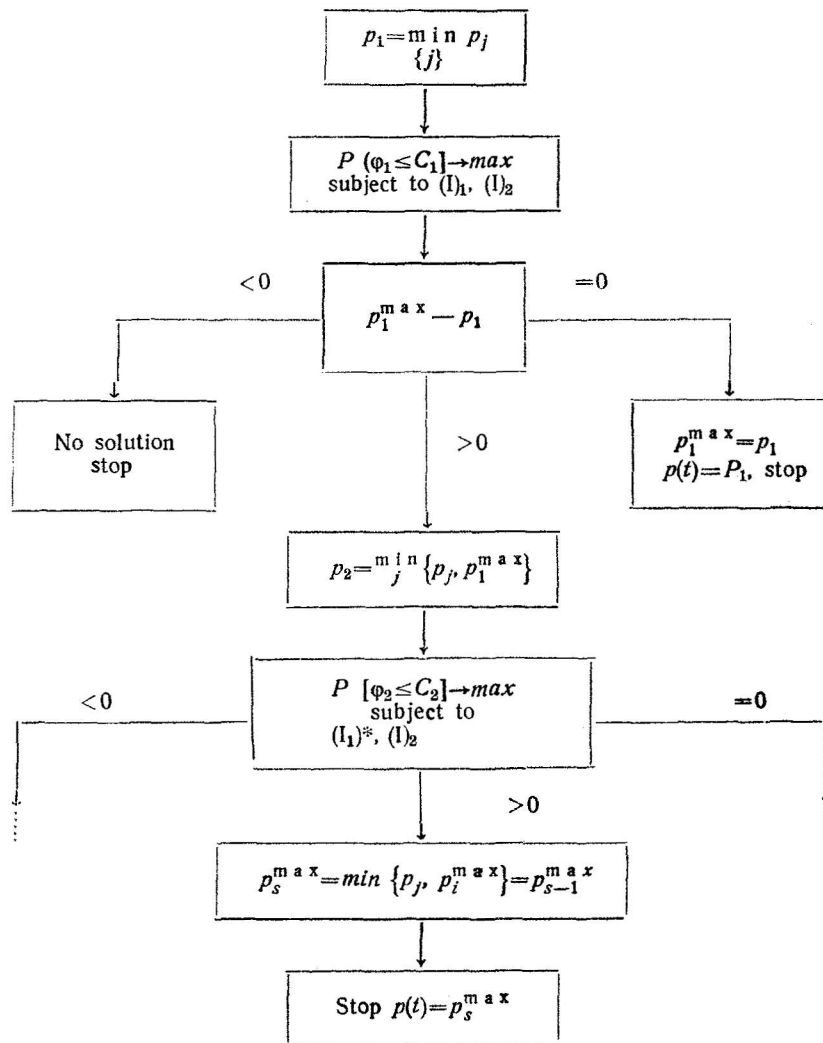
Hence we replace p_1 by $p_1^{\max}$ and the course goes on in the same way as step 1.

2. Remark: The above-mentioned probabilities p_j are chosen in advance. For weaker conditions, one can consider p_j as the unknowns of the problem.

$$p_j^0 \leq p_j \leq 1. \quad (j=1, 2, \dots, n)$$

Then we choose p_j^0 for p_j and the calculation course proceeds according to the above schema.

3. Block schema:



The volume of calculation for this schema depends essentially on solving the two problems (I) and (II). There are problems of stochastic programming for which the solving possibility is discussed in [6, 7]. In some contexts of mechanical problems, the problems (I) and (II) can be led to the apriori programming problems (nonlinear programming problem).

§4. Some Applications

1. Determining the probability of resistance of a system with a given limit force.

As we know, the limit force $p_{\max}$ in the problem of limit analysis [8] satisfying the following conditions:

$$L\sigma = P, \quad f(\sigma) \leq C, \quad \forall \sigma \in V.$$

In account to stochastic characteristics of the field condition there is a problem raising as follows:

What does equal to the probability of safety for some load P_0 ?

(P_0 can be chosen as a limit load in the case of not considering stochastic factors or expectation).

In other words, one must find

$$P[P_{\max} \geq P_0]$$

or, equivalently, one must consider the following problem:

$$P[p \leq p_0] \rightarrow \min$$

$$(III) \text{ subject to } \begin{cases} L\sigma = P \\ P[f(\sigma) \leq C] \geq q, \quad q \in [0, 1], \end{cases}$$

where σ , P are unknowns.

The shakedown problem is stated in the same way.

2. The problem of optimal design according to a given reliability.

$$MJ(\mathbf{x}, \theta) \rightarrow \max(\min)$$

(IV) subject to

$$P \left\{ \begin{matrix} P[\varphi_j \leq C_j] \geq p_j \\ j=1, n \\ \mathbf{x} \in X \end{matrix} \right\} = P_T,$$

where M is an expected operator, J is objective functional, P_T is a given probability.

We replace problem (IV) by equivariant problem as follows:

$$MJ(\mathbf{x}, \theta) \rightarrow \max(\min)$$

(IV)

$$\text{subject to } \begin{cases} P[\varphi_j \leq C_j] \geq p_j \\ j=1, n \\ P_T \leq p_j \leq 1 \\ \mathbf{x} \in X \end{cases}$$

(IV)' is a stochastic programming problem.

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