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# The Influence of Boundary Conditions on the Stress Singularities in a Biwedge

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## Stress Singularities in a Biwedge

Consider a composite body consisting of two homogeneous, elastic and isotropic wedges perfectly bonded along their common interface under conditions of plane strain or generalized plane stress. A system of polar coordinates is related to the body with the origin 0 at the vertex of the biwedge and the polar-axis along the bonded interface. The wedge numbered 1 occupies the sector between the lines $\vartheta=0°$ ano $\vartheta=\varphi_1$, while the wedge 2 occupies the sector between the lines $\vartheta=0$ and $\vartheta=\varphi_2$.

According to the complex formulation of the plane stress problem, the continuity of stresses and displacements along the bonded interface of the biwedge may be expressed by the following equations respectively:

$$\varphi_1(t)+\overline{t\varphi_1'(t)}+\overline{\psi_1(t)}=\varphi_2(t)+\overline{t\varphi_2'(t)}+\overline{\psi_2(t)} \tag{1}$$

$$\Gamma[u_1\varphi_1(t)-\overline{t\varphi_1'(t)}-\overline{\psi_1(t)}]=u_2\varphi_2(t)-\overline{t\varphi_2'(t)}-\overline{\psi_2(t)}, \tag{2}$$

where $\varphi(z)$, $\psi(z)$ are complex functions, $t$ is the complex variable along the interface, $u=(3-4\nu)$ for plane strain and $u=(3-\nu)/(1+\nu)$ for plane stress, $\nu$ is Poisson's ratio, $\Gamma=G_2/G_1$, and $G_{1,2}$ is the shear modulus of each material of the biwedge. Primes denote differentiation with respect to $t$ and indices 1 and 2 are used for the two materials of the biwedge.

For the case of the first fundamental problem when the two faces of the biwedge at the vicinity of the vertex are free from stresses the boundary conditions along these faces are expressed by successively zeroing the left and right members of relation (1) with $t=re^{i\varphi_1}$ and $t=re^{i\varphi_2}$ respectively. Similarly, for the case of the second fundamental problem when the two faces of the biwedge at the vicinity of the vertex are clamped we must put equal to zero the left and right members of relation (2) separately when we introduce to them the values $t=re^{i\varphi_1}$ and $t=re^{i\varphi_2}$, respectively. For the mixed fundamental problem when the face $\vartheta=\varphi_1$ is clamped and the face $\vartheta=\varphi_2$ is free, the boundary conditions are obtained by successively zeroing the first member of relation (2) with $t=re^{i\varphi_1}$ and the second member of relation (1) with $t=re^{i\varphi_2}$.

As we are interested for the particular behaviour of complex functions $\varphi(z)$ and $\psi(z)$ at the close vicinity of the vertex of the biwedge we put these functions in their asymptotic forms as follows:

(3) $$\varphi_1(z)=a_{11}z^{\lambda}+a_{21}z^{\overline{\lambda}},\ \psi_1(z)=b_{11}z^{\lambda}+b_{21}z^{\overline{\lambda}}$$

for material 1 and:

(4) $$\varphi_2(z)=a_{12}z^{\lambda}+a_{22}z^{\overline{\lambda}},\ \psi_2(z)=b_{12}z^{\lambda}+b_{22}z^{\overline{\lambda}}$$

for material 2.

Substituting these forms of the stress functions $\varphi_{1,2}(z)$ and $\psi_{1,2}(z)$ into the appropriate boundary conditions for each of the corresponding fundamental problem considered, and seeking a non-trivial solution of the derived homogeneous system, we obtain the following equations for the determination of the order of stress singularity:

(5) $$\Delta_1=\begin{vmatrix} 1-e^{2i\lambda\varphi_1} & \lambda(1-e^{2i\varphi_1}) & 1-e^{2i\lambda\varphi_2} & \lambda(1-e^{2i\varphi_2}) \\ \lambda(1-e^{-2i\varphi_1}) & 1-e^{-2i\lambda\varphi_1} & \lambda(1-e^{-2i\varphi_2}) & 1-e^{-2i\lambda\varphi_2} \\ \Gamma(u_1+e^{2i\lambda\varphi_1}) & -\Gamma\lambda(1-e^{2i\varphi_1}) & u_2+e^{2i\lambda\varphi_2} & -\lambda(1-e^{2i\varphi_2}) \\ -\Gamma\lambda(1-e^{-2i\varphi_1}) & \Gamma(u_1+e^{-2i\lambda\varphi_1}) & -\lambda(1-e^{-2i\varphi_2}) & u_2+e^{-2i\lambda\varphi_2} \end{vmatrix}=0$$

for the first fundamental problem,

(6) $$\Delta_2=\begin{vmatrix} 1+u_1e^{2i\lambda\varphi_1} & \lambda(1-e^{2i\varphi_1}) & 1+u_2e^{2i\lambda\varphi_2} & \lambda(1-e^{2i\varphi_2}) \\ \lambda(1-e^{-2i\varphi_1}) & 1+u_1e^{-2i\lambda\varphi_1} & \lambda(1-e^{-2i\varphi_2}) & 1+u_2e^{-2i\lambda\varphi_2} \\ \Gamma u_1(1-e^{2i\lambda\varphi_1}) & -\Gamma\lambda(1-e^{2i\varphi_1}) & u_2(1-e^{2i\lambda\varphi_2}) & -\lambda(1-e^{2i\varphi_2}) \\ -\Gamma\lambda(1-e^{-2i\varphi_1}) & \Gamma u_1(1-e^{-2i\lambda\varphi_1}) & -\lambda(1-e^{-2i\varphi_2}) & u_2(1-e^{-2i\lambda\varphi_2}) \end{vmatrix}=0$$

for the second fundamental problem, and

(7) $$\Delta_3=\begin{vmatrix} 1+u_1e^{2i\lambda\varphi_1} & \lambda(1-e^{2i\varphi_1}) & 1-e^{2i\lambda\varphi_2} & \lambda(1-e^{2i\varphi_2}) \\ \lambda(1-e^{-2i\varphi_1}) & 1+u_1e^{-2i\lambda\varphi_1} & \lambda(1-e^{-2i\varphi_2}) & 1-e^{-2i\lambda\varphi_2} \\ \Gamma u_1(1-e^{2i\lambda\varphi_1}) & -\Gamma\lambda(1-e^{2i\varphi_1}) & u_2+e^{2i\lambda\varphi_2} & -\lambda(1-e^{2i\varphi_2}) \\ -\Gamma\lambda(1-e^{-2i\varphi_1}) & \Gamma u_1(1-e^{-2i\lambda\varphi_1}) & -\lambda(1-e^{-2i\varphi_2}) & u_2+e^{-2i\lambda\varphi_2} \end{vmatrix}=0$$

for the mixed fundamental problem.

Developing the determinants (5) to (7) we obtain the following equations yielding $\lambda$ for the case of the first, second and third fundamental problems respectively:

(8) $$(1+u_2)^2H(\varphi_1,1,\lambda)+\Gamma^2(1+u_1)^2H(\varphi_2,1,\lambda)+4(\Gamma-1)F(\varphi_2,u_2,\lambda)H(\varphi_1,1,\lambda)$$
$$-4\Gamma(\Gamma-1)F(\varphi_1,u_1,\lambda)H(\varphi_2,1,\lambda)+\Gamma(1+u_1)(1+u_2)[H(\varphi_1-\varphi_2,1,\lambda)$$
$$-H(\varphi_1,1,\lambda)-H(\varphi_2,1,\lambda)]=0$$

(9) $$(1+u_1)^2H(\varphi_2,u_2^2,\lambda)+\Gamma^2(1+u_2)^2H(\varphi_1,u_1^2,\lambda)+4(\Gamma-1)F(\varphi_1,u_1,\lambda)H(\varphi_2,u_2^2,\lambda)$$
$$-4\Gamma(\Gamma-1)F(\varphi_2,u_2,\lambda)H(\varphi_1,u_1^2,\lambda)+\Gamma(1+u_1)(1+u_2)[H(\varphi_1-\varphi_2,u_1u_2,\lambda)$$
$$-H(\varphi_1,u_1u_2,\lambda)-H(\varphi_2,u_1u_2,\lambda)]=0$$

(10) $$(1+u_1)^2(1+u_2)^2-4(1+u_1)^2F(\varphi_2,u_2,\lambda)-4(1+u_2)^2F(\varphi_1,u_1,\lambda)$$
$$+4\Gamma(1+u_1)(1+u_2)[F(\varphi_1,u_1,\lambda)+F(\varphi_2,u_1,\lambda)-F(\varphi_1-\varphi_2,u_1,\lambda)]$$
$$+16\Gamma(\Gamma-1)H(\varphi_1,u_1^2,\lambda)H(\varphi_2,1,\lambda)-16(\Gamma-1)F(\varphi_1,u_1,\lambda)F(\varphi_2,u_2,\lambda)=0,$$

where we define the functions $F(x, \nu, \lambda)$ and $H(x, \nu, \lambda)$ by relations:

$$F(x, \nu, \lambda) = \nu \sin^2 \lambda x + \lambda^2 \sin^2 x$$
(11)
$$H(x, \nu, \lambda) = \nu \sin^2 \lambda x - \lambda^2 \sin^2 x.$$

For the special case, when $\Gamma = 0$, which condition is satisfied either if $G_1 \to \infty$, or if $G_2 = 0$, equations (8), (9) and (10) take the following forms respectively:

$$[(1+u_2)^2 - 4F(\varphi_2, u_2, \lambda)]H(\varphi_1, 1, \lambda) = 0 \tag{12}$$

for the first fundamental problem,

$$[(1+u_1)^2 - 4F(\varphi_1, u_1, \lambda)]H(\varphi_2, u_2^2, \lambda) = 0 \tag{13}$$

for the second fundamental problem, and

$$[(1+u_1)^2 - 4F(\varphi_1, u_1, \lambda)]\,[(1+u_2)^2 - 4F(\varphi_2, u_2, \lambda)] = 0 \tag{14}$$

for the mixed fundamental problem.

Similarly, for the case when $\Gamma \to \infty$ ($G_1 = 0$ or $G_2 \to \infty$) equations (8), (9) and (10) take the following forms respectively:

$$[(1+u_1)^2 - 4F(\varphi_1, u_1, \lambda)]H(\varphi_2, 1, \lambda) = 0 \tag{15}$$

for the first fundamental problem,

$$[(1+u_2)^2 - 4F(\varphi_2, u_2, \lambda)]H(\varphi_1, u_1^2, \lambda) = 0 \tag{16}$$

for the second fundamental problem, and

$$H(\varphi_1, u_1^2, \lambda)H(\varphi_2, 1, \lambda) = 0 \tag{17}$$

for the mixed fundamental porblem.

For the case of a semi-infinite crack lying along the interface of two bonded dissimilar half-planes, it is obtained from equations (8), (9) and (10) respectively that:

$$\sin^2 \lambda\pi = \frac{[(\Gamma u_1 + u_2) + (\Gamma + 1)]^2}{4(\Gamma u_1 + 1)(\Gamma + u_2)} \tag{18}$$

for the first fundamental problem,

$$\sin^2 \lambda\pi = \frac{[(\Gamma u_1 + u_2) + u_1 u_2(\Gamma + 1)]^2}{4u_1 u_2(\Gamma u_1 + 1)(\Gamma + u_2)} \tag{19}$$

for the second fundamental problem and

$$16u_1(\Gamma u_1 + 1)(\Gamma + u_2)\sin^4 \lambda\pi - 4[u_1(1+u_2)^2 + u_2^2(1+u_1)^2 + 2\Gamma u_1(1+u_1)(1+u_2)]\sin^2 \lambda\pi + (1+u_1)^2(1+u_2)^2 = 0 \tag{20}$$

for the mixed fundamental problem.

Equation (19), valid for the second fundamental problem, for the special case when the two half-planes have equal Poisson's ratios ($u_1 = u_2 = u$) coincides with equation (18) of the first fundamental problem.

Equations (18) and (19) for the case of a crack in a homogeneous and isotropic material give for the first and the second boundary value problems the well-known result $\lambda = 0.50$, whereas equation (20) for the mixed fundamental problem gives:

$$\lambda = \frac{1}{4} + \frac{\ln u}{4\pi} i \quad \text{or} \quad \lambda = \frac{3}{4} + \frac{\ln u}{4\pi} i, \tag{21}$$

which coincides with the result given by Muskhelishvili.

## Results for two particular geometries of the biwedge

We have studied in detail two particular geometrical configurations of the biwedge, i. e. I) the equal angle biwedge, and II) the case of an indenter of any angle acting on a half-plane. For each of these two cases we studied the variation of $\lambda$ with the geometry, the mechanical properties and the boundary conditions of the biwedge.

### a) The equal angle biwedge ($\varphi_1=\varphi$, $\varphi_2=-\varphi$)

The characteristic equations for the determination of the order of elastic stress singularity for the case of the equal angle biwedge and under

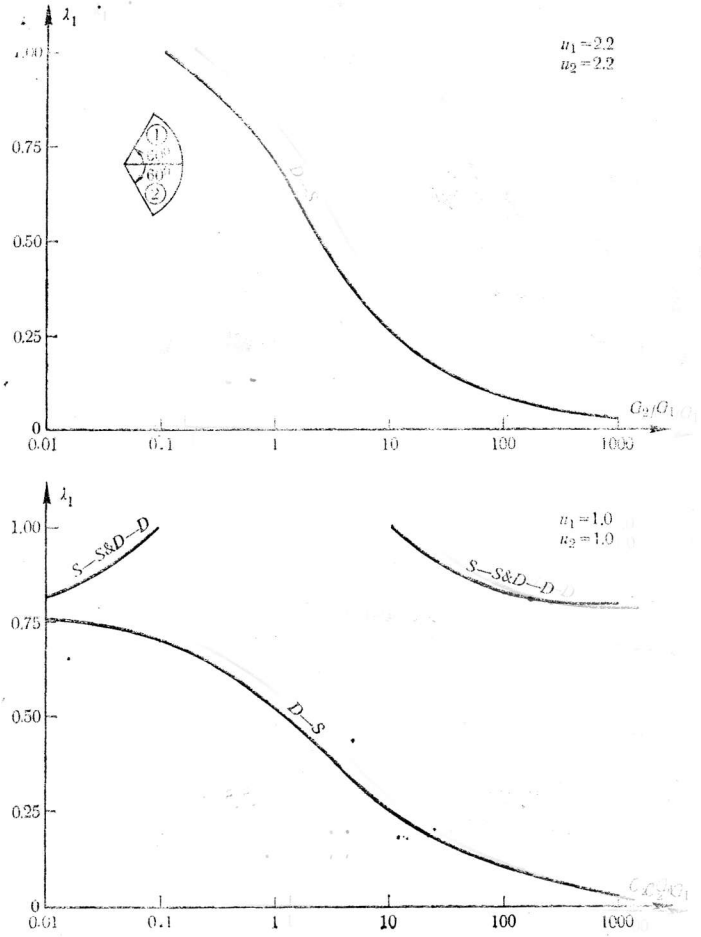



Fig. 1. Real $\lambda_1$ (continuous lines) and imaginary $\lambda_2$ (dashed lines) parts of $\lambda$ for the equal angle biwedge with $\varphi_1=-\varphi_2=60^0$ and $\mu_1=\mu_2=2.2$ (a), $\mu_1=\mu_2=1{,}0$ (b), $\mu_1=2.2$, $\mu_2=1.0$ (c) and $\mu_1=1{,}0$, $\mu_2=2.2$ (d), versus ratio $(G_2/G_1)$ of the shear moduli $G_1$ and $G_2$ under the first $(S-S)$, second $(D-D)$ and mixed $(D-S)$ fundamental problems

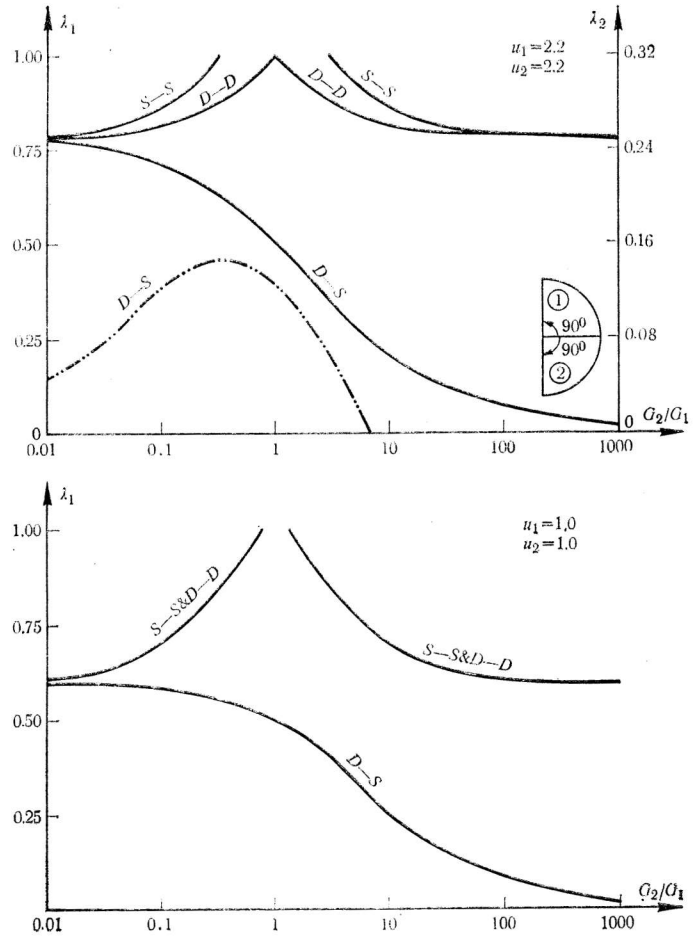



Fig. 2. As in Figure 1 with $\varphi_1=-\varphi_2=90°$

the first, second and mixed fundamental problems are obtained from equations (8), (9) and (10) respectively, by substituting into them the values $\varphi_1=\varphi$ and $\varphi_2=-\varphi$, where $2\varphi$ is the angle of the biwedge. These equations were solved numerically in a digital computer and the values of $\lambda$, corresponding to various angles $2\varphi$ of the biwedge with various material combinations of its constituent wedges were determined. From the roots of these equations we are interested only for those with the smallest real part of $\lambda$, since they give the strongest stress singularity expressed by $(1-\lambda)$. The real values $\lambda_1$ of $\lambda$ must lie into the interval $0<\lambda_1<1$, since the continuity of displacements at the vicinity of the vertex of the biwedge requires that $\lambda_1>0$, while for infinite stresses at the vertex of the biwedge $\lambda_1$ requires the condition $\lambda_1<1$.

Figures 1 to 3 present the values of $\lambda$ for angles of the biwedge $2\varphi=120°$, 180° and 360° and for $u_1=u_2=2.2$ ($\nu_1=\nu_2=0.2$ for plane strain and $\nu_1=\nu_2=0.25$ for plane stress) (case (a)), $u_1=u_2=1.0$ ($\nu_1=\nu_2=0.5$ for plane strain) (case (b)), $u_1=2.2$, $u_2=1.0$ (case (c)) and $u_1=1.0$, $u_2=2.2$ (case (d)), versus the ratio $(G_2/G_1)$ of shear moduli of the materials of the biwedge $(0.01<(G_2/G_1)<1000)$. In these figures the real $\lambda_1$ and imaginary $\lambda_2$ parts of $\lambda$ are presented by con-

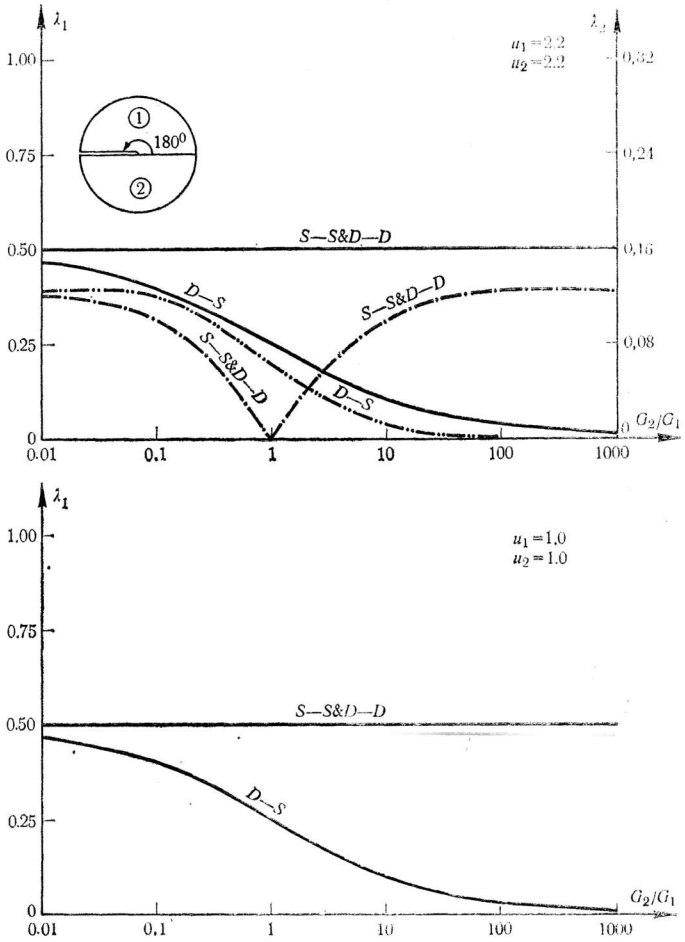



Fig. 3. As in Figure 1 with $\varphi_1=-\varphi_2=180°$

tinuous and dashed lines respectively under the first ($S-S$), the second ($D-D$) and the mixed ($D-S$) fundamental problems.

From figures 1 to 3 in combination with equations (8), (9) and (10) with $\varphi_1=\varphi$, $\varphi_2=-\varphi$ the following conclusions can be derived for the equal angle biwedge:

I) The singularity, expressed by $(1-\lambda)$, increases with angle $2\varphi$ of the biwedge.

II) The singularity becomes more severe for mixed boundary conditions. For this case, curves $\lambda=\lambda(G_2/G_1)$ have a sigmoid form, tending to $\lambda=0$, as the ratio $(G_2/G_1)$ tends to infinity. For this case the singularity becomes more severe as $(G_2/G_1)$ increases.

III) The singularities for the equal angle biwedge with $u_1=u_2=1$ for the first and second fundamental problems coincide (see equations (8) and (9)).

IV) For the first and second fundamental problems and for $u_1=u_2$ singularity curves are symmetric with respect to the point $(G_2/G_1)=1$, since symmetric positions correspond to identical biwedges, whereas this does not occur for the mixed fundamental problem, due to asymmetry in boundary conditions.

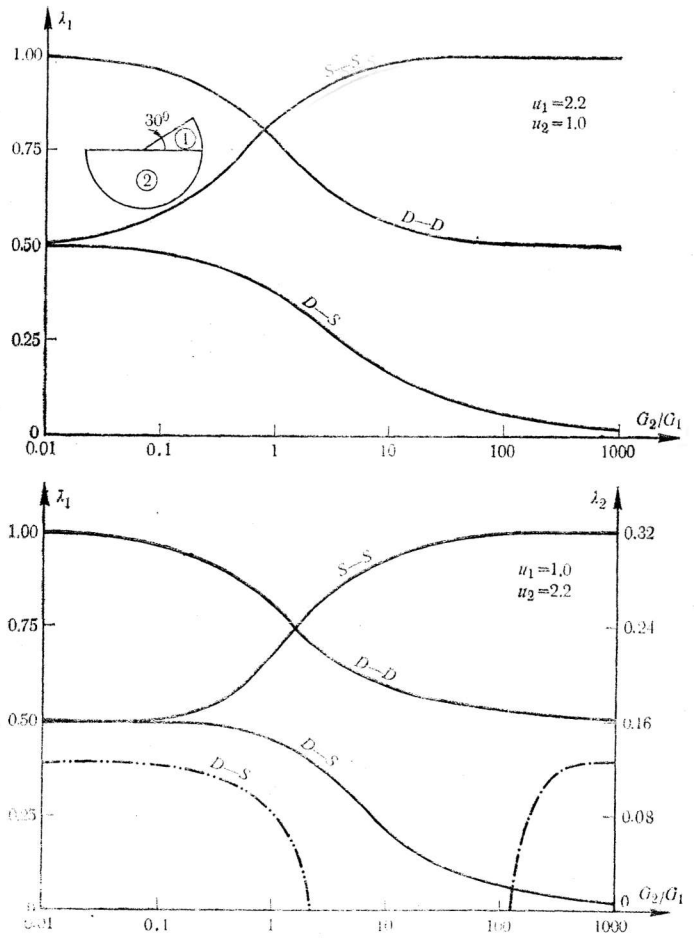



Fig. 4. Real $\lambda_1$ (continuous lines) and imaginary $\lambda_2$ (dashed lines) parts of $\lambda$ for the case of a plane indenter of angle $\varphi=30°$ acting on a half-plane with $u_1=u_2=2.2$ (a), $u_1=u_2=1.0$ (b), $u_1=2.2$, $u_2=1.0$ (c) and $u_1=1.0$, $u_2=2.2$ (d), versus ratio $(G_2/G_1)$ of the shear moduli $G_1$ and $G_2$ under the first $(S-S)$, second $(D-D)$ and mixed $(D-S)$ fundamental problems

V) The value of singularity for $u_1=u_2$ and for each angle $2\varphi$ of the biwedge and for all three boundary conditions studied increases as the value of $u_1=u_2=u$ decreases.

VI) For the first and second fundamental problems the singularity does not have imaginary values for angles $2\varphi$ of the biwedge less than 180° ($2\varphi<180°$), whereas for the mixed fundamental problem this occurs for an angle $2\varphi$ less than 90° ($2\varphi<90°$). As angle $2\varphi$ increases the imaginary parts $\lambda_2$ of $\lambda$ also increase and the greatest values of $\lambda_2$ occur for $2\varphi=360°$.

VII) Comparing the influence of the angle of the biwedge and the type of boundary conditions on the stress singularity at the vertex of the biwedge, it can be concluded that the boundary conditions of the biwedge are more critical than the influence of its angle and, furthermore, from all boundary conditions the most severe is that of the mixed type.

**b) The plane indenter of angle $\varphi$ acting on a half-plane**

The characteristic equations for the determination of the order of elastic stress singularity for the case of a plane indenter of angle $\varphi$ acting on a half-plane under the first, second and mixed fundamental problems are obtained by substituting into equations (8), (9) and (10) respectively $\varphi_1=\varphi$ and $\varphi_2=-\pi$. These equations were solved numerically in a computer for various values of angle $\varphi$ of the indenter and for various material combinations of the indenter and the half-plane.

Figures 4 to 6 present the real $\lambda_1$ (continuous lines) and imaginary $\lambda_2$ (dashed lines) parts of $\lambda$ for angles $\varphi$ of the indenter equal to $\varphi=30°$, 90° and 150° and for $u_1=u_2=2.2$ (case (a)), $u_1=u_2=1.0$ (case (b)), $u_1=2.2$, $u_2=1.0$ (case (c)) and $u_1=1.0$, $u_2=2.2$ (case (d)), versus the ratio $(G_2/G_1)$, while the case of $\varphi=180°$ is presented in Figure 3.

The following conclusions can be derived:

I) The singularity increases with angle $\varphi$ of the indenter.

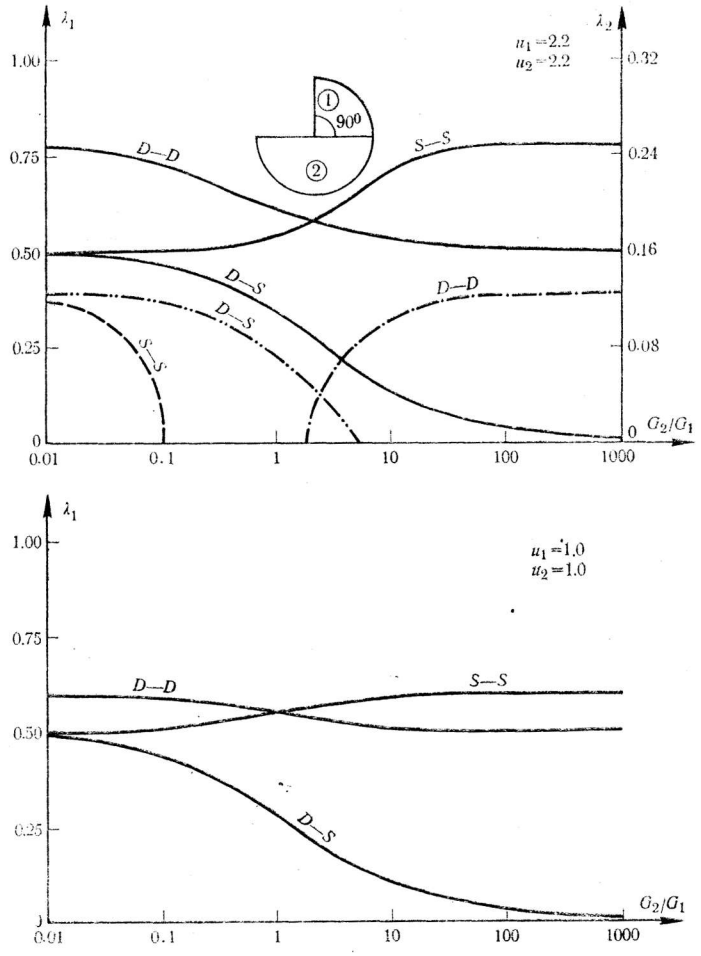



Fig. 5. As in Figure 4 with $\varphi=90°$

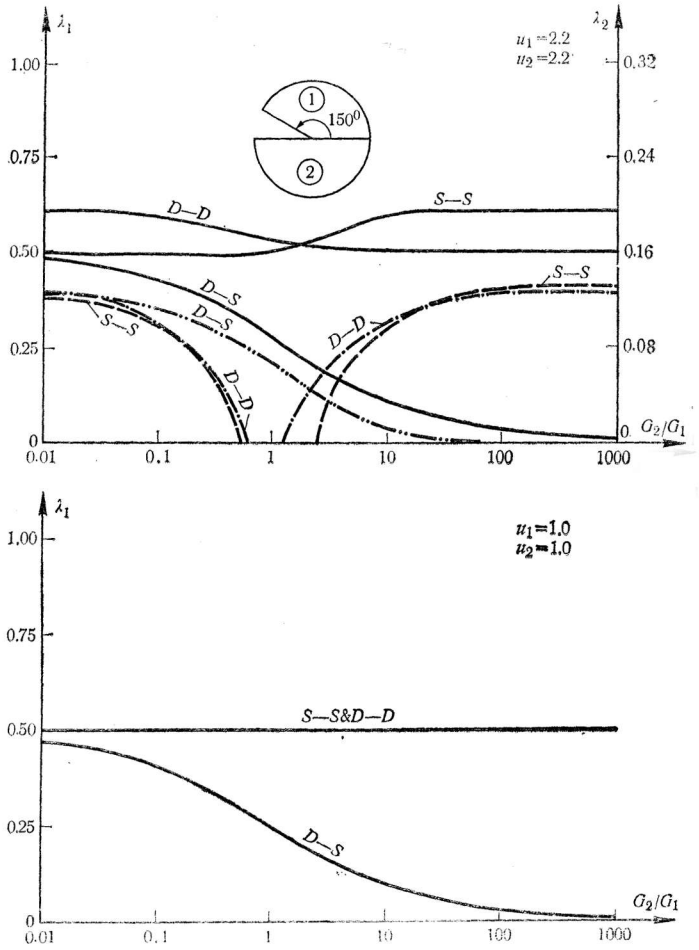



Fig. 6. As in Figure 4 with $\varphi=150°$

II) The singularity becomes more severe for mixed boundary conditions. For this case, as in the equal angle biwedge, the singularity increases with $(G_2/G_1)$ and $\lambda$ tends to zero as $(G_2/G_1)$ tends to infinity.

III) When the ratio $(G_2/G_1)$ tends to zero the real part $\lambda_1$ of the singularity at the vertex of the biwedge for the first and mixed fundamental problems tends to $\lambda_1=0.50$, corresponding to a half-plane with mixed boundary conditions, whereas for the second fundamental problem it tends to that corresponding to a single wedge of angle $\varphi$ with mixed boundary conditions.

IV) When the ratio $(G_2/G_1)$ tends to infinity the real part $\lambda_1$ of the singularity for the first fundamental problem tends to that of a single wedge of angle $\varphi$ under mixed boundary conditions, while for the second fundamental problem it tends to $\lambda_1=0.50$ and for the mixed fundamental problem it tends to zero.

V) The variation of $\lambda$ versus the ratio $(G_2/G_1)$ forms increasing sigmoid curves for the first fundamental problem and decreasing sigmoid curves for the second fundamental problem.

VI) For $u_1=u_2=u$ the stress singularity decreases for all angles $\varphi$ of the indenter with $u$ for all fundamental problems.

VII) For the case when $u_1=u_2=1$ ($\nu=0.5$) of each angle $\varphi$ of the indenter the singularity for the first fundamental problem equals that corresponding to the second fundamental problem when the materials of the indenter and the half-plane are interchanged.

VIII) Conclusion No. VII for the equal-angle biwedge is also valid for the case of a plane indenter acting on a half-plane.

## Conclusions

A simple and versatile method for the determination of the order of the elastic singularity $\lambda$ at the vertex of a biwedge based on the theory of complex potentials was presented. The characteristic equations for the determination of $\lambda$ for the cases of the first, second and mixed fundamental problems when the two wedges of the biwedge are perfectly bonded were derived. Numerical examples for two particular geometrical configurations of the biwedge, that is, the equal angle biwedge and the case of a plane indenter acting on a half-plane were given. Interesting laws for the dependence of the order of singularity on the geometry, the mechanical properties and the boundary conditions of the biwedge were disclosed.

*Rec. 4. V. 1977*