

An Equation Describing the Evolution of Non-Stationary Geometrically Thin Accretion Disc

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Accretion discs play important role in quasar, active galactic nuclei and binary X-ray source modelling. Regardless of their large application, the main problem within the disc theory—that of the viscosity nature is still unresolved. This problem cannot be solved with theoretical considerations alone. It is necessary to use observational data and to compare them with available theoretical models. Therefore, accretion discs should be studied both in stationary [1, 2] and non-stationary situation [3, 4, 5, 6].

An equation on the evolution of a geometrically thin accretion disc is developed in this paper under determined generalization of viscosity and transparency laws of the disc plasma. This equation presents a logical generalization of the equation obtained by Lynden — Bell and Pringle in 1974 [3]. The aim is to obtain an equation, the solution of which should be sensitive to the physical processes in the disc. This equation allows for a wide class of invariant solutions of the so called self-similar solutions of I-st order, but the problem will not be treated in this paper.

Basic Equations and Magnitudes

Here we shall review the basic equations and magnitudes, used for the description of the geometrically thin accretion disc behaviour [1, 2]. We shall introduce two more new generalizations, presenting the logical continuation of the available approximations of the viscosity processes in the disc [1, 2, 3, 4, 6, 7, 8] and the transparency processes in the plasma. Similar to [1, 2, 3, 4] a cylindrical coordinate system (R, ϕ, z) is used for the description of the disc behaviour as well as the following magnitudes:

- (1) a) $h = (GMR)^{1/2}$ — specific rotation momentum for the case of Keplerian accretion disc
- (2) b) $\Omega_k = \sqrt{GM/R^3}$ — Keplerian angular velocity
- (3) c) $F = 2\pi W_{R\phi} R^2$ — friction force momentum between two cylindrical layers in the disc, averaged by the z coordinate, where $W_{R\phi}$ is (R, ϕ) components of the tensor tension of the disc viscosity forces
- (4) d) $P = \frac{3}{10} \cdot \frac{\Sigma}{H} V_s^2$ — equation for the state of completely ionized hydro-

gen plasma, where $\Sigma = \int_0^\infty \rho dz$ is surface disc density, H — disc thickness, V_s —

acoustic velocity in the plasma and ρ — the plasma density

(5) e) $W_{R\Phi} = k \Sigma^{a_0} T^{b_0} H^{c_0}$ — viscosity tension under the assumption that in the most generalized form may be expressed as a product of the local disc parameters Σ , T and H powers, where T is the disc temperature averaged by z

(6) f) $V_s = \Pi_1 \Omega_k H$ — equation of hydrostatic balance in z -direction of the disc where Π_1 is a numerical constant close to the unit

(7) g) $Q^+ = \frac{3}{4} W_{R\Phi} \Omega_k$ — equation for energy emission in the disc based on viscosity

(8) h) $\varepsilon_i = k_1 T^4 = \frac{\pi_2 \kappa \Sigma Q^+}{c}$ — equation of beam transfer in the disc, where κ is the plasma transparence, k_1 is the Boltzman constant, Π_2 — numerical constant, c — the light velocity

(9) i) $\kappa = \Pi_3 \Sigma^{a_1} T^{b_1} H^{c_1}$ — plasma transparence law in a generalized form, where Π_3 — numerical constant, and the powers will be respectively at: free-free transitions $a_1=1$, $b_1=-3.5$, $c_1=-1$; Thompson scattering — $a_1=0$, $b_1=0$, $c_1=0$.

(10) j) $V_s^2 = R_g T$ — ideal gas equation where R_g is the universal gas constant

Using these equations we may obtain expressions on the basic disc parameters in function of the
Acoustic velocity

$$(11) \quad V_s = V_{s_0} \Sigma^{-\left(\frac{a_0+a_1+b_0+1}{c_0+c_1+b_0+2b_1-8}\right)} \left(\frac{c_0+c_1-b_0-1}{c_0+c_1+b_0+2b_1-8}\right) \Omega_k$$

$$V_{s_0} = \pi \left[\frac{4}{3} \left(\frac{10}{3}\right)^{b_0} \frac{\Pi_1^{2(4-b_1-b_0)} \Pi_2}{\Pi_3} \cdot \frac{ck_1}{kR_g^{(4-b_1)}} \right] \frac{1}{c_0+c_1+b_0+2b_1-8}$$

Disc plasma temperature

$$T = T_0 \Sigma^{-2\left(\frac{a_0+a_1+b_0+1}{c_0+c_1+b_0+2b_1-8}\right)^2} \left(\frac{c_0+c_1-b_0-1}{c_0+c_1+b_0+2b_1-8}\right) \Omega_k$$

$$(12) \quad T_0 = \left[\frac{4}{3} \left(\frac{10}{3}\right)^{b_0} \frac{\Pi_1^{c_0+c_1-b_0} \Pi_2}{\Pi_3} \cdot \frac{ck_1}{kR_g \frac{c_0+c_1+b_0}{2}} \right] \frac{2}{c_0+c_1+b_0+2b_1-8}$$

Pressure in the disc

$$P = P_0 \Sigma^{-\left(\frac{9+a_0+a_1-c_0-c_1-2b_1}{c_0+c_1+b_0+2b_1-8}\right)} \left(\frac{9-2c_0-2c_1-2b_1}{c_0+c_1+b_0+2b_1-8}\right) \Omega_k$$

$$(13) \quad P_0 = \left[\frac{4}{3} \left(\frac{10}{3}\right)^{(8-c_0+c_1-2b_1)} \frac{\Pi_1^{2(c_0+c_1+b_1-4)} \Pi_2}{\Pi_3} \cdot \frac{ck_1}{kR_g^{4-b_1}} \right] \frac{1}{c_0+c_1+b_0+2b_1-8}$$

$W(R, \Omega_k)$ component of viscous tension in the disc

$$(14) \quad W_{R\Phi_0} = W_{R\Phi_0} \left(\frac{a_0 c_1 - c_0 a_1 + 2a_0 b_1 - c_0 - 8a_0 + b_0 c_1 + 2b_0 b_1 - a_1 b_0 - 9b_0}{c_0 + c_1 + b_0 + 2b_1 - 8} \right)$$

$$\Sigma \left(\frac{2b_0 b_1 + 2b_0 c_1 + 7c_0 - 9b_0 - 2b_1 c_0}{c_0 + c_1 + b_0 + 2b_1 - 8} \right) \Omega_k$$

$$W_{R\Phi_0} = k \left[\left(\frac{4}{3} \right)^{b_0 + c_0} \left(\frac{10}{3} \right)^{b_0 (8 - c_1 - 2b_1)} \frac{\Pi_1^2 (b_0 c_1 + b_0 b_1 + 4b_0 + 4c_0 - b_1 c_0) \Pi_2^{b_0 + c_0}}{\Pi_3} \right.$$

$$\left. \cdot \left(\frac{ck_1}{kR_g^{4-b_1}} \right)^{b_0 + c_0} \right] \frac{1}{c_0 + c_1 + b_0 + 2b_1 - 8}.$$

Yield of the basic equation

Using the above formulae for the basic disc parameters and substituting Ω_k with $h = \sqrt{GMR}$ we obtain the following expression for the friction force momentum between two cylinders:

$$(15) \quad F = \Pi_4 \Sigma^{N_1} h^{N_2},$$

where

$$N_1 = \frac{a_0 c_1 - c_0 a_1 + 2a_0 b_1 - c_1 - 8a_0 + b_0 c_1 + 2b_0 b_1 - a_1 b_0 - 9b_0}{c_0 + c_1 + b_0 + 2b_1 - 8},$$

$$(16) \quad N_2 = \frac{31b_0 + 6b_1 c_0 + 4c_1 + 8b_1 - 17c_0 - 6b_1 b_1 - 6b_0 c_1 - 32}{c_0 + c_1 + b_0 + 2b_1 - 8},$$

$$\Pi_4 = 2\pi k \left[\left(\frac{4}{3} \right)^{b_0 + c_0} \left(\frac{10}{3} \right)^{b_0 (8 - c_1 - 2b_1)} \Pi_2^{2(b_0 c_1 + b_0 b_1 + 4b_0 + 4c_0 - b_1 c_0)} \right.$$

$$\left. \cdot \left(\frac{\Pi_2}{\Pi_3} \cdot \frac{ck_1}{kR_g} (4 - b_1) \right)^{b_0 + c_0} \right] \frac{1}{c_0 + c_1 + b_0 + 2b_1 - 8} (GM)^{\frac{(2c_0 - 20b_0 + 4b_0 b_1 + 4b_0 c_1 - 4b_1 c_0 - 2c_1 - 4b_1 + 16)}{c_0 + c_1 + b_0 + 2b_1 - 8}}.$$

Thus we can express Σ as a function of F and h using (15)

$$(17) \quad \Sigma = \left[\frac{F}{\Pi_4 h^{N_2}} \right]^{\frac{1}{N_1}}.$$

Let us remember the basic equation of non-stationary disc accretion dynamics [4, 6].

Continuum equation

$$(18) \quad \frac{\partial \Sigma}{\partial t} = - \frac{1}{2\pi R} \frac{\partial M}{\partial R}.$$

where $\dot{M} = 2\pi R \Sigma V_R$ — accretion velocity, V_R — radial velocity of disc plasma. Substituting R with h we obtain

$$(19) \quad \frac{\partial \Sigma}{\partial t} = - \frac{(GM)^2}{2\pi h^3} \frac{\partial M}{\partial h}.$$

Momentum transfer equation

$$(20) \quad \frac{M \Omega_k R}{2} = - 2\pi \frac{\partial}{\partial R} (W_{R\Phi} R^2).$$

Using $F = 2\pi W_{R\Phi} R^2$ and substituting R with h we obtain

$$(21) \quad \dot{M} = - \frac{\partial E}{\partial h}.$$

And finally combining (19) and (21) we have

$$(22) \quad \frac{\partial F}{\partial t} = D \frac{F^m}{h^n} \frac{\partial^2 F}{\partial h^2},$$

where

$$(23) \quad \begin{aligned} m &= \frac{N_1 - 1}{N_1}, \\ n &= \frac{3N_1 - N_2}{N_1}, \\ D &= \frac{N_1}{2\pi} \Pi_4 \frac{1}{N_1} (GM)^2. \end{aligned}$$

As it is seen from the type of equation (22) at $m=0$ it is transformed into the linear case, described in [3] under the assumption that the viscosity is a power law of the radius.

Let us make the following substitution:

$$(24) \quad h = \left\{ \left(\frac{n+2}{1} \right) y \right\}^{\frac{2}{n+2}}$$

which transforms equation (23) into the following type, easy to be analysed

$$(25) \quad \frac{\partial F}{\partial t} = D F^m \frac{\partial^2 F}{\partial y^2} + \frac{8D(n-4)}{(n+2)} \cdot \frac{F^m}{y} \frac{\partial F}{\partial y}.$$

It is seen in expression (25) that the non-stationary disc accretion is effected by the interaction between two non-linear processes: non-linear isotropic diffusion and non-linear transfer.

In the present form, equation (25) is very close to the Burgers' classical equation [10, 11]. If we introduce the following smoothing factor

$$(26) \quad \varepsilon = \int_{y_0}^{y_\Phi} \frac{\partial^2 \varphi}{\partial y^2} \frac{1}{y^{1-m}} dy / \int_{y_0}^{y_\Phi} \frac{1}{y} \cdot \frac{\partial \varphi}{\partial y} \frac{1}{y^{1-m}} dy,$$

where $\varphi = F^{1-m}$ and using that $0 \leq m < 1$ are always all types of viscosity and transparency occurring in accretion discs and if $\varepsilon > 1$ then the major role in the non-stationary accretion will be that of the isotropic non-linear diffusion and the reverse.

Discussion and Conclusion

Equation (22) obtained here is a generalization of the equations yielded thus far for the description of a non-stationary geometrically thin disc. Therefore, its analysis will result into a more detailed investigation of the behaviour of disc-like configurations around gravity centres.

As it is known, [12, 13] equations of this type (22) allow for a wide class of invariant solutions. This in its turn, provides for their resolution. The problems attributed to this area will be the subject of future considerations

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