

O. C h r i s t o v

Perturbations of a Mechanical System with Two Degrees of Freedom

1. Introduction

The theory of Hamiltonian systems is of great interest to many branches of science. The question of integrability of Hamiltonian systems is one of the oldest problems of classical mechanics [1]. It has inspired H. P o i n c a r e to produce some of his greatest results though most of the Hamiltonian systems are not integrable. So, the main problem of dynamics is to study Hamiltonian systems which are close to the integrable ones. The most powerful approach to such systems is the KAM - theory. Before giving a brief account of the KAM - theory we shall recall the structure of the integrable Hamiltonian systems.

The phase space of the generic integrable Hamiltonian systems with n - degrees of freedom is foliated into invariant manifolds, the typical fibre being an n - dimensional torus, on which the motion is quasiperiodic. A natural question is whether small perturbations destroy these tori, the KAM - theory gives conditions for the integrable systems which guarantee the survival of most of the invariant tori. The conditions are given in terms of the so-called action - angle variables $I_1, I_2, \dots, I_n$; $\varphi_1, \varphi_2, \dots, \varphi_n$. Without going into details, we remind that the action - angle variables can be introduced for any integrable system locally near a fixed torus and they have the property that $I = (I_1, I_2, \dots, I_n)$ maps a neighbourhood of a fixed torus on an open subset of $\mathbb{R}^n$. The functions $\varphi_1, \varphi_2, \dots, \varphi_n$ are the co - ordinates on any of the nearby tori. Moreover the first integrals become functions of the action variables $I_1, I_2, \dots, I_n$. At last to any fixed torus there corresponds an invariant torus the motion on it being quasiperiodic with frequencies $(\omega_1(I), \dots, \omega_n(I)) = (\partial H / \partial I_1, \dots, \partial H / \partial I_n)$ (see [2] for details).

One condition, stated by Kolmogorov (see [2], app.8 and literature there), on the Hamiltonian of the integrable system that ensures the survival of most of the invariant tori under small perturbations, is that the frequency map

$$I \rightarrow \left[\omega_1(I), \omega_2(I), \dots, \omega_n(I) \right]$$

should be non - degenerate. Analytically this means that the Hesseian

$$(1.1) \quad \det \left(\frac{\partial^2 H}{\partial I_j \partial I_k} \right) \quad j, k = 1, \dots, n,$$

does not vanish. We should note that the measure of the surviving tori decreases with the increase of both perturbation and measure of the set when the hessian is too close to zero.

Another condition of this type, stated by V. Arnold and J. Moser (see [2, app.8], [3]) is that of isoenergetical non-degeneracy, which can be explained as follows. Fix an energy level $H_0 = h_0$. If the Hamiltonian H_0 is written in action variables then define the following map F_{h_0} from the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into the projective space $\mathbb{P}^{n-1}$:

$$F_{h_0} : I \rightarrow \left[\omega_1(I) : \omega_2(I) : \dots : \omega_n(I) \right].$$

Then the system is isoenergetically non-degenerate if the map F_{h_0} is a homeomorphism. Analytically the isoenergetical non-degeneracy is tantamount to nonvanishing of the determinant

$$(1.2) \quad \det \begin{bmatrix} \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H_0}{\partial I} \\ \frac{\partial H_0}{\partial I} & 0 \end{bmatrix}$$

Knorrer [8] found a method for checking Kolmogorov's condition by reducing the number of degrees of freedom. Using this method, he proved that for several systems including the geodesic flow on the ellipsoid and K. Neumann's system Kolmogorov's condition is fulfilled almost everywhere.

In recent paper Horozov [4] proved that for the system describing the spherical pendulum condition (1.1) is satisfied everywhere out of the bifurcation diagram of the energy-momentum map. The crucial role in [4] is played by certain algebraic curves and Abelian integrals on them.

The purpose of this paper is to check the KAM - theory conditions (1.1) and (1.2) for a mechanical system with two degrees of freedom. We explore some of the important auxiliary results of [4,5], but don't concern the techniques used there like algebraic geometry methods (Picard - Fuchs equations) and elements of Picard - Lefschetz theory (see [4] for details).

2. Action - angle variables

Let us consider the following mechanical system [6]. Two particles with masses m_1 and m_2 are connected with flexible weightless thread, passing through the hole on the smooth plane. The particle with mass m_1 is moving on the plane and m_2 is below the plane. We suppose that m_2 moves strictly along the vertical (Fig. 1).

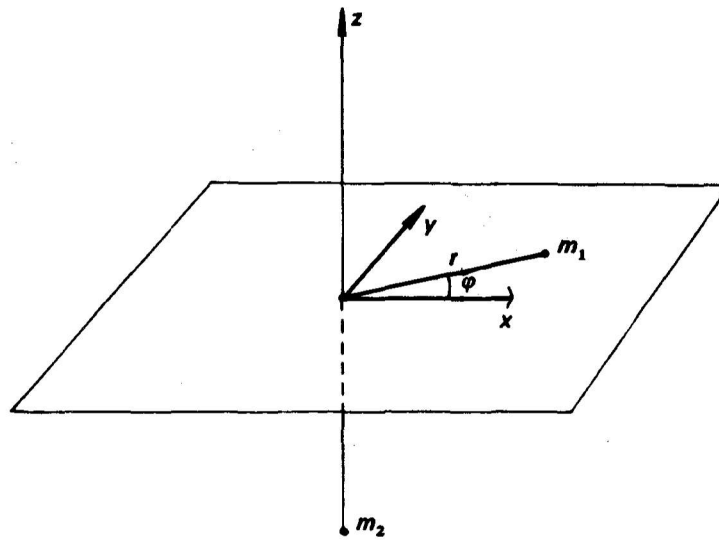


Fig. 1

The natural co-ordinates in which the system takes the simplest form are polar co-ordinates. The kinetic and the potential energy are

$$T = (m_1 + m_2)\dot{r}^2/2 + m_1 r^2 \dot{\varphi}^2/2, \quad \Pi = m_2 g r$$

So, the Lagrangian of the system is

$$L = (m_1 + m_2)\dot{r}^2/2 + m_1 r^2 \dot{\varphi}^2/2 - m_2 g r$$

where $r \in [0, \infty]$ and $\varphi \in [0, 2\pi]$. Let introduce the following variables

$$r_1 = r \sqrt{m_1}, \quad M = m_2/m_1, \quad g_1 = g \sqrt{m_1}.$$

Denote $s = M g_1 > 0$ and let write r again for simplicity. Then the corresponding

Hamiltonian system is defined via Hamiltonian

$$H = \frac{p_r^2}{2(1+M)} + \frac{p_\varphi^2}{2r^2} + sr,$$

where

$$p_r = \partial L / \partial \dot{r} = (1+M)\dot{r}, \quad p_\varphi = \partial L / \partial \dot{\varphi} = \dot{\varphi} r^2.$$

The symplectic structure is $\omega = d\sigma$, where

$$(2.1) \quad \sigma = p_r dr + p_\varphi d\varphi.$$

The system is integrable with obvious integrals

$$(2.2) \quad \begin{aligned} F = p_\varphi &= f = \text{const}, \\ H &= \frac{p_r^2}{2(1+M)} + \frac{f^2}{2r^2} + sr = h = \text{const}. \end{aligned}$$

The values of H and F for which the real movement takes place define the set

$$U = \left\{ (h, f) : h \geq 0, \quad 0 \leq f^2 \leq \frac{8h^3}{27s^2} \right\},$$

In order to introduce the action - angle variables we need to exclude from U the critical values of the energy - momentum map (H, F) . It is easy to calculate that these points are the boundaries of U i.e. the points satisfying the equations

$$f = 0, \quad f^2 = \frac{8h^3}{27s^2}.$$

Denote by U_r the set of regular values of the energy - momentum map

$$U_r = \left\{ (h, f) : h > 0, \quad 0 < f^2 < \frac{8h^3}{27s^2} \right\}. \quad (\text{Fig. 2})$$

For the points $(h, f) \in U_r$ the level surface determined by the equations $H = h$, $F = f$ is a torus $T_{h,f}$. Choose a basis γ_1, γ_2 of the homology group $H_1(T_{h,f}, \mathbb{Z})$ with the following representatives. For γ_1 take the curve on $T_{h,f}$ defined by fixing r, p_r, p_φ and letting φ run through $[0, 2\pi]$. For γ_2 fix φ and let r, p_r make one circle on the curve given by the equation

$$\frac{p_r^2}{2(1+M)} + \frac{f^2}{2r^2} + sr = h.$$

Now we can define the action co-ordinates I_1, I_2 by the formula

$$I_j = \int_{\gamma_j} \sigma, \quad j = 1, 2,$$

where σ is canonical one-form (2.1). Trivial computations give

$$(2.3) \quad I_1 = 2\pi f,$$

$$(2.4) \quad I_2 = \int_{\gamma_2} p_r dr = 2 \int_{r_-}^{r_+} \sqrt{(1+M)(2h - 2sr - f^2/r^2)} dr,$$

where $r_+ > r_-$ are the two roots of the equation

$$h - sr - f^2/2r^2 = 0.$$

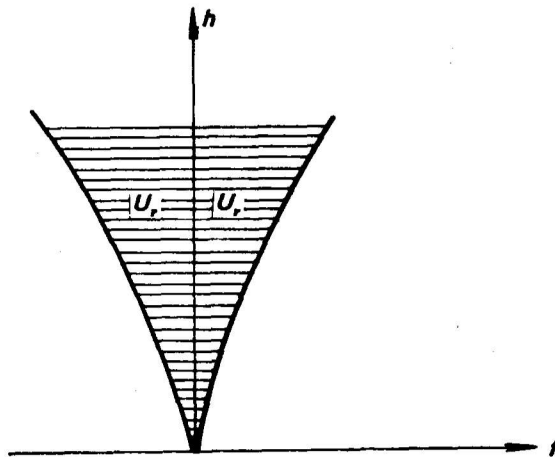


Fig. 2

Put $y^2 = 2hr^2 - 2sr^3 - f^2$. Denote the oval of the curve

$$\Gamma_{h,f} = \left\{ (y,r) : y^2 = 2hr^2 - 2sr^3 - f^2 \right\} \text{ by } \gamma. \text{ Then we have}$$

$$(2.5) \quad \psi(h,f) \stackrel{\text{def}}{=} I_2 = \sqrt{(1+M)} \int_{\gamma} \frac{ydr}{r}.$$

3. Statement of the results

Denote by $\tilde{H}(I_1, I_2)$ the Hamiltonian of the considered system in action-angle co-ordinates. We state the theorems which are the aim of this paper.

THEOREM 1. For $(h,f) \in U_r$ the following determinant

$$(3.1) \quad \det \begin{bmatrix} \frac{\partial^2 \tilde{H}}{\partial I_1^2} & \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} \\ \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial^2 \tilde{H}}{\partial I_2^2} \end{bmatrix} \neq 0.$$

THEOREM 2. For $(h,f) \in U_r$ the following determinant

$$(3.2) \quad \det \begin{bmatrix} \frac{\partial^2 \tilde{H}}{\partial I_1^2} & \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial \tilde{H}}{\partial I_1} \\ \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial^2 \tilde{H}}{\partial I_2^2} & \frac{\partial \tilde{H}}{\partial I_2} \\ \frac{\partial \tilde{H}}{\partial I_1} & \frac{\partial \tilde{H}}{\partial I_2} & 0 \end{bmatrix} \neq 0.$$

The conditions (3.1) and (3.2) are introduced by Kolmogorov and by Arnold and Moser respectively and are crucial in the KAM - theory (see [2], [3] and references there).

We shall give conditions (3.1) and (3.2) in an explicite form in terms of Abelian integrals of the second kind. Using the expressions (2.3) and (2.5) for

I_1 and I_2 we can determine $\tilde{F}$ and $\tilde{H}$ implicitly from the equations

$$I_1 = 2\pi \tilde{F},$$

$$I_2 = \psi(\tilde{F}, \tilde{H}).$$

The following lemma is proved in [4], but we give the proof for completeness and reader's convenience.

LEMMA 1 (Horozov [4]).

$$(2\pi)^2 (\partial\psi/\partial h)^4 \det \begin{bmatrix} \frac{\partial^2 \tilde{H}}{\partial I_1^2} & \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} \\ \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial^2 \tilde{H}}{\partial I_2^2} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial^2 \psi}{\partial h^2} & \frac{\partial^2 \psi}{\partial h \partial f} \\ \frac{\partial^2 \psi}{\partial h \partial f} & \frac{\partial^2 \psi}{\partial f^2} \end{bmatrix}.$$

Proof : Differentiating equality $I_2 = \psi(I_1/2\pi, \tilde{H}(I_1, I_2))$ by I_1 and I_2 , we obtain

$$0 = \psi_f \frac{1}{2\pi} + \psi_h \frac{\partial \tilde{H}}{\partial I_1},$$

$$1 = \psi_h \frac{\partial \tilde{H}}{\partial I_2}.$$

From here we find

$$\frac{\partial \tilde{H}}{\partial I_1} = -\frac{1}{2\pi} \frac{\psi_f}{\psi_h},$$

$$\frac{\partial \tilde{H}}{\partial I_2} = \frac{1}{\psi_h}.$$

Differentiating again these equalities we have

$$\frac{\partial^2 \tilde{H}}{\partial I_1^2} = -\frac{1}{(2\pi)^2} \frac{\psi_{ff}\psi_h^2 - 2\psi_h\psi_f\psi_{hf} + \psi_f^2\psi_{hh}}{\psi_h^3},$$

$$\frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} = -\frac{1}{2\pi} \frac{\psi_{fh} - \psi_{hh}(\psi_f/\psi_h)}{\psi_h^2},$$

$$\frac{\partial^2 \tilde{H}}{\partial I_2^2} = - \frac{\psi_{hh}}{\psi_h^3}.$$

Consequently

$$\det \begin{bmatrix} \frac{\partial^2 \tilde{H}}{\partial I_1^2} & \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} \\ \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial^2 \tilde{H}}{\partial I_2^2} \end{bmatrix} = \frac{1}{(2\pi)^2} \frac{1}{\psi_h^4} (\psi_{hh} \psi_{ff} - \psi_{hf}^2).$$

Thus the proof of lemma 1 is completed.

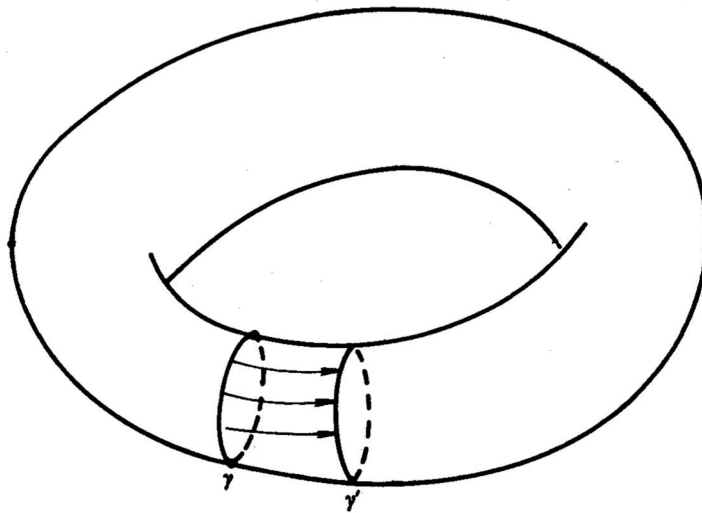


Fig.3 Deformation of the cycle γ

Similarly, we have
LEMMA 2.

$$(2\pi)\psi_h^3 \det \begin{bmatrix} \frac{\partial^2 \tilde{H}}{\partial I_1^2} & \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial \tilde{H}}{\partial I_1} \\ \frac{\partial^2 \tilde{H}}{\partial I_1 \partial I_2} & \frac{\partial^2 \tilde{H}}{\partial I_2^2} & \frac{\partial \tilde{H}}{\partial I_2} \\ \frac{\partial \tilde{H}}{\partial I_1} & \frac{\partial \tilde{H}}{\partial I_2} & 0 \end{bmatrix} = \psi_{ff}.$$

The proof of Lemma 2 is a straightforward computation using Lemma 1. It is easy to be seen that

$$\psi_h = \int_{\gamma} \frac{r dr}{y} \neq 0$$

in U_r [9]. So, instead of theorems 1 and 2 we shall prove the equivalent to them.

THEOREM 3. For the $(h,f) \in U_r$ the determinant

$$D = \det \begin{bmatrix} \frac{\partial^2 \psi}{\partial h^2} & \frac{\partial^2 \psi}{\partial h \partial f} \\ \frac{\partial^2 \psi}{\partial h \partial f} & \frac{\partial^2 \psi}{\partial f^2} \end{bmatrix}$$

does not vanish.

THEOREM 4. For the $(h,f) \in U_r$ the expression $D_1 = \psi_{ff}$ does not vanish.

The proofs will be done in section 4.

Next we would like to show that the entries of D (and D_1) can be represented as elliptic integrals. If we differentiate (2.5) twice formally, we get the following expressions

$$\begin{aligned}
 (3.3) \quad \frac{\partial^2 \psi}{\partial h^2} &= -\sqrt{(1+M)} \int_{\gamma} \frac{r^3 dr}{y^3}, \\
 \frac{\partial^2 \psi}{\partial h \partial f} &= f \sqrt{(1+M)} \int_{\gamma} \frac{r dr}{y^3}, \\
 \frac{\partial^2 \psi}{\partial f^2} &= -\sqrt{(1+M)} \left[\int_{\gamma} \frac{dr}{ry} + f^2 \int_{\gamma} \frac{dr}{ry^3} \right].
 \end{aligned}$$

(We have used $y \partial y / \partial f = -f$, $y \partial y / \partial h = r^2$ which follows by differentiating the curve $\Gamma_{h,f}$.)

The differential forms containing y^{-3} have poles along γ . Standart way to get rid of the poles on the integration path is to consider $\Gamma_{h,f}$ as a complex curve. Topologically it is a torus from which one point is removed [9]. The differential form $yr^{-1}dr$ is holomorphic on γ . If we deform cycle γ into cycle γ' homological to γ (Fig.3) on which functions y and r have no poles, nor zeroes then by Cauchy's theorem [9] the function $\psi(h,f)$ can be defined by the integral (2.5) on γ' instead of γ . After these notes it's clear that the derivatives (3.3) are well defined. We again denote γ' by γ .

4. Proof of the results

The proofs will be done with the help of several lemmas. First, we shall need the functions

$$(4.1) \quad w_j = \int_{\gamma} \frac{r^j dr}{y^3}, \quad j = 0, 1.$$

The next lemma gives a representation of D as a quadratic form in w_0, w_1 .

LEMMA 3. The determinant D has the following representation

$$(4.2) \quad D = -\left[1 + M\right] f^2 w_1 \left[w_1 - 2hw_0/3s\right].$$

PROOF :We have to express derivatives (3.3) by the integrals w_0, w_1 . Obviously

$\psi_{hf} = \sqrt{(1+M)} f w_1$. For ψ_{ff} we make the following transformations .

$$\frac{\partial^2 \psi}{\partial f^2} = -\sqrt{(1+M)} \left[\int_{\gamma} \frac{dr}{ry} + f^2 \int_{\gamma} \frac{dr}{ry^3} \right] =$$

$$= -\sqrt{(1+M)} \int \frac{y^2 + f^2}{ry^3} dr = -\sqrt{(1+M)} \int \frac{2hr - 2sr^2}{y^3} dr.$$

So,

$$\psi_{ff} = -2h\sqrt{(1+M)} w_1 + 2s\sqrt{(1+M)} \int \frac{dr^2}{y^3}.$$

Now, we express integrals $\int \frac{r^2 dr}{y^3}$ and $\int \frac{r^3 dr}{y^3}$ via w_0 and w_1 in

the following way :

$$\begin{aligned} \int \frac{r^2 dr}{y^3} &= \frac{1}{3} \int \frac{dr^3}{y^3} = -\frac{1}{6s} \int \frac{d(-2sr^3)}{y^3} = \\ &= -\frac{1}{6s} \int \frac{d(y^2 - 2hr^2 + f^2)}{y^3} = -\frac{1}{3s} \left[\int \frac{dy}{y^2} - 2h \int \frac{r dr}{y^3} \right] = \\ &= (2h/3s) w_1. \end{aligned}$$

Similarly,

$$\begin{aligned} \int \frac{r^3 dr}{y^3} &= -\frac{1}{6s} \int \frac{rd(-2sr^3)}{y^3} = -\frac{1}{6s} \int \frac{rd(y^2 - 2hr^2 + f^2)}{y^3} = \\ &= -\frac{1}{3s} \left[\int \frac{r dy}{y^2} - 2h \int \frac{r^2 dr}{y^3} \right]. \end{aligned}$$

Integrating first summand by parts, we obtain

$$\int \frac{r^3 dr}{y^3} = (f^2/s) w_0.$$

Consequently

$$\psi_{hh} = -\sqrt{(1+M)} (f^2/s) w_0, \quad \psi_{ff} = -\sqrt{(1+M)} (2h/3) w_1,$$

from where we obtain the representation (4.2). Thus the proof of lemma 3 is

completed.

Earlier in [7] was proved that $w_0 < 0$. We introduce the function $\delta = w_1/w_0$. Then D and D_1 have the following form

$$(4.3) \quad D = - (1 + M) f^2 w_0^2 \delta \left[\delta - (2h/3s) \right],$$

$$(4.4) \quad D_1 = - (2/3)h \sqrt{(1+M)} w_0 \delta.$$

Next, we need some other functions for the study of δ . In order to introduce them, we put the family of elliptic curves $\Gamma_{h,f}$ into the normal form :

$$(4.5) \quad \Gamma_p = \left\{ (u,v) \in \mathbb{C}^2, v^2 = 2(u^3 - 3u + p) \right\}$$

by the translation $r = x + (h/3s)$ and the rescaling $y = \alpha v$, $x = \beta u$, where

$$(4.6) \quad \beta = -h/(3s), \quad \alpha^2 = -s\beta^3.$$

If we put

$$(4.7) \quad p(h,f) = \frac{4h^3 - 27s^2f^2}{2h^3},$$

we get (4.5). In these variables the integrals w_0 and w_1 become

$$w_0 = \frac{\beta}{\alpha^3} \int_{\gamma(p)} \frac{du}{v^3}, \quad w_1 = \frac{\beta}{\alpha^3} \int_{\gamma(p)} \frac{(\beta u + h/3s) du}{v^3}.$$

We introduce the new functions

$$\theta_0(p) = \int_{\gamma(p)} \frac{du}{v^3}, \quad \theta_1(p) = \int_{\gamma(p)} \frac{udu}{v^3}$$

and their ratio

$$\rho(p) = \theta_1(p)/\theta_0(p).$$

In these notations we have

$$\delta = \beta\rho + h/3s = (1 - \rho)h/3s.$$

Then, in these variables expressions D and D_1 become

$$(4.8) \quad D = -(1+M)f^2 \frac{\beta^2}{\alpha^6} \frac{h^2}{9s} \theta_0^2(\rho+1)(\rho-1),$$

$$(4.9) \quad D_1 = -2 \sqrt{(1+M)} \frac{\beta}{\alpha^3} \frac{h^2}{9s} \theta_0(1-\rho)$$

Note that $p|_{f=0} = 2$ and $p|_{f=8h^3/27s^2} = -2$.

The following result from [4] is crucial for the proofs of the theorems.

LEMMA 4. (Horozov, [4]) (i) The function $\rho(p)$ is strictly monotonous decreasing in the interval $[-2, 2]$,

(ii) $\rho(-2) = 7/5$, $\rho(2) = 1$.

PROOF of the theorems 3 and 4 .

For the D and D_1 we have expressions (4.8) and (4.9)

$$D = -(1+M)f^2 \frac{\beta^2}{\alpha^6} \frac{h^2}{9s} \theta_0^2(\rho+1)(\rho-1),$$

$$D_1 = -2 \sqrt{(1+M)} \frac{\beta}{\alpha^3} \frac{h^2}{9s} \theta_0(1-\rho).$$

Lemma 4 shows that D and D_1 vanishes on the boundary of the set U_r . Thus the proof of the theorems is completed.

REMARK 1. The condition of isoenergetical non-degeneracy guarantees stability of action variables in the sense that they remain always near to their initial values when the perturbations are small (see [2, app. 8]).

REMARK 2. Our results are valid also for the system consisting of a particle moving in the central field with potential $V=ar$, $a > 0$.

Acknowledgements

Acknowledgements are due to E.Horozov and A.Yanovsky for the valuable discussion.

R e f e r e n c e s

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Received on October 28, 1991