

GENERAL MECHANICS

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Theory of Lyapunov's Characteristic Exponents for Impulsive Differential Systems. III. Regular Systems*

1. Introduction

In the papers [1] and [2] for the linear homogeneous impulsive system

$$(1) \quad \begin{aligned} \frac{dx}{dt} &= A(t)x, \quad t \neq \tau_k, \\ \Delta x &= B_k x, \quad t = \tau_k, \quad k = 1, 2, \dots \end{aligned}$$

some fundamental results of the theory of Lyapunov's characteristic exponents are exposed. The proofs of these results are based on the property of linearity of the space L comprising solutions of system (1). The space L is linear if the following conditions (A) hold:

A1. $B_k \in C^{n \times n}$, $\det(E + B_k) \neq 0$ ($k = 1, 2, \dots$).

A2. $\tau_0 = 0 < \tau_1 < \tau_2 < \dots$, $\lim_{k \rightarrow +\infty} \tau_k = +\infty$.

A3. The matrix-valued function $A: \mathbb{R}_+ \rightarrow C^{n \times n}$ is continuous for $t \in \mathbb{R}_+$, $t \neq \tau_k$, and for $t = \tau_k$ it has a discontinuity of the first kind and is continuous from the left.

Under these conditions in the present paper we shall introduce the notion of regularity for the impulsive system (1) and investigate the relation of this property to the properties of normality of the basis and to the reducibility. For this purpose first we shall consider some properties of the conjugate to each other linear impulsive systems.

2. Conjugate impulsive systems

With system (1) we associate the system

$$(2) \quad \begin{aligned} \frac{dy}{dt} &= -A^*(t)y, \quad t \neq \tau_k, \\ \Delta y &= -(E + B_k^*)^{-1} B_k^* y, \quad t = \tau_k \quad (k = 1, 2, \dots). \end{aligned}$$

Definition 1 [3]. System (2) is said to be conjugate to system (1).

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In view of the identity

$$(E - B(E + B)^{-1})^{-1}B(E + B)^{-1} = B \quad (B \in \mathbb{C}^{n \times n})$$

it is easy to check that system (1) is conjugate to system (2), i. e. we can say that systems (1) and (2) are conjugate to each other.

The matrices $-A^*(t)$ and $F_k = -(E + B_k^*)^{-1}B_k^*$ as well as $A(t)$ and B_k satisfy conditions (A), moreover,

$$(3) \quad \det(E + F_k) = \det(E - (E + B_k^*)^{-1}B_k^*) = \det(E + B_k^*)^{-1} = \frac{1}{D_k^*},$$

where $D_k = \det(E + B_k)$.

Hence, the space M of all solutions of system (2) defined in $\mathbb{R}_+$ is an n -dimensional linear space.

Lemma 1 [3]. Let conditions (A) hold. Then:

1. For any two solutions $x(t)$ and $y(t)$ of the conjugate to each other systems (1) and (2) the following identity is valid:

$$(x(t) | y(t)) = c \quad (t \in \mathbb{R}_+),$$

where $c = (x(0) | y(0))$ is a constant.

2. The fundamental matrices $X(t)$ and $Y(t)$ of systems (1) and (2) satisfy the identity

$$(4) \quad Y^*(t)X(t) = C \quad (t \in \mathbb{R}_+),$$

where $C \in \mathbb{C}^{n \times n}$ is a constant matrix.

3. If relation (4) holds, where $C \in \mathbb{C}^{n \times n}$ is a constant matrix and $X(t)$ is a fundamental matrix of system (1), then $Y(t)$ is a fundamental matrix of system (2).

Remark 1. If $X(t)$ and $Y(t)$ are fundamental matrices of two conjugate to each other systems and $Y^*(0)X(0) = E$, then from Lemma 1 it follows that $Y = (X^*)^{-1}$. For the solutions

$$\begin{aligned} x_1(t), \dots, x_n(t), \\ y_1(t), \dots, y_n(t) \end{aligned}$$

corresponding to these fundamental matrices the relations

$$(5) \quad (x_i(t) | y_j(t)) = \delta_{ij}$$

hold, where δ_{ij} is Kronecker's symbol.

3. Regular systems

The fundamental matrix

$$(6) \quad X(t) = \{x_1(t), \dots, x_n(t)\}$$

of solutions of system (1) is said to be ordered if

$$\chi[x_1] \leq \chi[x_2] \leq \dots \leq \chi[x_n]$$

and the fundamental matrix

$$(7) \quad Y(t) = \{y_1(t), \dots, y_n(t)\}$$

of solutions of system (2) is said to be ordered if

$$\chi[y_1] \geq \chi[y_2] \geq \dots \geq \chi[y_n].$$

By Remark 1 to each fundamental matrix (6) there corresponds a unique fundamental matrix (7) such that relations (5) hold. Such two fundamental matrices are said to be reciprocal.

Two reciprocal fundamental matrices not always are ordered at the same time but from (5) and [1], property 5 it follows that

$$\chi[x_i] + \chi[y_i] \geq 0 \quad (i=1, \dots, n).$$

That is why the number

$$\gamma' = \max_{i=1, \dots, n} (\chi[x_i] + \chi[y_i])$$

is nonnegative. This number is called defect of the two reciprocal fundamental matrices.

Definition 2. The minimal defect

$$\gamma = \min \gamma',$$

where the minimum is taken over all possible reciprocal fundamental matrices, is called coefficient of irregularity of systems (1) and (2).

It is obvious that the coefficient of irregularity does not depend on the choice of the two reciprocal fundamental matrices but it is completely determined by the coefficients of systems (1) and (2). Since system (1) determines uniquely system (2), we can only relate the coefficient of irregularity with system (1).

Definition 3. System (1) is said to be regular if its coefficient of irregularity equals zero, $\gamma=0$.

Another coefficient characterizing the regularity of a system is Perron's coefficient. We define it as follows:

Let the characteristic exponent of system (1) (with account of their multiplicities) be arranged in an increasing order

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

and those of system (2) be arranged in a decreasing order

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n.$$

Definition 4. We call Perron's coefficient for system (1) the number (or the symbol $+\infty$)

$$\pi = \max_{i=1, \dots, n} (\lambda_i + \mu_i).$$

In the definition of Perron's coefficient we assume that $(-\infty) + (+\infty) = 0$ and $\lambda + (+\infty) = \infty$ if $\lambda \neq -\infty$.

From Definition 4 it follows that Perron's coefficient depends eventually only on the coefficients of system (1).

The following theorem of Perron for impulsive systems is valid.

Theorem 1. Let conditions (A) hold.

System (1) is regular iff Perron's coefficient π equals zero, i. e. if the characteristic exponents of systems (1) and (2) (with account of their multiplicities) are respectively,

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$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n,$$

$$-\lambda_1 \geq -\lambda_2 \geq \dots \geq -\lambda_n,$$

i. e. if for any $i=1, 2, \dots, n$

$$\lambda_i + \mu_i = 0.$$

Moreover, if $X(t) = \{x_1(t), \dots, x_n(t)\}$ and $Y(t) = \{y_1(t), \dots, y_n(t)\}$ are reciprocal fundamental matrices, then from the normality of one of them there follows the normality of the other one and

$$\chi[y_i] = -\chi[x_i] \quad (i=1, \dots, n).$$

Theorem 1 is proved as in [4].

We shall consider one more criterion of regularity which is defined for systems with finite spectrum.

Let system (1) have a finite spectrum

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

Definition 5. We call Lyapunov's coefficient for impulsive systems the number

$$\sigma = \sigma_{x_0} - \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right),$$

where $\sigma_{x_0} = \sum_{i=1}^n \lambda_i$ is the sum of the characteristic exponents of system (1).

Lemma 2. Let conditions (A) hold and let the spectrum of system (1) be finite. Then Lyapunov's coefficient σ equals zero iff the following conditions hold:

1. There exists the limit

$$(8) \quad S = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right).$$

2. The equality $\sigma_{x_0} = S$ holds.

Proof. The sufficiency is obvious.

Necessity. Let $\sigma = 0$, i. e.

$$(9) \quad \sigma_{x_0} = \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right).$$

But from the inequality of Lyapunov we have that

$$(10) \quad \sigma_{x_0} \geq \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right).$$

From (9) and (10) it follows that the limit (8) exists and equals σ_{x_0} .

Theorem 2. Let conditions (A) hold and let the spectrum of system (1) be finite. System (1) is regular iff its Lyapunov's coefficient σ equals zero.

Proof. Necessity. Let system (1) be regular and let $X_0(t)$ and $Y_0(t)$ be two reciprocal normal fundamental matrices of (1) and (2). Then from Theorem 1 it follows that

$$(11) \quad \sigma_{x_0} + \sigma_{y_0} = 0.$$

In view of (3) and the fact that $\operatorname{Re} \operatorname{Tr}(-A^*(s)) = -\operatorname{Re} \operatorname{Tr} A(s)$, then Lyapunov's inequality for system (2) takes the form

$$(12) \quad \sigma_{y_0} \geq -\liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right).$$

From Lyapunov's inequality (10) for system (1) it follows that

$$(13) \quad \begin{aligned} \sigma &= \sigma_{x_0} - \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right) \\ &\geq \sigma_{x_0} - \limsup_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right) \geq 0 \end{aligned}$$

and from (12) and (11) it follows that

$$(14) \quad \sigma = \sigma_{x_0} - \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right) \leq \sigma_{x_0} + \sigma_{y_0} \leq 0.$$

Comparing (13) and (14), we obtain that $\sigma = 0$.

Sufficiency. Let $\sigma = 0$ and $X(t) = \{x_{ij}(t)\}$ be a normal fundamental matrix corresponding to the solutions

$$x_1(t), \dots, x_n(t)$$

for which $\chi[x_k] = \lambda_k$,

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

Then by Lemma 1

$$(15) \quad Y(t) = [X^{-1}(t)]^* = \{y_{ij}(t)\}$$

is a fundamental matrix of the conjugate system (2). Let the solutions $y_1(t), \dots, y_n(t)$ correspond to $Y(t)$ and $\chi[y_k] = \mu_k$ ($k = 1, \dots, n$).

From (15) it follows that $(x_j(t) | y_j(t)) \equiv 1$ ($j = 1, \dots, n$) and by [1], property 5 we have

$$(16) \quad \lambda_i + \mu_i \geq 0 \quad (i = 1, \dots, n).$$

If we denote by $X_{ij}(t)$ the cofactor of the entry $x_{ij}(t)$ of the determinant $\det X(t)$ then for the entry $y_{ij}(t)$ of $Y(t)$ we obtain

$$(17) \quad y_{ij}(t) = \left[\frac{X_{ij}(t)}{\det X(t)} \right]^*,$$

where $\det X(t) = \det X(0) \prod_{\tau_k < t} D_k \exp \left(\int_0^t \operatorname{Tr} A(s) ds \right) \neq 0$.

From (17) by Definition 1 and property 2 of [1] it follows that

$$(18) \quad \chi[y_{ij}(t)] \leq \chi[X_{ij}^*(t)] - \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right).$$

Since $\sigma = 0$, by Lemma 2 there exists the limit

$$(19) \quad \liminf_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right) = \sigma_{x_0}.$$

Taking into account, when finding the cofactor $X_{ij}(t)$, that we omit the j -th column containing the components of the solution x_j , we conclude that

$$(20) \quad \chi[X_{ij}^*(t)] = \chi[X_{ij}(t)] \leq \sigma_{x_0} - \lambda_j.$$

Finally, from (18), (19) and (20) it follows that

$$\chi[y_{ij}(t)] \leq (\sigma_{x_0} - \lambda_j) - \sigma_{x_0} = -\lambda_j$$

or

$$\mu_j = \max_{i=1, \dots, n} \chi[y_{ij}(t)] \leq -\lambda_j,$$

i. e.

$$\lambda_j + \mu_j \leq 0 \quad (j=1, \dots, n)$$

and in view of (16) we obtain that

$$\lambda_j + \mu_j = 0 \quad (j=1, \dots, n).$$

Then by Theorem 1 system (1) is regular.

Theorem 3. Let conditions (A) hold and let the spectrum of system (1) be finite. Then if system (1) is reducible, it is regular.

Proof. By Lyapunov's transformation $x = L(t)y$ let system (1) be transformed into the system

$$(21) \quad \frac{dy}{dt} = Ay$$

where $A \in \mathbb{C}^{n \times n}$.

Let $X(t)$ be a normal fundamental matrix of system (1) and let $Y(t)$ be a fundamental matrix of system (21) for which

$$(22) \quad X(t) = L(t)Y(t),$$

From (22) it follows that $\det X(t) = \det L(t) \det Y(t)$ or

$$\det X(0) \prod_{\tau_k < t} D_k \exp \left(\int_0^t \operatorname{Tr} A(s) ds \right) = \det L(t) \det Y(0) \exp(t \operatorname{Tr} A),$$

i. e.

$$(23) \quad \prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) = K |\det L(t)| \exp(t \operatorname{Re} \operatorname{Tr} A),$$

where $K = |\det Y(0) X^{-1}(0)| \neq 0$.

Since $\chi[K|\det L(t)|]=0$, from (23) we obtain that there exists the limit

$$(24) \quad S = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln \left(\prod_{\tau_k < t} |D_k| \exp \left(\int_0^t \operatorname{Re} \operatorname{Tr} A(s) ds \right) \right) \\ = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln (\exp (t \operatorname{Re} \operatorname{Tr} A)) = \operatorname{Re} \operatorname{Tr} A.$$

Let σ_x and σ_y be the sums of the characteristic exponents of the solutions of the fundamental matrices $X(t)$ and $Y(t)$, respectively. Since Lyapunov's transformation preserves the characteristic exponents, then

$$(25) \quad \sigma_x = \sigma_y.$$

Moreover, the normality of $X(t)$ implies the normality of $Y(t)$. Since for the system without impulse effect (21) we have

$$(26) \quad \sigma_y = \operatorname{Re} \operatorname{Tr} A$$

from (24), (25) and (26) it follows that

$$\sigma_x = S.$$

Consequently, by Lemma 2 and Theorem 2 system (1) is regular.

References

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