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# Computation of Stabilized Turbulent Fluid Flow and Heat Transfer in Circular Smooth Pipes for Moderate Prandtl Numbers. Part 1.—Fluid Flow and Eddy Diffusivity

M. Mikhailov, N. Vulchanov, V. Zimparov

The understanding of stabilized turbulent momentum and heat transfer in circular pipes is fundamental for various types of applied problems, such as the design of shell-and-tube heat exchangers, etc. A typical approach towards the mathematical and computer simulation of the solution of the momentum and heat transfer problem is the decoupling of the momentum and heat transfer mechanisms. This enables one to find first the velocity and the eddy diffusivity distributions and then, inserting these into the energy balance equation, to determine the heat transfer characteristics of the process. The formal description of this approach results in the formulation of the so-called Graetz type problems [1].

The purpose of Part 1 of this study is to derive a model for the turbulent momentum transfer and the eddy diffusivity combining elements of previously published models. A computational procedure for the velocity distribution will be described and some illustrative results will be reported.

It will be assumed that the turbulent flow is hydrodynamically fully developed. The physical properties of the fluid are constant while for a prescribed Reynolds number the velocity distribution and the transport coefficients are assumed to be functions of the radial coordinate only. All axial and dissipation effects will be neglected. From physical considerations the flow can be considered as consisting of two regions of quite different nature [2], [3]—the wall region and the core of the flow. The wall region is characterized by a steep gradient of the velocity. It comprises the major component of the thermal resistance and influences significantly the transport phenomena considered. In view of the continuous character of the problem it is not reasonable to expect that there will be a sharp boundary between the wall region and the core. It is usually assumed that the location of this boundary is known [4] or that the transport coefficients are expressed by one and the same law from the axis to the wall [5]. Our major assumption is that the boundary is not known a priori and we calculate it for a prescribed Reynolds number from the condition that at the boundary the eddy viscosity is a continuous function of the radial coordinate.

As basic constitutive hypotheses we shall use the "stress-rate or deformation" relation

(1a) $$\tau/\tau_w=\frac{1}{R^+}(1+\varepsilon_m/\nu)\frac{dU^+}{d\eta}$$

and the "heat flux — temperature gradient" relation

(1b) $$q=-(1+P_r\varepsilon_H/\nu)\frac{d\theta}{dR},$$

where the following notation has been used: $R$ — the dimensionless radial coordinate, $\eta=1-R$ — the wall coordinate, $R^+$ — the dimensionless hydraulic radius of the pipe, $U^+(\eta)$ — the dimensionless velocity distribution. Length and velocity scale factors are assumed to be $(\nu^2\rho/\tau_w)^{1/2}$ and $(\tau_w/\rho)^{1/2}$, respectively, where as usually $\nu[m^2\,s^{-1}]$ is the kinematic viscosity and $\rho$ [kg m$^{-3}$] is the specific gravity of the fluid and $\tau_w$[N m$^{-2}$] is the (unknown) wall shear stress. $Pr$ is the conventional Prandtl number, $\tau$[N m$^{-2}$] is the shear stress, $q$ and $\theta(R)$ are the dimensionless heat flux and temperature, respectively, while $\varepsilon_m/\nu$ and $\varepsilon_H/\nu$ denote the dimensionless eddy viscosity and eddy diffusivity.

Although $\varepsilon_m/\nu$ and $\varepsilon_H/\nu$ "do nothing to illuminate the nature of turbulence, they provide a means to calculate momentum and heat transfer rates" [4], therefore their correct definition is of primary importance for the simulation described in what follows. These quantities are related through

(2) $$\frac{\varepsilon_H}{\nu}=\frac{1}{Pr_t}\,\frac{\varepsilon_m}{\nu},$$

where $Pr_t$ is known as the turbulent Prandtl number [2]—[5]. Often it has been assumed that $Pr_t=$const or even 1 [4], [5] but practice has shown that this results in incorrect heat transfer data [4]. Because the properties of the wall region and the core differ significantly, the definitions of $\varepsilon_m/\nu$ and $\varepsilon_H/\nu$ in both regions will be discussed separately.

Unlike some classical concepts [6] we shall not divide the wall region into subregions of different properties because this has lead to incorrect heat transfer results [4]. To obtain continuous eddy viscosity and velocity profiles, we shall use the "mixing length" type definition [7] for $\varepsilon_m/\nu$ in the wall region:

(3a) $$\frac{\varepsilon_m}{\nu}=\left(\frac{l^+}{R^+}\right)^2\frac{dU^+}{d\eta},\qquad 0<\eta<\eta_B,$$

where $\eta_B$ denotes the (unknown) boundary between the wall region and the core depending only on the Reynolds number $Re$. The "mixing length" $(l^+/R^+)$ is determined by [7]:

(3b) $$\left(\frac{l^+}{R^+}\right)=0.4\eta\{1-\exp(-\eta R^+/26)\}.$$

To define $\varepsilon_H/\nu$ one has to know the turbulent Prandtl number $Pr_t$. This quantity accounts for the influence of the Prandtl number on the eddy diffusivity thus depending on $\eta$, $Re$ and $Pr$. We have chosen to utilize a definition [2] based also on the "mixing length" concept with constants determined through data fitting of experimental results for $0.02\leq Pr\leq 15$. Thus

(4a) $$Pr_t=0.909\,\frac{1-\exp(-\eta R^+/26)}{1-\exp(-\eta R^+/B^+)},\qquad 0<\eta<\eta_B,$$

(4b) $$Pr_t=0.909B^+/26,\qquad \eta=0,$$

(4c) $$B^+=(Pr)^{-1/2}\sum_{i=1}^{5} C_i(\log_{10} Pr)^{i-1},$$

(4d) $$C_1=34.96,\quad C_2=28.79,\quad C_3=33.95,\quad C_4=6.33,\quad C_5=-1.186.$$

In the core region $\eta_B<\eta<1$ the "mixing length" hypothesis is considered invalid [3]. We shall use a correlation for $\varepsilon_m/\nu$ of the form [5]:

(5a) $$\frac{\varepsilon_m}{\nu}=0.07044R^+\{1-(1-\eta)^2\}\{1+2.345(1-\eta)^2\},\quad \eta_B<\eta<1$$

valid in the range $18000\leqq Re\leqq 100000$ and correctly predicting nonzero and almost constant diffusivity close to the axis of the pipe. Due to the negligible influence of the core on the temperature distribution we expect that eq. (5a) can be applied for $Re>100000$, too. Having in mind that for large $\eta$ eq. (4a) can be written as

(5b) $$Pr_t=0.909,\quad \eta_B<\eta\leqq 1,$$

for $\varepsilon_H/\nu$ one has

(5c) $$\frac{\varepsilon_H}{\nu}=0.0755R^+\{1-(1-\eta)^2\}\{1+2.345(1-\eta)^2\},\quad \eta_B<\eta\leqq 1.$$

At the boundary between the wall and the core regions $-\eta=\eta_B-$ a continuity condition on the eddy viscosity is imposed, thus

(6a) $$\left(\frac{\varepsilon_m}{\nu}\right)_{\text{wall}}=\left(\frac{\varepsilon_m}{\nu}\right)_{\text{core}},\quad \eta=\eta_B,$$

where eqs. (3a) and (5a) are used to define the left and right hand sides of eq. (6a). Theoretically, one has to satisfy a similar condition for $\varepsilon_H/\nu$:

(6b) $$\left(\frac{\varepsilon_H}{\nu}\right)_{\text{wall}}=\left(\frac{\varepsilon_H}{\gamma}\right)_{\text{core}},\quad \eta=\eta_T,$$

where in general $\eta_T\neq\eta_B$ and $\eta_T$ will define a "thermal" wall region. We shall presently leave this question open and the final discussion will be given a little later.

Thus we have accomplished the formulation of the model for the transport coefficients for the problem under consideration. Now, making use of the linear distribution of the shear stress over the radius of the pipe (which is a particular case of the momentum balance equation), written as

(7) $$\tau/\tau_w=R,$$

and recalling that $\eta=1-R$, one can combine eqs. (1a), (3a) and (7) for the wall region and eqs. (1a), (5a) and (7) for the core. In view of the axial symmetry of the problem, to find the velocity distribution $U^+(\eta)$ (or equivalently $-U^+(R)$) one has to solve the following initial-value problem:

(8a) $$\frac{dU^+}{d\eta}=\begin{cases}2(1-\eta)\left\{\frac{1}{R^+}+\left[\left(\frac{1}{R^+}\right)^2+4(1-\eta)\left(\frac{l^+}{R^+}\right)^2\right]^{1/2}\right\}^{-1}, & 0<\eta<\eta_B,\\ (1-\eta)\left\{\frac{1}{R^+}\left(1+\frac{\varepsilon_m}{\nu}\right)\right\}^{-1}, & \eta_B<\eta\leqq 1,\end{cases}$$

(8b) $$U^+=0,\quad \eta=0.$$

As far as closed form solution of eqs. (8) is not available, this problem has to be solved numerically. In addition to this, an iterative procedure has to be formulated to yield simultaneously the velocity profile $U^+(\eta)$, $R^+$ and $\eta_B$.

This procedure starts with an initial approximation $R^+_{(1)}$ for $R^+$, taken in the form [5]

(9a) $$R^+_{(1)} = 0.094\, Re^{0.875}.$$

For $k=1, 2, \ldots, n$ one solves the initial value problem, eqs. (8), by standard fourth order variable step Runge—Kutta method. The stepsize is controlled so that the increment over one step of $U^+$ does not exceed 0.5 and the maximum stepsize is 0.00667. The motivation of the latter restrictions will be discussed in Part 2 of this study [8]. Next, one calculates

(9b) $$F_{(k)} = 0.5\left\{\int_0^1 R U^+_{(k)}(R)\, \mathrm{d}R\right\}^{-2}, \quad k=1, 2, \ldots, n,$$

(9c) $$Re_{(k)} = 2R^+_{(k)}[F_{(k)}/2]^{-1/2}, \quad k=1, 2, \ldots, n,$$

and checks whether

(9d) $$\frac{|Re - Re_{(k)}|}{Re} \leqq \varepsilon,$$

where $\varepsilon = 0.0001$ for the numerical results reported. If eq. (9d) is not satisfied, one sets

(9e) $$R^+_{(k+1)} = 0.5\, Re\left\{\frac{F_{(k)}}{2}\right\}^{1/2},$$

and goes back to integrate eqs. (8). The value of $\eta_B$ is determined by means of eq. (6a). In our numerical experiments we tried to determine $\eta_T$, too. Its values were close to those of $\eta_B$ and for this reason the results reported here are obtained under the assumption that $\eta_T = \eta_B$.

In the range of $10000 \leqq Re \leqq 1000000$ convergence was achieved in 6—11 iterations depending on the value of $Re$. The algorithm was programmed in BASIC and all computations were performed on an ISOT 0310 minicomputer (relative machine precision $1.19\ 10^{-7}$).

To test the procedure for the computation of the velocity distribution we compared computed numerical values for $U^+(\eta)$ to tabulated experimental data reported in [9] for $10000 \leqq Re \leqq 1000000$. The maximum deviation was less than 10 % and it occurred for small $Re$ close to the wall. The average deviation observed was 3—4 %.

Table 1 exhibits values for the Fanning factor $F$ reported in other studies and our results. The data from the table have been obtained by means of the following correlations:

[3] $$-F = 0.0791\, Re^{-0.25}, \quad Re < 100000;$$

[4] — data are given in a table, $10000 \leqq Re \leqq 1000000$;

[9] — $F = 0.0008 + 0.05525\, Re^{-0.237}$. $Re > 100000$;

[10] — $F = \{1.76 \ln(ReF^{1/2}) - 0.4\}^{-2}, \quad Re \leqq 1000000$;

[11] — $F=0.079\,Re^{-0.25}$, $Re<100000$;

$F=0.046Re^{-0.2}$, $100000\leq Re\leq 10000000$.

Table 1

| $Re$ $10^{-4}$ | $F \cdot 10^3$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | [9] exp. | [9] $Re>10^5$ | [3] $Re<10^5$ | [10] $Re<10^6$ | [11] $Re<10^7$ | [4] $Re<10^6$ | Our results |
| 1.00 | — | — | 0.7910 | 0.7548 | 0.7900 | 0.7625 | 0.8584 |
| 1.67 | 0.6950 | — | 0.6958 | 0.6614 | 0.6949 | — | 0.7285 |
| 2.00 | — | — | 0.6651 | 0.6325 | 0.6643 | 0.6450 | 0.6908 |
| 4.32 | 0.5685 | — | 0.5487 | 0.5274 | 0.5480 | — | 0.5618 |
| 5.00 | — | — | 0.5290 | 0.5103 | 0.5283 | 0.5225 | 0.5426 |
| 10.00 | — | 0.4408 | — | 0.4393 | 0.4600 | 0.4500 | 0.4623 |
| 10.60 | 0.4500 | 0.4359 | — | 0.4340 | 0.4547 | — | 0.4574 |
| 20.00 | — | 0.3862 | — | 0.3818 | 0.4005 | 0.3925 | 0.3990 |
| 39.60 | 0.3450 | 0.3404 | — | 0.3353 | 0.3493 | — | 0.3487 |
| 72.50 | 0.3070 | 0.3056 | — | 0.3006 | 0.3095 | — | 0.3117 |

The numerical results reported here indicate that this model for the hydrodynamic problem and the transport coefficients is a reliable basis for the solution of the heat transfer problem to be discussed in Part 2 of this study [8].

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